MAC 2313-0010
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Test 4
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Print Name Answer Key

Signature

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $[a, b] \times [c, d]$ denotes the rectangle $\{a \le x \le b; c \le y \le d\}$ in xy plane.
- Keep Calm and Enjoy the Computations!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) The positive orientation of a closed curve is clock-wise.

False. Counter-clockwise

(2) (5pts) $\{1 \le x^2 + y^2 \le 4\}$ is simply-connected.

False: it has a hole in the middle

(3) (5pts) A vector field $\vec{F}$ is conservative if and only if $div\vec{F} = 0$.

False. $\vec{F}$ is conservative iff curl $\vec{F} = \vec{0}$.

(4) (5pts) The line integral of a conservative vector field is independent of path.

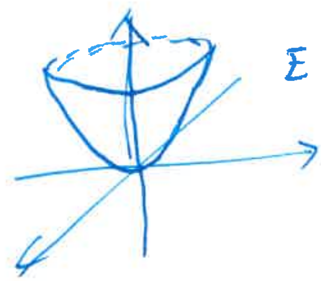
True.

2. Integrals

2.1. **(15 pts).** Express the following Riemann sum over the solid E as a triple integral, and compute the integral.

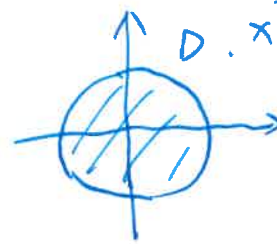
$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{x_i^2+y_j^2}\,\Delta V.$$

Here E is enclosed by $z = x^2 + y^2$ and $z = 4$.



E

$$\iiint_E x^2+y^2\,dV \qquad E:\ z=x^2+y^2 \text{ and } z=4$$



D . $x^2+y^2\le 4$

$$= \iint_D \int_{x^2+y^2}^{4} (x^2+y^2)\,dz\,dA$$

$$= \iint_D (x^2+y^2)(4-x^2-y^2)\,dA$$

$$= \int_0^2\int_0^{2\pi} r^2(4-r^2)\cdot r\,dr\,d\alpha$$

$$= 2\pi\cdot \int_0^2 4r^3-r^5\,dr$$

$$= 2\pi\cdot \left(r^4-\frac{r^6}{6}\right)\Big|_0^2$$

$$= 2\pi\left(16-\frac{64}{6}\right) = 2\pi\left(\frac{16}{3}\right) = \frac{32}{3}\pi$$

2.2. **(10 pts).** Find the triple integral over the solid E enclosed by $1 \leq x^2+y^2+z^2 \leq 4$ and $z \geq 0$.

$$\iiint_E z dV$$

Use spherical coordinates

$$\begin{cases} x = r\cos\alpha\cos\beta \\ y = r\cos\alpha\sin\beta \\ z = r\sin\alpha \end{cases}$$

$\alpha: 0 \to \pi$

$\beta: 0 \to 2\pi$

$r: 1 \to 2$

$J = r^2 \sin\alpha$

$$\iiint_E z dV = \int_1^2 \int_0^{\pi} \int_0^{2\pi} (r\sin\alpha)(r^2 \sin\alpha)\, d\beta\, d\alpha\, dr$$

$$= 2\pi \int_1^2 \int_0^{\pi} r^3 \sin^2\alpha\, d\alpha\, dr$$

$$= 2\pi \int_1^2 r^3 dr \int_0^{\pi} \frac{1-\cos 2\alpha}{2} d\alpha$$

$$= 2\pi \cdot \left(\frac{r^4}{4}\Big|_1^2\right)\left(\frac{\pi}{2} - \frac{1}{4}\int_0^{2\pi} \cos u\, du.\right)$$

$$= \pi^2 \cdot \frac{2^4 - 1}{4} = \frac{15\pi^2}{4}$$

2.3. **(10 pts).** [Hint: set $u = xy$, $v = xy^2$]

Compute the following integral by changing variables.

$$\iint_R y^2 dA.$$

Here R is enclosed by $1 \le xy \le 2$ and $1 \le xy^2 \le 2$.

$$u = xy, \quad v = xy^2 \Rightarrow \begin{cases} x = \frac{u^2}{v} \\ y = \frac{v}{u}. \end{cases} \Rightarrow J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$= \frac{2}{v} - \frac{1}{v} = \frac{1}{v}$$

$$R: \begin{cases} 1 \le xy \le 2 \\ 1 \le xy^2 \le 2 \end{cases} \longrightarrow \begin{cases} 1 \le u \le 2 \\ 1 \le v \le 2 \end{cases} = \hat{R}.$$

$$\text{so } \iint_R y^2 dA = \iint_{\hat{R}} \left(\frac{v}{u}\right)^2 \cdot \frac{1}{v}\, d\hat{R} = \int_1^2 \int_1^2 \frac{v}{u^2}\, du\, dv$$

$$= \int_1^2 v\, dv \cdot \int_1^2 \frac{1}{u^2}\, du$$

$$= \left(\frac{1}{2}v^2\Big|_1^2\right)\left(-\frac{1}{u}\Big|_1^2\right) = \left(\frac{3}{2}\right) \cdot \left(\frac{1}{2}\right) = \frac{3}{4}$$

3. Vector Calculus

3.1. **(10 pts).** Let C be a piece of space helix given by $\vec{r}(t) =< \cos t, \sin t, t >$, $0 \le t \le \pi/4$. Let $\vec{F} =< x, y, xy >$ be a smooth vector field. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

C : $\vec{r}(t) = < \cos t, \sin t, t >$. $t : 0 \longrightarrow \frac{\pi}{4}$

$\dot{\vec{r}}(t) = < -\sin t, \cos t, 1 >$

$\vec{F}(t) = < \cos t, \sin t, \sin t \cos t >$

$\int_C \vec{F} \, d\vec{r} = \int_0^{\frac{\pi}{4}} < \cos t, \sin t, \sin t \cos t > \cdot < -\sin t, \cos t, 1 > dt$

$= \int_0^{\frac{\pi}{4}} \sin t \cos t \, dt$

$= \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin 2t \, dt = \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin u \, du = \frac{1}{4} \cos u \Big|_{\frac{\pi}{2}}^{0}$

$= \frac{1}{4}$

3.2. **(20 pts).** A vector field $\vec{F}$ is given by $< yz, xz, xy + 2z >$.

(1) (5 pts) Show that $\vec{F}$ is a conservative vector field.

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+2z \end{vmatrix} = \vec{0}.$$

and the observation that $\vec{F}$ is defined everywhere in $\mathbb{R}^3$.

(2) (10 pts) Find a differentiable function $f(x, y, z)$ such that $\nabla f = \vec{F}$.

$$* \begin{cases} f_x = yz \\ f_y = xz \\ f_z = xy + 2z \end{cases} \longrightarrow f = xyz + f(y,z) \Rightarrow \begin{cases} f_y = xz + f_y \\ f_z = xy + f_z \end{cases}$$

$$\Rightarrow \begin{cases} f_y = 0 \\ f_z = 2z \end{cases} \Rightarrow f(y,z) = z^2.$$

so $f(x,y,z) = xyz + z^2$

(3) (5 pts) Find $\oint_C \vec{F} \cdot d\vec{r}$. Here C is a smooth curve from $(0, 0, -1)$ to $(0, 0, 1)$.

By the Fundamental Theorem of Calculus

$$\int_C \vec{F}\, d\vec{r} = f(0,0,1) - f(0,0,-1)$$

$$= 0 - 0 = 0$$

3.3. **(10 pts).** For any differentiable functions $f(x, y, z)$ and $g(x, y, z)$ with continuous second prtial derivatives, compute and show that $div(\nabla f \times \nabla g) = 0$.

$\nabla f = \langle f_x, f_y, f_z \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle$

$$\nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \langle f_y g_z - f_z g_y, \; f_z g_x - f_x g_z, \; f_x g_y - f_y g_x \rangle$$

so $div(\nabla f \times \nabla g)$

$= \frac{\partial}{\partial x}(f_y g_z - f_z g_y) + \frac{\partial}{\partial y}(f_z g_x - f_x g_z) + \frac{\partial}{\partial z}(f_x g_y - f_y g_x)$

$= \underline{f_{yx} g_z}_{①} + \underline{f_y g_{zx}}_{②} - \underline{f_{zx} g_y}_{③} - \underline{f_z g_{yx}}_{④}$

$+ \underline{f_{zy} g_x}_{⑤} + \underline{f_z g_{xy}}_{④} - \underline{f_{xy} g_z}_{①} - \underline{f_x g_{zy}}$

$+ _{③}\underline{f_{xz} g_y} + \underline{f_x g_{yz}} - \underline{f_{yz} g_x}_{⑤} - \underline{f_y g_{zx}}_{②}$

$= 0$ because $f_{xy} = f_{yx}$ $f_{yz} = f_{zy}$ $f_{xz} = f_{zx}$

and

$g_{xy} = g_{yx}$ $g_{yz} = g_{zy}$ $g_{xz} = g_{zx}$

3.4. **(10 pts).** Find the following line integral

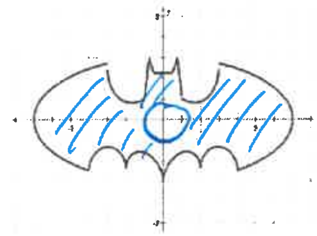
$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Step 1:

Let D be the region after we punch a ~~hole~~ hole in the middle.

* $\vec{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ is defined everywhere on D
* $\vec{F}$ is conservative on D

Here C is the closed cirve given by the figure:



[Hint: 'Poled' Green's Theorem.]

Step 2: Apply "Poled" Green's Theorem

Let C' be the small hole in the middle.

$$\iint_S \text{curl}\vec{F}\cdot\vec{k}\,dA = \oint_C \vec{F}d\vec{r} - \oint_{C'} \vec{F}d\vec{r}$$

$$\iint_S 0\,dA = 0$$

$\Rightarrow$ This shows that $\oint_C \vec{F}d\vec{r} = \oint_{C'} \vec{F}d\vec{r}$.

so we can replace C by the unit circle.

Step 3. Let C be the unit circle.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$. $t: 0 \to 2\pi$

$\vec{F}(t) = \langle -\sin t, \cos t \rangle$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle \cdot dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$