MAC 2313-0001
Xiping Zhang
Test 3
03/23/2017

Print Name Answer Key

Signature Xiping Zhang

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $[a, b] \times [c, d]$ denotes the rectangle $\{a \le x \le b; c \le y \le d\}$ in xy plane.
- Keep Calm and Enjoy the Computations!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) $\int_a^b \int_c^d f(x, y)\,dx\,dy = \int_c^d \int_a^b f(x, y)\,dy\,dx$ for any continuous function $f(x, y)$.

True : Fubini's Theorem

(2) (5pts) By rotating the graph of $y = f(x)$ along the x-axis one will get a surface with equation $y^2 + z^2 = f(x)^2$.

True.

(3) (5pts) Polar transform maps a disk with center at $(1, 0)$ in xy plane to a rectangle in $r\theta$ plane.

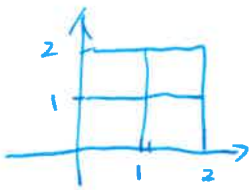
False. Only the circle with center $(0, 0)$ can be transfored to a rectangle under polar coordinates.

(4) (5pts) The double integral $\iint_D f(x, y) - g(x, y)\,dA$ represents the volume of the solid enclosed by surfaces $z = f(x, y)$ and $z = g(x, y)$ over the region D.

True if $f(x, y) \geq g(x, y)$.

2. 'Riemann Sum'

2.1. **(10pts).** Estimate the volume of the solid that lies over the square $R : [0, 2] \times [0, 2]$ and below the surface $z = xy$ via **Riemann Sum**. Here we divide R into **four** equal squares and use the **upper-right** corner as sample points.



test pts : (1. 1), (2. ~~0~~). (1. 2), (2. 2)

$R = [(1\times1) + (2\times1) + (1\times2) + (2\times2)] \times (1\times1$

$= 9$

2.2. **(15pts).** Express the following Riemann Sum over the rectangle $[0,\pi]\times[0,1]$ as a double integral, and then compute it.

$$\lim_{m,n\to\infty}\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{\sin(x_i^*)}{1+(y_i^*)^2}\Delta x_i\Delta y_j$$

(1) (5 pts) [Express the Riemann sum as a Double Integral and sketch the integration region]

$$\int_0^{\pi}\int_0^{1}\frac{\sin x}{1+y^2}\,dy\,dx$$

(2) (10 pts)[Compute the above integral]

$$\int_0^1\int_0^{\pi}\frac{\sin x}{1+y^2}\,dx\,dy = \int_0^1\frac{1}{1+y^2}\,dy\cdot\int^{\pi}\sin x\,dx$$

$$= -\cos x\Big|_0^{\pi}\cdot \arctan y\Big|_0^1$$

$$= 2\times\frac{\pi}{4} = \frac{\pi}{2}$$

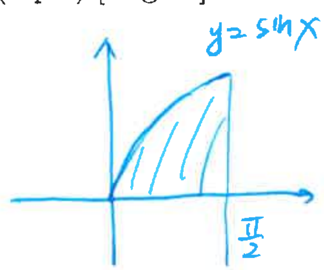
3. Integrals and Applications

3.1. **(15pts).** Sketch the region and evaluate the double integral

$$\iint_R \cos(x)\, dA$$

where R is the region in xy plane enclosed by $x = 0$, $x = \pi/2$ and $y = \sin(x)$.

(1) (5 pts) [Region]



(2) (10 pts)[Compute the above integral]

$$\iint_R \cos x\, dA = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x\, dy\, dx = \int_0^{\frac{\pi}{2}} \cos x \cdot \left(y \Big|_0^{\sin x} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \cos x\, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x\, dx = \frac{1}{4} \cos 2x \Big|_{\frac{\pi}{2}}^{0}$$

$$= \frac{1}{2}$$

3.2. **(15pts).** Find the area of the surface $z = x^2 + 2y$ that lies above the triangle T in xy plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$f(x,y) = z = x^2 + 2y$

$f_x = 2x \quad f_y = 2.$

$$A = \iint_R \sqrt{1+4x^2+4}\, dA$$

$$= \int_0^1 \int_0^x \sqrt{5+4x^2}\, dy\, dx$$

$$= \int_0^1 x\sqrt{5+4x^2}\, dx$$

$u = 5 + 4x^2 \quad du = \cancel{8dx}\ 8x\, dx$

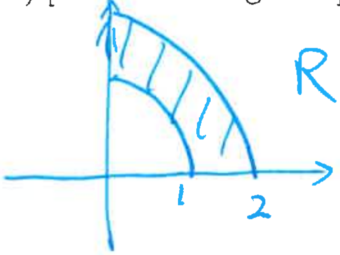
$x: 0 \to 1 \to u: 5 \to 9$

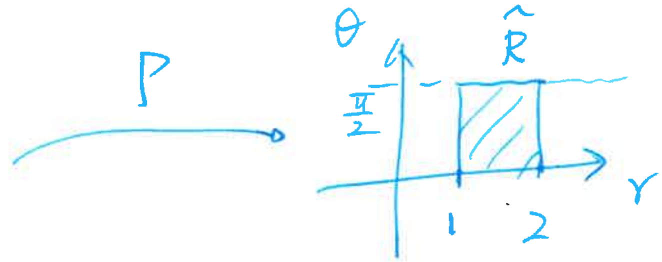
$$A = \int_5^9 \frac{1}{8}\sqrt{u}\, du = \frac{1}{8}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}}\Big|_5^9$$

$$= \frac{1}{12}\left(9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

3.3. **(15 pts).** Find the volume of the solid under the surface $z = xy$ and above the region R in the xy plane. Here R is the **Upper Half Plane** enclosed by $\{x \geq 0\}$, $\{y \geq 0\}$ and $\{1 \leq x^2 + y^2 \leq 4\}$.

(1) (5 pts) [Sketch the Region R]





(2) (10 pts)[Find the volume]

use polar transform.

$$V = \iint_R xy\, dV = \iint_{\hat{R}} r\cos\alpha \cdot r\sin\alpha \cdot r\, d\hat{V} = \iint_{\hat{R}} r^3 \sin\alpha\cos\alpha\, d\hat{V}$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 r^3 \sin\alpha\cos\alpha\, dr\, d\alpha = \int_0^{\frac{\pi}{2}} \sin\alpha\cos\alpha\, d\alpha \int_1^2 r^3\, dr$$

$$= -\frac{1}{4}\cos 2\alpha \Big|_0^{\frac{\pi}{2}} \cdot \frac{r^4}{4}\Big|_1^2$$

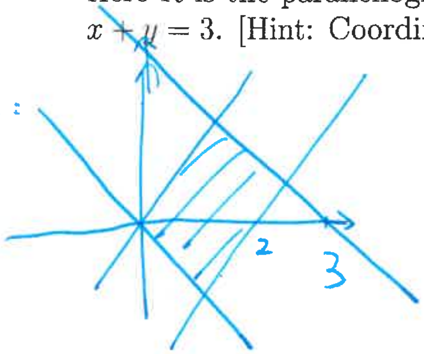
$$= -\frac{1}{4}\cos\beta \Big|_0^{\pi} \cdot \frac{r^4}{4}\Big|_1^2 = \frac{1}{2} \cdot \frac{15}{4} = \frac{15}{8}$$

3.4. **(10)pts.** Sketch the region, and evaluate the following double integral.

$$\iint_R (x+y)e^{x^2-y^2}\,dA$$

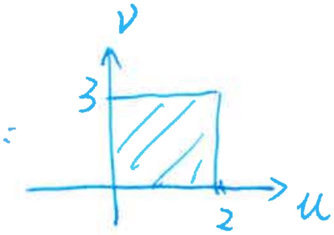
Here R is the parallellogram enclosed by the lines $y=x$, $y=x-2$, $x+y=0$, and $x+y=3$. [Hint: Coordinate Change]

R :



use coordinate change

$$u = x+y \qquad \tilde{R}:$$
$$v = x-y$$



so $\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

so $\displaystyle\iint_R (x+y)e^{x^2-y^2}\,dA = \iint_{\tilde{R}} ue^{uv}\cdot -\frac{1}{2}\,d\hat{A} = \int_0^2\int_0^3 ue^{uv}\left(-\frac{1}{2}\right)dv\,du$

$$= \int_0^2 -\frac{1}{2}\cdot\left(e^{uv}\Big|_{v=0}^{v=3}\right)du = \int_0^2 -\frac{1}{2}\left(e^{3u}-1\right)du$$

$$= \int_0^2 -\frac{1}{2}e^{3u}\,du - -\frac{1}{2}\int_0^2 du$$

$$= -\frac{1}{6}e^{3u}\Big|_0^2 + 1 = -\frac{1}{6}\left(e^6-1\right)+1$$

$$= -\frac{1}{6}e^6 + \frac{7}{6}$$

Scratch Here