# PRACTICE FOR FINAL EXAM

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## 1. All the Previous Tests

## 2. Vector Operations and Equations of planes and surfaces

**2.1.** Let $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{c} = \vec{j} - 5\vec{k}$.

(1) Find $\vec{a} \times (\vec{b} + \vec{c})$.
(2) Find $\vec{a} \cdot \vec{b}$.
(3) Area of the parallelogram generated by $\vec{a}$ and $\vec{b}$. What does it mean if the area is 0 ?
(4) Volume of the parallelpiped generated by $\vec{a}$, $\vec{b}$ and $\vec{c}$. What does it mean if the volume is 0 ?
(5) Angel between $\vec{a}$ and $\vec{b}$.
(6) Find the projection of $\vec{a}$ onto $\vec{b}$.

**2.2. Find the equations of following lines or planes.**

(1) Angle between 2 lines.
(2) Angle between 2 planes.
(3) The line that passes through 2 points.
(4) The plane that passes through 3 points.
(5) The line that passes through a given point and orthogonal to a given plane.
(6) The plane that is orthogonal to a given line and contains a point.
(7) The line cut by 2 planes.
(8) The point cut by 3 planes.

**2.3. Find the following distances.**

(1) Distance between a point to a line.
(2) The distance between two lines. **Here you need to determine what is the relatuion of these two lines, do they intersect, or are they parallel, or are they skew lines?**

*Date*: April 28, 2017.

(3) Distance between a point and a plane.
(4) Distance between a point to a general surface. (See Max/Min problrms)

## 3. Vector Functions and Space Curves

**3.1.** Let $\vec{r}(t) =< \ln(x^2 - x - 2), \frac{x^2-4}{x-2}, 2xe^{x^2} >$ be a vector function.

(1) Find the domain of $\vec{r}(t)$
(2) Find the 1st derivetive $\vec{r}'(t)$.
(3) Find the following limits: $\lim_{t\to 2} \vec{r}(t)$ and $\lim_{t\to 1} \vec{r}(t)$.
(4) Find $\int \vec{r}'(t)dt$

**3.2.** Let $\vec{r}(t) =< 2\cos t, \sin t, t >$ be a space curve.

(1) Find the equation of the tangent line at $(2, 0, 0)$.
(2) Find the arc length of the curve from $(2, 0, 0)$ to $(-2, 0, \pi)$.
(3) Find the frenet frames, i.e., the unite tangent vector $\vec{\mathbf{T}}$, the normal vector $\vec{\mathbf{N}}$, and the binomal vector $\vec{\mathbf{B}}$.
(4) Find the curvature function $\kappa(t)$. At which points does the curve reaches the maximal/minal curvature ?
(5) Find the equations of the osculating plane and the normal plane of the curve.

## 4. Function of Several Variables

**4.1.** Show that

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$$

doesn't exist

**4.2. Maximal and Minimal Problem.** Find the Absolute Max and Absolute Min of the following function $f(x, y)$ with the domain $D$.

$$f(x, y) = x^2 + y^2 - 4x - 2y$$

and $D$ is the closed disk $x^2 + y^2 = 20$.

**4.3. Implicit Differantion.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ if

$$x^2 + y^2 + z^2 = z.$$

Then find the equation of the tangent plane to the surface at point $(0, 0, 1)$.

## 5. Multiple Integrals

### 5.1. Sketch, REMEMBER, and PARAMETRIZE the following surfaces.

(1) $x + y + z = 1$, $z = 1 + x$
(2) $x^2 + y^2 + z^2 = 1$, $2x^2 + 3y^2 + z^2 = 1$.
(3) $x^2 + y^2 + z^2 = 2x$.
(4) $z = 1 - x^2 - y^2$, $z = x^2 + y^2$, $z = \sqrt{x^2 + y^2}$.
(5) $x^2 + y^2 = 1$, $z = x^2$, $x^2 - y^2 = z$.
(6) THe 'Icercam': $x^2 + y^2 + z^2 = z$ intersect with $z = \sqrt{x^2 + y^2}$.

### 5.2. Sketch the Regions and Compute the following integrals.

(1) $\iint_R xydA$, $R = \{1 \leq y \leq 1; y^2 \leq x \leq y + 2\}$.
(2) $\iint_R xydA$, $R$ is the region in the first quadrant bounded by $x = y^2$ and $x = 8 - y$.
(3) $\iint_R (x^2 + y^2)^{3/2} dA$, $R$ is the region in the first quadrant bounded by $y = 0$, $y = \sqrt{3}x$, and the circle $x^2 + y^2 = 1$.
(4) $\iint_R (x^2+y^2)^{3/2} dA$, $R$ is the region in the upper half plane bounded by $x^2+y^2 = 1$ and $x^2 + y^2 = 4$.
(5) Find the areas of the abouve regions.

### 5.3. Compute the following integral by suitable change of coordinates.

(1) $\iint_R ydA$, $R$ is bounded by $y \geq 0$, $y^2 = 4-4x$, $y^2 = 4+4x$. [Hint: $x = u^2 - v^2$, $y = 2uv$.]
(2) $\iint_R e^{\frac{x+y}{x-y}} dA$, $R$ is the trapezodial region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, $(0, -1)$.
(3) $\iint_R \cos \frac{x+y}{x-y} dA$, $R$ is the trapezodial region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, $(0, 1)$.
(4) $\iint_R (x + y)e^{x^2-y^2} dA$, $R$ is the rectangle enclosed by $x - y = 0$, $x - y = 2$, $x + y = 0$, $x + y = 3$.

### 5.4. Sketch the Regions and Compute the following integrals.

(1) $\iiint_E xydV$, $E$ is the solid tetrahedron cut by plane $3x + y + z = 1$ with coordinate lines.
(2) $\iiint_E xydV$, $E$ is bounded by $x = 1 - y^2 - y^2$ and $x = 0$.
(3) $\iiint_E zdV$, $E$ is bounded by $y = 0$, $z = 0$, $x + y = 2$, and the cylinder $x^2 + y^2 = 1$.

(4) $\iiint_E z^3\sqrt{x^2+y^2+z^2}dV$, $E$ is the solid bounded by $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ and $z\geq 0$.
(5) Find the volumes of the above solids.

## 6. Line and Surface Integrals

### 6.1. Evaluate the following line integrals.

(1) $\int_C xds$, $C$ is the arc of $y=x^2$ from $(0,0)$ to $(1,1)$.
(2) $\int_C ydx+(x+y^2)dy$, $C$ is the the ellipse $4x^2+9y^2=36$ with **clockwise** orientation.
(3) $\int_C y^3dx+x^2dy$, $C$ is the arc of $x=1-y^2$ from $(0,-1)$ to $(0,1)$.
(4) $\int_C \vec{F}\cdot d\vec{r}$, here $\vec{F} =< e^z, xz, (x+y) >$, $C$ is given by $\vec{r}(t) =< t^2, t^3, -t >$, $0\leq t\leq 1$.

**6.2.** Let $\vec{F} =< e^y, xe^y+e^z, ye^z >$ be a vector field.

(1) Show that $\vec{F}$ is a conservative vector field.
(2) Find such a function $f(x,y,z)$ such that $\nabla f=\vec{F}$.
(3) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ for $C$ given by **any curve** from $(0,2,0)$ to $(4,0,3)$.
(4) Use the Fundamental Theorem of Calculus to verify your result.
(5) Find the arc length of the above curves.

### 6.3. Verify thet Green's Theorem is true for the following line integral.

$$\oint_C xy^2dx - x^2ydy$$

Here $C$ consists of the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$, and the line segment from $(1,1)$ to $(-1,1)$ .

### 6.4. Find the following line integral.

$$\int_C \sqrt{1+x^3}dx+2xydy.$$

Here $C$ is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

**6.5. Evaluate the following Surface integrals.**

(1) $\iint_S z dS$, $S$ is the part of $z = x^2 + y^2$ the lies under $z = 4$.
(2) $\iint_S (x^2 z + y^2 z) dS$, $S$ is the part of $4 + x + y - z = 0$ the lies inside $x^2 + y^2 = 1$.
(3) $\iint_S \vec{F} \cdot d\vec{S}$, here $\vec{F} =< xz, -2y, 3x >$, and $S$ is the sphere $x^2 + y^2 + z^2 = 4$ with **inside orientation**.
(4) $\iint_S \vec{F} \cdot d\vec{S}$, here $\vec{F} =< x^2, xy, z >$, and $S$ is the part of the paraboloid $z = x^2 + y^2$ below $z = 1$ with **upward orientation**.

## 7. IMPORTANT

**During the test I won't tell you whether you should use Green's Theorem or Stoke's Theorem or Divergence Theorem or other ways to compute the surface integrals/line integrals, you need to decide by yourself.**

## 8. Stoke's Theorem and the Divergence Theorem

**8.1.** Verify that the Stoke's Theorem is true for the vector field $\vec{F} =< x^2, y^2, z^2 >$, where $S$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the $xy$ plane. $S$ is oriented **downward**.

**8.2.** Use Stoke's Theorem to compute $\iint_S curl \vec{F} d\vec{S}$, where $\vec{F} =< x^2 yz, yz^2, z^3 e^{xy} >$, $S$ is the part of $x^2 + y^2 + z^2 = 5$ that lies above $z = 1$.

**8.3.** Use Stoke's Theorem to compute $\int_C \vec{F} d\vec{r}$. $\vec{F} =< xy, yz, zx >$, and $C$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented **clockwise**.

**8.4.** Use the Divergence Theorem to calculate $\iint_S \vec{F} d\vec{S}$. Here $\vec{F} =< x^3, y^3, z^3 >$, and $S$ is the boundary surface of the solid $E$ bounded by $x^2 + y^2 = 1$, $z = x + y$ and $z = 0$.

**8.5.** Use the Divergence Theorem to calculate $\iint_S \vec{F} d\vec{S}$. Here $\vec{F} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$, and $S$ is the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

**8.6.** Use the Divergence Theorem to calculate $\iint_S \vec{F} d\vec{S}$. Here $\vec{F} =< x^2 y, 1/3y^3 + \tan z^{1997}, xy^2 >$, and $S$ is the upper half semi-shpere $z = \sqrt{1 - y^2 - z^2}$.