MAC 2313-0001
Xiping Zhang
Test 1
09/15/2016

Print Name XP

Signature XP

△: fundamental problems
□: challenge problems

INSTRUCTIONS:

- Write answer in the paper provided.
- Clearly show all work and circle/box answer.
- $\mathbb{R}$ denotes the real line.
- $\mathbb{R}^3$ denotes the three dimensional real space.

KEEP CALM AND DO SOME CALCULUS !

1.(20pts). 'Trick or Treat'
Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

△ (1) (5pts) Any three points P, Q, R in $\mathbb{R}^3$ are coplanar.
□ (2) (5pts) For any vectors $\vec{u}$ and $\vec{v}$, $|\vec{u}+\vec{v}| = |\vec{u}| + |\vec{v}|$.
△ (3) (5pts) If $\vec{r(t)}$ is a differentiable function, then it is continuous.
△ (4) (5pts) $\vec{u} \perp \vec{v}$ if and only if $\vec{u} \times \vec{v} = 0$.

2.(30pts). Let $\vec{a} = \vec{i} + 2\vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{c} = \vec{j} - 2\vec{k}$.

△ (1) (5pts) Find $|\vec{b}|$.
△ (2) (5pts) Find the angle between $\vec{a}$ and $\vec{c}$.
△ (3) (5pts) Find the projection of $\vec{a}$ on $\vec{c}$.
△ (4) (5pts) Find $\vec{b} \times \vec{c}$.
△ (5) (5pts) Find the area of the parallelogram generated by $\vec{b}$ and $\vec{c}$.
△ (6) (5pts) Find the volume of the parallelepiped generated by $\vec{a}, \vec{b}$ and $\vec{c}$.

△ **3.(10pts).** Find the distance between origin to the plane $3x + 2y - z = 5$.

4.(15pts). The line L_1 is perpendicular to the plane $x - y + 2z = 2016$ and passes through origin. The line L_2 is cut by two planes $x + y + z = 3$ and $x - y + z = 1$.

△ (1) (10 pts) Write down the equations of L_1 and L_2.
□ (2) (5 pts) Describe the relative position of L_1 and L_2 and **explain why**. Are they parallel, intersect, or skew lines ?

KEEP CALM AND DO SOME CALCULUS !

△ **5.(10pts).** $r(\vec{t}) =< \cos t, \sin t, t >$ is a smooth space curve. Find the equation of the tangent line to the curve at $(0, 1, \pi/2)$.

△ **6.(10pts).** Find the equation of the plane that passes through $(1, 2, 3)$ and contains the x-axis.

☐ **7.(10pts).** Prove the following statement:
For any vector $\vec{u}$, if we denote α to be the angle between $\vec{u}$ and $\vec{i}$, β to be the angle between $\vec{u}$ and $\vec{j}$, and γ to be the angle between $\vec{u}$ and $\vec{k}$, then we have:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Here $\vec{i}, \vec{j}, \vec{k}$ are coordinate vectors.

(Hint: Set $\vec{u} =< a, b, c >$, then express those cosine's using a, b, c)

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Answer Sheet

1. True or False

① $\boxed{\text{True}}$. Any three pts generate a plane.

② $\boxed{\text{False}}$ Consider $\vec{u} = -2\vec{v}$. $|\vec{u}+\vec{v}| = |\vec{v}|$, but $|\vec{u}| + |\vec{v}| = 3|\vec{v}|$

③ $\boxed{\text{True}}$ $\vec{u} \perp \vec{v}$ iff $\vec{u} \cdot \vec{v} = 0$, not $\vec{u} \times \vec{v}$.

④ $\boxed{\text{False}}$

2. ① $|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+1+4} = \sqrt{14}$

② Denote the angle by α. the angle between $\vec{a}$ & $\vec{c}$.

$$\cos\alpha = \frac{\vec{u} \cdot \vec{c}}{|\vec{a}| \cdot |\vec{c}|} = \frac{\langle 1, 2, -2 \rangle \cdot \langle 0, 1, -2 \rangle}{\sqrt{1^2 + 2^2 + (-2)^2} \cdot \sqrt{1^2 + (-2)^2}} = \frac{2+4}{\sqrt{9} \cdot \sqrt{5}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}}.$$

$\alpha = \arccos \frac{2}{\sqrt{5}}$.

③ length of the projection: $|\vec{u}| \cdot \cos\alpha = 3 \cdot \frac{2}{\sqrt{5}} = \frac{6}{\sqrt{5}}$.

direction vector = unit vector of $\vec{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\langle 0, 1, -2 \rangle}{\sqrt{5}}$

Projection vector is: $\frac{6}{\sqrt{5}} \cdot \frac{\langle 0, 1, -2 \rangle}{\sqrt{5}} = \frac{6}{5} \cdot \langle 0, 1, -2 \rangle = \langle 0, \frac{6}{5}, \frac{-12}{5} \rangle$.

④ $$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \vec{i} \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix}$$

$= 3\vec{i} + 6\vec{j} + 3\vec{k} = \langle 3, 6, 3 \rangle$

⑤ Area $= |\vec{b} \times \vec{c}| = \sqrt{3^2 + 6^2 + 3^2} = 3\sqrt{1^2 + 2^2 + 1^2} = 3\sqrt{6}$

⑥ volume $= |\vec{a} \cdot (\vec{b} \times \vec{c})| = |\langle 1, 2, -2 \rangle \cdot \langle 3, 6, 3 \rangle|$

Math Rocks !

$= |3 + 12 - 6| = 9$.

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Answer Sheet

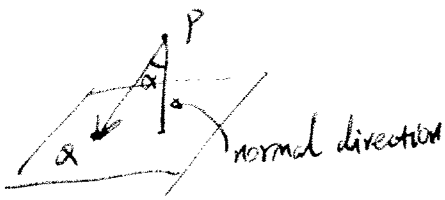
3. P = origin = $(0, 0, 0)$

Pick any pt Q on the plane. (set $z=0$, find a solution for x, y)

$(1, 1, 0)$

$\overrightarrow{PQ} = \langle 1, 1, 0 \rangle$, $\vec{n}$ = normal vector of the plane $= \langle 3, 2, -1 \rangle$

distance $= |\overrightarrow{PQ}| \cdot \cos\alpha = \dfrac{\overrightarrow{PQ} \cdot \vec{n}}{|\vec{n}|} = \dfrac{\langle 1,1,0 \rangle \cdot \langle 3, 2, -1 \rangle}{\sqrt{3^2+2^2+(-1)^2}} = \dfrac{5}{\sqrt{14}}$



4. 1) ① equation of L_1: $\begin{cases} \text{pt: } (0, 0, 0) \\ \text{direction: } \langle 1, -1, 2 \rangle \end{cases} \Rightarrow L_1: \langle 0+t, 0-t, 0+2t \rangle = \langle t, -t, 2t \rangle$

② equation of L_2: $\begin{cases} x+y+z=3 \\ x-y+z=1 \end{cases}$ set $z=t$ be the free parameter $\begin{cases} x+y = 3-t \\ x-y = 1-t \end{cases}$

$\Rightarrow \begin{cases} x = 2-t \\ y-1 \end{cases} \Rightarrow L_2: \begin{cases} x = 2-t \\ y=1 \\ z=t \end{cases} = \langle 2-t, 1, t \rangle$

2) ① L_1 is Not parallel to L_2 because direction vectors of them are Not multiple of each other. ($\langle 1, -1, 2 \rangle$ and $\langle -1, 0, 1 \rangle$)

② L_1: $\langle t, -t, 2t \rangle$

L_2. $\langle 2-s, 1, s \rangle$

$\begin{cases} t = 2-s \\ -t = 1 \\ 2t = s \end{cases} \Rightarrow$ No solution. So they do Not intersect.

③ Hence they are skew lines

Math Rocks !

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Answer Sheet

5. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

pt is $(0, 1, \frac{\pi}{2})$, Need to find what t value

$\vec{r}(t) \begin{cases} \cos t = 0. \\ \sin t = 1 \\ t = \frac{\pi}{2} \end{cases} \Rightarrow t = \frac{\pi}{2}$.

so the tangent line has direction:

$\vec{r}'(\frac{\pi}{2}) = \langle -1, 0, 1 \rangle$

equation: $\langle 0 - t, 1 + 0t, \frac{\pi}{2} + 1t \rangle$
$= \langle -t, 1, \frac{\pi}{2} + t \rangle$.

6. equation of a plane: need a pt & normal vector.

to get normal vector: need two vectors on the plane, then cross product.

first vector: line in the plane: x-axis. $= \langle t, 0, 0 \rangle \Rightarrow \vec{u} = \langle 1, 0, 0 \rangle$

second vector: try two other pts: $P(1, 2, 3)$ and any pt on the line. $(1, 0, 0)$ Q.

$\vec{PQ} = \langle 0, -2, -3 \rangle$

normal vector: $\vec{PQ} \times \vec{u} = \langle 1, 0, 0 \rangle \times \langle 0, -2, -3 \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -2 & -3 \end{vmatrix} = -(-3\vec{j} + 2\vec{k}) = 3\vec{j} - 2\vec{k}$

equation: $0(x-1) + 3(y-2) + (-2)(z-3) = 0$
$3(y-2) - 2(z-3) = 0$.

7. $\cos\alpha = \frac{\vec{u} \cdot \vec{i}}{|\vec{u}| |\vec{i}|} = \frac{\langle a, b, c \rangle \cdot \langle 1, 0, 0 \rangle}{\sqrt{a^2+b^2+c^2} \cdot 1} = \frac{a}{\sqrt{a^2+b^2+c^2}}$

$\cos\beta = \frac{\vec{u} \cdot \vec{j}}{|\vec{u}| \cdot |\vec{j}|} = \frac{b}{\sqrt{a^2+b^2+c^2}}$ $\quad \cos\gamma = \frac{\vec{u} \cdot \vec{k}}{|\vec{u}| \cdot |\vec{k}|} = \frac{c}{\sqrt{a^2+b^2+c^2}}$

so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2} = 1$.