* Let $\vec{OP}$ be a vector in $\mathbb{R}^3$. Denote α, β, γ to be the angles between $\vec{OP}$ with x-axis, y-axis and z-axis respectively.

Then: 1) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

2) $\alpha + \beta + \gamma = 180^\circ$ iff $\vec{OP}$ lies in the coordinate plane.
i.e., one of them is 90°.

<u>Proof</u>: 1) We have shown in class.

2) Assume that $\alpha + \beta + \gamma = 180^\circ$

$\Rightarrow \gamma = 180^\circ - \alpha - \beta = \pi - \alpha - \beta$

so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \overbrace{\cos^2(\pi - \alpha - \beta)}^{\cos^2(\alpha+\beta)} = 1$

$\Rightarrow \dfrac{1+\cos 2\alpha}{2} + \dfrac{1+\cos 2\beta}{2} + \dfrac{1+\cos(2\alpha+2\beta)}{2} = 1$

$\Rightarrow 1 + \cos 2\alpha + \cos 2\beta + \cos(2\alpha + 2\beta) = 1$

$\Rightarrow \boxed{(\cos 2\alpha + 1)(\cos 2\beta + 1) = \sin 2\alpha \sin 2\beta}$

$\cos(2\alpha + 2\beta) = \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

<u>Case 1</u>. $1 + \cos 2\alpha = 0$ or $1 + \cos 2\beta = 0$

$\Rightarrow 2\alpha = \pi$ or $2\beta = \pi$ $\Rightarrow \alpha = \frac{\pi}{2}$ or $\beta = \frac{\pi}{2}$.

<u>Case 2</u>. $1 + \cos 2\alpha \neq 0 \neq \cos 2\beta \Rightarrow \alpha \neq 0,\ \alpha \neq \frac{\pi}{2},\ \beta \neq 0,\ \beta \neq \frac{\pi}{2}$

then $\boxed{\dfrac{\sin 2\alpha}{1+\cos 2\alpha} \cdot \dfrac{\sin 2\beta}{1+\cos 2\beta} = 1}$ Use double-angle formula

$\Rightarrow \dfrac{2\sin\alpha\cos\alpha}{2\cos^2\alpha} \cdot \dfrac{2\sin\beta\cos\beta}{2\cos^2\beta} = 1 \Rightarrow \tan\alpha \tan\beta = 1$

$\Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \gamma = \frac{\pi}{2}$

Q.E.D.