MAC 2313-0010
Spring 2017
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Test 1
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Print Name: Answer Key

Signature: ____________

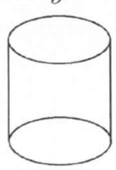
INSTRUCTIONS:

- Write answer in the space provided.
- Clearly show all work and circle/box answer.
- $\mathbb{R}$ denotes the real line.
- $\mathbb{R}^3$ denotes the three dimensional real space.
- There are 8 pages and 5 major problems in total.

KEEP CALM AND DO SOME CALCULUS !

1.(20pts). 'Trick or Treat'
Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) The equation $y^2 = 10 - x^2$ defines a surface in $\mathbb{R}^3$ that has the following



cylinder shape.

$x^2+y^2=10$

Yes, this is a cylinder of radius $\sqrt{10}$.

(2) (5pts) A space curve $\vec{r}(t)$ is a straight line if and only if $\dot{\vec{r}}(t)$ is a constant vector.

Yes. $\dot{\vec{r}}(t) = \text{constant} \iff \kappa(t) = 0 \iff$ straight line.

(3) (5pts) If $\vec{a} \neq \vec{0}$, and $\vec{a} \cdot \vec{c} = \vec{a} \cdot \vec{b}$, then $\vec{b} = \vec{c}$.

False. Take $\vec{c} = \vec{0}$ and $\vec{b} \perp \vec{a}$.

(4) (5pts) If $\vec{a} \neq \vec{0}$, and $\vec{a} \times \vec{c} = \vec{a} \times \vec{b}$, then $\vec{b} = \vec{c}$.

False. Take $\vec{c} = \vec{0}$ and $\vec{b} = \vec{a}$

2. (25pts). Let $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$, $\vec{b} = < 3, -2, 1 >$, $\vec{c} = \vec{j} - 2\vec{k}$.

(1) (5pts) Find $|\vec{a}|$.

$\vec{a} = <1, 2, -2>$

$|\vec{a}| = \sqrt{1^2 + 2^2 + (-2)^2} = 3.$

(2) (5pts) Find the angle between $\vec{a}$ and $\vec{c}$.

$\vec{a} \cdot \vec{c} = <1, 2, -2> \cdot <0, 1, -2>$

$= 1 \times 0 + 2 \times 1 + (-2) \times (-2) = 6$

$\cos\alpha = \dfrac{\vec{a} \cdot \vec{c}}{|\vec{a}| \cdot |\vec{c}|} = \dfrac{6}{\sqrt{1^2 + 2^2 + (-2)^2} \cdot \sqrt{0^2 + 1^2 + (-2)^2}} = \dfrac{6}{3\sqrt{5}} = \dfrac{2}{\sqrt{5}}$

$\alpha = \arccos \dfrac{2}{\sqrt{5}}.$

(3) (5pts) Find the projection of $\vec{a}$ on $\vec{c}$.

$\text{Proj}_{\vec{c}} \vec{a} = |\vec{a}| \cdot \cos\alpha \cdot \dfrac{\vec{c}}{|\vec{c}|} = |\vec{a}| \cdot \dfrac{\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{c}|} \cdot \dfrac{\vec{c}}{|\vec{c}|}$

$= \dfrac{\vec{a} \cdot \vec{c}}{|\vec{c}|^2} \vec{c} = \dfrac{6}{(\sqrt{5})^2} \cdot <0, 1, -2>$

$= <0, \dfrac{6}{5}, \dfrac{-12}{5}>$

(4) (5pts) Find the area of the parallelogram generated by $\vec{b}$ and $\vec{c}$.

Area $= |\vec{b} \times \vec{c}|$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \langle 3, 6, 3 \rangle$$

so $A = \sqrt{3^2+6^2+3^2} = 3\sqrt{1^2+2^2+1^2} = 3\sqrt{6}$

(5) (5pts) Find the volume of the parallelepiped generated by $\vec{a}, \vec{b}$ and $\vec{c}$.

Volume $= |\vec{a} \cdot (\vec{b} \times \vec{c})|$

$= |\langle 1, 2, -2 \rangle \cdot \langle 3, 6, 3 \rangle|$

$= |3 + 2 \times 6 + (-2) \times 3|$

$= |3 + 12 - 6|$

$= 9$

3.(~~35pts~~ 30). There are three lines in $\mathbb{R}^3$, L_1 is the intersection of two planes H_1: $x+y+z=1$ and H_2: $2x-y-4z=2$; L_2 is the line that passes through the point $(2,0,1)$ and perpendicular to the plane $2017+x+y+z=0$; L_3 is the line that lies in the xy plane given by equation $y=x$.

(1) (~~15~~ 15 pts) Find the equations of L_1, L_2, and L_3.

L_1: $\begin{cases} x+y+z=1 \\ 2x-y-4z=2 \end{cases}$ Set $y=t$. $\begin{cases} x+z=1-t \\ 2x-4z=2+t \end{cases} \Rightarrow \begin{cases} x=1-\frac{1}{2}t \\ z=\frac{-1}{2}t \end{cases}$

$\vec{r_1}(t)=\langle 1-\frac{1}{2}t, t, \frac{-1}{2}t\rangle = \langle 1,0,0\rangle + t\langle \frac{-1}{2}, 1, \frac{-1}{2}\rangle$

L_2: $\vec{r_2}(t)=\langle 2,0,1\rangle + t\langle 1,1,1\rangle = \langle 2+t, t, 1+t\rangle$

L_3: $\langle t, t, 0\rangle = \langle 0, 0, 0\rangle + t\langle 1, 1, 0\rangle = \vec{r_3}(t)$

(2) (5 pts) Find the intersection of L_1 and L_2.

L_1. $\vec{r_1}(t) = \cancel{\langle 2+t, t, 1+t\rangle}\ \langle 1-\frac{1}{2}t, t, \frac{-1}{2}t\rangle$

L_2: $\vec{r_2}(s) = \langle 2+t, t, 1+t\rangle$

so put different parameters

$\begin{cases} 1-\frac{1}{2}t = s+2 \\ t = s \\ \frac{-1}{2}t = 1+s \end{cases} \Rightarrow t=s=\frac{-2}{3}.$

so they intersect @ $\left(\frac{4}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

(3) 5 (7 pts) What is the equation of the plane generated by L_1 and L_2 ?

point $= \left(\frac{4}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ is the intersection point

normal vector = (direction vector of L_1) $\times$ (direction vector of L_2)

$= \langle -1, 2, -1 \rangle \times \langle 1, 1, 1 \rangle$

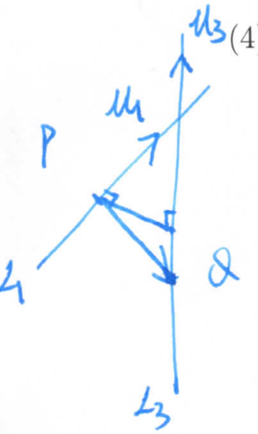
$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle 3, 0, -3 \rangle$

so equation: $3\left(x - \frac{4}{3}\right) + 0\left(y + \frac{2}{3}\right) + (-3)\left(z - \frac{1}{3}\right) = 0$

$3x - 4 - (3z - 1) = 0 \Rightarrow \boxed{x - z - 1 = 0}$

$3x - 3z - 3 = 0$

(4) 5 (8 pts) Find the distance between the skew lines L_1 and L_3.



distance $= \dfrac{|\vec{PQ} \cdot (\vec{u_1} \times \vec{u_3})|}{|\vec{u_1} \times \vec{u_3}|}$ ~~Here~~ Here P, Q are random points on L_1 & L_3.

set $t=0$ on $L_1 \Rightarrow P = (1, 0, 0)$
set $t=0$ on $L_3 \Rightarrow Q = (0, 0, 0)$ $\Rightarrow \vec{PQ} = \langle -1, 0, 0 \rangle$

$\vec{u_1} = \langle -1, 2, -1 \rangle$. $\vec{u_3} = \langle 1, 1, 0 \rangle$

$\vec{u_1} \times \vec{u_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \langle +1, -1, -3 \rangle$

$= \langle 1, -1, -3 \rangle$

so $d = \dfrac{1}{\sqrt{1^2 + (-1)^2 + (-3)^2}} = \dfrac{1}{\sqrt{11}}$

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4.(15pts). The vector function $\vec{r}(t) = < \ln(2-t), e^{t-1}, \sqrt{t} >$ describes a smooth space curve.

(1) 10 (5pts) Find the domain of $\vec{r}(t)$.

$x(t) = \ln(2-t) \Rightarrow t < 2.$

$y(t) = e^{t-1} \Rightarrow t > -\infty$ & $t < \infty$

$z(t) = \sqrt{t} \Rightarrow t \geq 0$

so Domain is $[0, 2)$

(2) (10 pts) Find the equation of the tangent line to the curve at $(0, 1, 1)$.

$\dot{\vec{r}}(t) = \langle \frac{-1}{2-t}, e^{t-1}, \frac{1}{2\sqrt{t}} \rangle$

@ $(0,1,1) \Rightarrow \begin{cases} \ln(2-t) = 0 \\ e^{t-1} = 1 \\ \sqrt{t} = 1 \end{cases} \Rightarrow t = 1$

so $\dot{\vec{r}}(1) = \langle -1, 1, \frac{1}{2} \rangle$

equation of the tangent line is

$\vec{r}(t) = \langle 0, 1, 1 \rangle + t \langle -1, 1, \frac{1}{2} \rangle$

Bonus

5.(10pts). For any differentiable vector function $\vec{r}(t)$. Prove that if $\vec{r}(t)$ is a unit vector for any t, then $\dot{\vec{r}}(t)$ is orthogonal to $\vec{r}(t)$.

Since $\vec{r}(t)$ is a unit vector

$\Rightarrow |\vec{r}(t)| = 1$

$\Rightarrow |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t) = 1$

so $\frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{d}{dt}(1)$

By Product Rule for dot product

$\dot{\vec{r}}(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$

$2\vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$

$\vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$

$\Rightarrow |\vec{r}(t)| \cdot |\dot{\vec{r}}(t)| \cos\alpha = 0$

$\Rightarrow \cos\alpha = 0$

so $\alpha = \frac{\pi}{2}$

i.e. $\vec{r}(t) \perp \dot{\vec{r}}(t)$.