MAC 2313-0010
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Test 2
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Print Name Answer Key

Signature

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $\mathbb{R}^n$ denotes the n dimensional real space.
- Keep Calm and Enjoy Calculus!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) If a space curve $\vec{r}(t)$ has constant binormal vector $\vec{B}$, then it's a plane curve.

Yes, this is true.

(2) (5pts) $f_{xy} = f_{yx}$ for any continuous function $z = f(x, y)$.

False. This only holds true for $f(x,y)$ that has continuous 1st partial derivatives.

(3) (5pts) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{1}{2}$.

False. $\left\{\begin{array}{l}\text{Along } x=0\text{, the limit is } 0\\ \text{Along } x=y \text{ the limit is } \frac{1}{2}\end{array}\right\}$ different limits indicate that the limit doesn't exist.

(4) (5pts) The curvature of a unit circle is 1.

True. unit circle means radius $=1$

$\Rightarrow \kappa(t) = \frac{1}{r} = 1$.

2. 'Space Curve'

Let C be a space curve parametrized by

$$\vec{r}(t) = <\sqrt{3}\cos t, -\cos t, 2\sin t>$$

(1) (15pts) Find the unit tangent vector $\vec{\mathbf{T}}$, the normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$ for the curve.

$\vec{r}'(t) = <-\sqrt{3}\sin t, \sin t, 2\cos t> \Rightarrow |\vec{r}'(t)| = \sqrt{(3+1)\sin^2 t + 4\cos^2 t} = 2$

so $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = <\frac{-\sqrt{3}}{2}\sin t, \frac{1}{2}\sin t, \cos t>$

$\dfrac{d\vec{T}}{dt} = <\frac{-\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t>$

$\vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \dfrac{<\frac{-\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t>}{\sqrt{(\frac{3}{4}+\frac{1}{4})\cos^2 t + \sin^2 t}} = <\frac{-\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t>$

$B = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sqrt{3}}{2}\sin t & \frac{1}{2}\sin t & \cos t \\ \frac{-\sqrt{3}}{2}\cos t & \frac{1}{2}\cos t & -\sin t \end{vmatrix} = <-\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0>$

so $\vec{T} = <\frac{-\sqrt{3}}{2}\sin t, \frac{1}{2}\sin t, \cos t>$

$\vec{N} = <\frac{-\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t>$

$\vec{B} = <\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0>$

(2) (10pts) Find the curvature function $\kappa(t)$ of C.

$$K(t) = \frac{|\vec{T}'(t)|}{\left|\frac{ds}{dt}\right|} = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}'(t)|} = \frac{\left|\left\langle -\frac{\sqrt{3}}{2}\cos t, \frac{1}{2}\cos t, -\sin t\right\rangle\right|}{2} = \frac{1}{2}.$$

(3) (5pts) Find equation of the osculating plane **H** to the curve C at point $(0, 0, 2)$.

* osculating plane has normal vector $\vec{T} \times \vec{N} = \vec{B}$.

so $\vec{n} = \vec{B} = \left\langle -\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0\right\rangle$

so the equation of the plane H is:

$$(x-0)\left(-\frac{1}{2}\right) + (y-0)\left(-\frac{\sqrt{3}}{2}\right) + (z-2)0 = 0$$

$$\Rightarrow \quad -x - \sqrt{3}y = 0$$

$$\Rightarrow \quad x + \sqrt{3}y = 0$$

3. 'Functions'

3.1. **(20pts).** Let $w = x\sin(y-z)$ be a differentiable function of three variables.

(1) (10pts) Find the directional derivative $\nabla_u f$ of $w = f(x,y,z)$ along the direction $\vec{u} =< 1,1,1 >$ at $(3, 2017\pi, 2016\pi)$.

$\nabla f = \langle \sin(y-z),\ x\cos(y-z),\ -x\cos(y-z) \rangle$

$\nabla f \big|_{(3, 2017\pi, 2016\pi)} = \nabla f(3, 2017\pi, 2016\pi) = \langle \sin(\pi),\ 3\cos(\pi),\ -3\cos(\pi) \rangle$

$= \langle 0, -3, 3 \rangle$

$D_{\vec{u}} f(3, 2017\pi, 2016\pi) = \nabla f \cdot \vec{u} \big| (3, 2017\pi, 2016\pi)$

$= \langle 0, -3, 3 \rangle \cdot \langle 1, 1, 1 \rangle$

$= 0$

(2) (10pts) Consider the level surface S in $\mathbb{R}^3$ given by $w = 0$. Find the equation of the tangent plane to S at point $(3, 2017\pi, 2016\pi)$.

* The normal vector to the tangent plane of $f(x, y, z) = k$ is ∇f *

@ $(3, 2017\pi, 2016\pi)$

$\vec{n} = \nabla f(3, 2017\pi, 2016\pi) = \langle 0, -3, 3 \rangle$

equation of the tangent plane is:

$0(x-3) + (-3)(y - 2017\pi) + 3(z - 2016\pi) = 0$

$-y + 2017\pi + z - 2016\pi = 0$

$\pi - y + z = 0$

15 pts

3.2. ~~(10pts)~~. Let $f(x, y) = \sin x + \sin y + \cos(x + y)$ be a function of two variables, where $0 \le x, y \le 2\pi$.

(1) (5 pts) Find the critical point(s) of the function.

$\nabla f = \langle \cos x - \sin(x+y),\ \cos y - \sin(x+y) \rangle = \vec{0}$

$\Rightarrow \begin{cases} \cos x = \sin(x+y) \\ \cos y = \sin(x+y) \end{cases} \Rightarrow \begin{cases} \cos x = \cos y & ① \\ \cos x = \sin(x+y) & ② \end{cases}$

① $\Rightarrow x = y$ or $x = 2\pi - y$.

if $x = y \overset{②}{\Rightarrow} \cos x = \sin(2x) = 2\sin x \cos x \Rightarrow \cos x(2\sin x - 1) = 0 \Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2}$

so $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} = y$

if $x = 2\pi - y \Rightarrow x + y = 2\pi$. so ② $\Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} = 2\pi - y$

so critical pts are $(\frac{\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{3\pi}{2}), (\frac{\pi}{6}, \frac{\pi}{6})$
$(\frac{\pi}{2}, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2}), (\frac{5\pi}{6}, \frac{5\pi}{6})$

(2) (5 pts) For each of the critical points you found, decide whether it is a local maximum, or local minimum, or saddle point.

$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\sin x - \cos(x+y), & -\cos(y+x) \\ -\cos(y+x) & -\sin y - \cos(x+y) \end{bmatrix}$

@ $(\frac{\pi}{2}, \frac{\pi}{2}) \to H_f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to$ saddle point.

@ $(\frac{3\pi}{2}, \frac{3\pi}{2}) \to H_f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \to$ local min

@ $(\frac{\pi}{2}, \frac{3\pi}{2}) \to H_f = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} \to$ ~~local max~~ saddle point

$(\frac{3\pi}{2}, \frac{\pi}{2}) \to H_f = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \to$ saddle point

@ $(\frac{\pi}{6}, \frac{\pi}{6}) \to H_f = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \to$ local max

@ $(\frac{5\pi}{6}, \frac{5\pi}{6}) \to H_f = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \to$ local max.

15 pts

3.3. (~~10pts~~). Let $f(x,y) = x^2 + y^2 - 4x - 2y$ be a differentiable function defined on the the curve C: $x^2 + y^2 = 20$. Find the **Absolute Maximum** and **Absolute Minimum** of the function restricted on C.

$$L(x,y,\lambda) = (x^2+y^2-4x-2y) - \lambda(x^2+y^2-20)$$

$$\nabla L = \langle 2x - 2x\lambda - 4,\ 2y - 2 - 2\lambda y,\ -(x^2+y^2-20)\rangle = \vec{0}$$

$$\Rightarrow \cancel{\begin{cases} x-\lambda = 2 \\ y-\lambda = 1 \\ x^2+y^2 = 20 \end{cases} \Rightarrow x = y+1 \Rightarrow (y+1)^2 + y^2 = 20}$$

$$\begin{cases} (1-\lambda)x = 2 \\ (1-\lambda)y = 1 \\ x^2+y^2 = 20 \end{cases} \Rightarrow x = 2y \Rightarrow (2y)^2 + y^2 = 20 \Rightarrow y = \pm 2$$

so critical pts of L are: $(4, 2)$ & $(-4, -2)$

@ $(4, 2)$. $f(4, 2) = 20 - 16 - 4 = 0$

@ $(-4, -2)$ $f(x,y) = f(-4, -2) = 20 + 16 + 4 = 40$

so $f(x,y)$ has Global Max 40 @ $(-4, -2)$

& Global Min 0 @ $(4, 2)$

[BONUS]

3.4. **(10pts).** Prove that the following function is continuous **Everywhere**.

$$f(x,y) = \begin{cases} \frac{\sin(x^2-y^3)xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if}(x,y) = (0,0). \end{cases}$$

(Hint: Some trig functions have bounds.)

* The domain of $f(x,y)$ is everywhere. When $(x,y) \neq (0,0)$, the function $\frac{\sin(x^2-y^3)xy}{\sqrt{x^2+y^2}}$ is continuous because this is the composition of simple functions.

* @ $(0,0)$, as long as we can show $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$, the function will be continuous everywhere.

* To prove $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, we use ε-δ language.

For $\forall \varepsilon > 0$, we choose $\delta = 2\varepsilon$. , then we have, for any (x,y) such that $\sqrt{(x-0)^2+(y-0)^2} < \delta$, we have

$$\left| \frac{\sin(x^2-y^3)xy}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|\sin(x^2-y^3)|\,|xy|}{\left|\sqrt{x^2+y^2}\right|} \le \frac{|xy|}{\left|\sqrt{x^2+y^2}\right|} \le \frac{1}{2} \cdot \frac{|x^2+y^2|}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\sqrt{x^2+y^2} \le \frac{1}{2}\delta = \varepsilon .$$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Q.E.D