VECTOR AND LINEAR GEOMETRY

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1. Why do we talk about vectors

Vector can be realized as an extension of a real number into higher dimensional spaces. When we talk about a real number, we are actually talking about two things: the magnitude and the sign. On a real line, the magnitude tells us how far the point is away from the origin, and the sign tells us whether it's on the left side or on the right side. In other words, a number with direction. So vectors should be thought of, at a first approximation, as "numbers with direction". We can think a vector as a light beam, the direction shows where it goes, and the magnitude shows ho long the beam is.

In real life, especially in physics, lots of the common terminologies are defined as numbers with direction, such as velocity, force, acceleration, displacement, etc. And for some other terminologies such as work W, are defined using both the magnitudes and the relative position between the force and the displacement. This requires us to find a way to compute directly using the vectors, rather than just numbers.

To simplify notations, from now on we will restrict our space to be 3-dimensional without further notice.

2. Vectors and Basic Operations.

Vectors. A *vector* in $\mathbb{R}^3$ is composed by a magnitude and a direction. A vector doesn't have fixed starting or ending point, which means that you can move it around. Like numbers, we want to compare them, add them, subtract them, and multiply them, this requires an algebra system to describe vectors, and the natural idea is to move the vectors to a common starting point, then a vector is uniquely determined by the ending point. Talking about a common initial, which point could be better than the origin ? Then the coordinate of the ending point provides a measure for the vector.

Let $P = (a_1, b_1, c_1) \in \mathbb{R}^3$ be a point, then we give the vector $\vec{OP}$ a coordinate form $< a_1, b_1, c_1 >= a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$. For another point $Q = (a_2, b_2, c_2)$, the vector

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$\vec{PQ} = \vec{OQ} - \vec{OP}$ shall be given by $< a_2 - a_1, b_2 - b_1, c_2 - c_1 >$. This coordinate form guarantees us the following properties:

Prop 1. *Let $\vec{a} =< a_1, b_1, c_1 >$ and $\vec{b} =< a_2, b_2, c_2 >$ be two vectors. Let λ be a real number.*

(1) $\vec{a} = \vec{b}$ iff $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$.
(2) $\vec{a} \pm \vec{b} =< a_1 \pm a_2, b_1 \pm b_2, c_1 \pm c_2 >$.
(3) $\lambda \cdot \vec{a} =< \lambda a_1, \lambda b_1, \lambda c_1 >$.

Products. Of course we are not satisfied with the easy operations, we want to multiply vectors ! However, things get a little complicated on multiplication: there are two multiplications, depending on what you need.

Let $\vec{a} =< a_1, b_1, c_1 >$ and $\vec{b} =< a_2, b_2, c_2 >$ be two vectors in $\mathbb{R}^3$.

Product 1 We define the *Dot Product* of $\vec{a}$ and $\vec{b}$ to be a Real Number:

$$\vec{a} \cdot \vec{b} := a_1 a_1 + b_1 b_2 + c_1 c_2$$

Product 2 We define the *Cross Product* of $\vec{a}$ and $\vec{b}$ to be a Vector:

$$\vec{a} \times \vec{b} := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} =< b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1 >$$

Both products carry down the nice linear properties, however, there are major differences between them: the dot product gives you a scalar, while the cross product gives a vector; also the dot product is commutative, while the cross product is skew commutative:

Prop 2. *Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$, and λ be a real number.*

(1) $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda\vec{b})$.
(2) $(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}) = \vec{a} \times (\lambda\vec{b})$.
(3) $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$.
(4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.
(5) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
(6) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Notice that the cross product is only defined for vectors in $\mathbb{R}^3$, the dot product, however, is defined for all dimensional vectors. One way to generalize the cross product is to use the wedge product, and the dual space. This definition extends to arbitrary dimension spaces, and shares the same properties as the cross product.

The products also contain rich information about the geometry: they involve lengths of the vectors, together with the angle between them.

Prop 3. *Let $\vec{a}$ and $\vec{b}$ be two vectors in $\mathbb{R}^3$, and denote the angles between them to be α, then*

(1) $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\alpha$.
(2) $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin\alpha$.
(3) The direction of $\vec{a} \times \vec{b}$ comes from the Right Hand Rule from $\vec{a}$ to $\vec{b}$.

Hence playing with plane geometry we have the following results:

(1) The projection of $\vec{a}$ onto $\vec{b}$ is:

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \cdot \vec{b}$$

(2) The length of $\vec{a} \times \vec{b}$ equals the area of the parallelogram generated by $\vec{a}$ and $\vec{b}$.
(3) the absolute value of the triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$ equals the volume of the parallelepiped generated by $\vec{a}$, $\vec{b}$ and $\vec{c}$.
(4) To compute the triple product, one has the following shortcut:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) := \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

providing that $\vec{a} =< a_1, b_1, c_1 >$, $\vec{b} =< a_2, b_2, c_2 >$, and $\vec{c} =< a_3, b_3, c_3 >$.

3. Lines and Planes

Another benefit of introducing vectors is, they provide a simple way of writing equations of linear objects in space, such as lines and planes.

Lines. A line is the trace of a particle moving straight in space, hence is uniquely determined by a position and a direction. One can pick a random point on the path, and assign it to time 0, then along the moving direction, use the time t to describe the position. Notice that the direction of a line is nothing but a fixed vector, hence the *Equation of a Line* is given by:

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}$$

In terms of coordinate form, the points on the line obey the following equation:

$$\vec{r}(t) =< x(t), y(t), z(t) >=< x_0 + ta, y_0 + tb, z_0 + tc >$$

providing that the point is (x_0, y_0, z_0) and $\vec{u} =< a, b, c >$.

Planes. A plane can be realized as the complement of a line in $\mathbb{R}^3$, i.e., all the vectors that is perpendicular to a given line. Here the direction vector of the line is called the **Normal Vector**. Assume that $\vec{n} =< a, b, c >$, and given a point $Q = (x_0, y_0, z_0)$ on the plane, we have the following relation:

$$\vec{PQ} \cdot \vec{n} = (\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

Here $\vec{r} = \vec{OP} =< x, y, z >$ is any point on the plane, and $\vec{r}_0 = \vec{OQ}$ In terms of coordinate form:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Generally any equation of the form

$$ax + by + cz + d = 0$$

describes a plane, with normal vector $\vec{n} =< a, b, c >$.

Intersections

Intersection of Lines. Two lines could have three possible positions: parallel, intersect with each other, or skew lines. If they intersect, then they intersect at a point. Notice that if we parametrize the lines by time t, they don't necessarily intersect at the same time, hence we should parametrize two lines using different parameters: Assume that the two lines have equations $r_1(t) =< x_1(t), y_1(t), z_1(t) >$ and $r_2(s) =< x_2(s), y_2(s), z_2(s) >$, then the intersection point comes from the following linear system:

$$\begin{cases} x_1(t) = x_2(s) \\ y_1(t) = y_2(s) \\ z_1(t) = z_2(s) \end{cases}$$

Intersection of planes. Two planes will intersect at a line L, and the direction vector of L comes from the cross product of the normal vectors. Assume that plane A has normal vector $\vec{n}_1$, and plane B has normal vector $\vec{n}_2$. Then the direction vector of L equals $\vec{n}_1 \times \vec{n}_2$. In order to find the equation of L, one just need to find a point on the line, which can easily be done by putting the plane equations together, and set $x = 0$.

A line intersect with a plane. A line can either intersect, or be parallel to the plane. If they intersect, then the intersection is a point. To find the point, just replace the x, y, z in the equation of the plane by $x(t)$, $y(t)$, $z(t)$, then the equation becomes an equation with t only. Solve for t.

Intersection of three planes. Three planes will intersect at one point. Assume that the planes have equations $a_ix + b_iy + c_iz = d_i$, for $i = 1, 2, 3$, then to find the point is equivalent to finding the solution to the following linear system:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Distance

Distance from a point to a line. Let P be a point, L is a line with direction vector $\vec{u}$ and contains a point Q, then the distance between P to the line L is:

$$d = |PQ| \sin \alpha = \frac{|\vec{PQ} \times \vec{u}|}{|\vec{u}|}$$

Here Q can be any point on the line.

Distance from a point to a plane. Let P be a point, H is a plane with normal vector $\vec{n}$ and contains a point Q. Then the distance between P to the plane H is:

$$d = |PQ||\cos \alpha| = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

Here Q can be any point on the plane.

Actually, there is a nicer formula with coordinate form. Let $P = (x_0, y_0, z_0)$, and H has the equation $ax + by + cz + d = 0$, then the distance between P to H is:

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Try to prove this formula from the precious vector form formula.

Distance from a line to another line. As we know already, there are three possible relative positions between two lines: parallel, intersecting, or skew. We will discuss accordingly.

Case 1 Intersecting Lines
: If two lines intersect with each other, the the distance, of course, equals 0.

Case 2 Parallel Lines:
: If two lines L_1 and L_2 are parallel, then the distance between is the distance between P to L_2, where P is any point on L_1.

Case 3 Skew Lines:

If L_1 and L_2 are skew lines with direction vectors $\vec{u}_1$ and $\vec{u}_2$, then the distance is:

$$d = \frac{|\vec{PQ} \cdot (\vec{u}_1 \times \vec{u}_2)|}{|\vec{u}_1 \times \vec{u}_2|}$$

Here P is ANY point on L_1, and Q is ANY point on L_2.

Distance from a line to a plane. If they intersect, then the distance $d = 0$; otherwise the distance is given by the distance from any point on the line to the plane.

Distance from a plane to another plane. If they intersect, then the distance $d = 0$; otherwise the distance is given by the distance from any point on the first plane to another plane.