MAC 2313-0001
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Test 4
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Signature Answer Key

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $[a, b] \times [c, d]$ denotes the rectangle $\{a \leq x \leq b; c \leq y \leq d\}$ in xy plane.
- Keep Calm and Enjoy the Computations!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) The positive orientation of a closed curve is clock-wise.

False.

(2) (5pts) $\{1 \leq x^2 + y^2 \leq 4\}$ is simply-connected.

False

(3) (5pts) A vector field $\vec{F}$ is conservative if and only if $div\vec{F} = 0$.

False

(4) (5pts) Our final exam is on Wednesday.

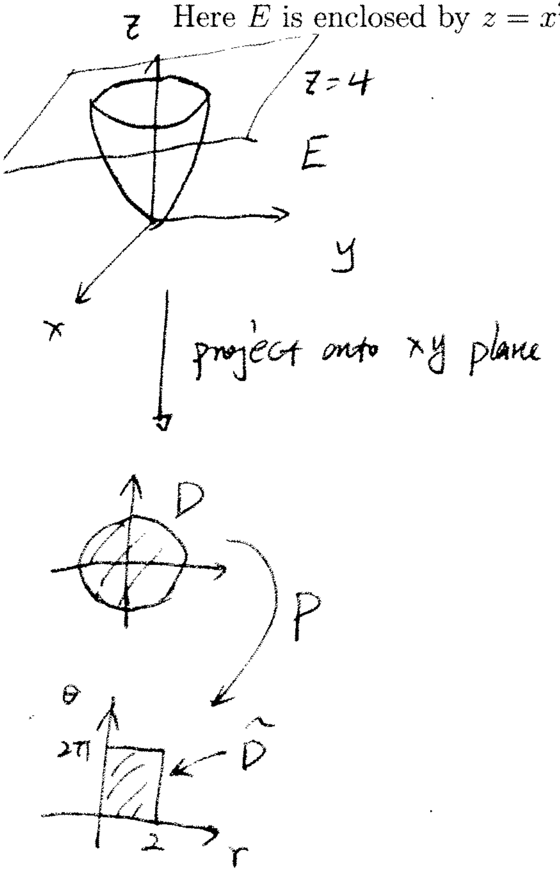
False

2. Integrals

2.1. **(15 pts).** Express the following Riemann sum over the solid E as a triple integral, and compute the integral.

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{x_i^2+y_j^2}\,\Delta V.$$

Here E is enclosed by $z=x^2+y^2$ and $z=4$.



$$\iiint_E x^2+y^2\,dV = \iint_D \int_{x^2+y^2}^{4}\sqrt{(x^2+y^2)}\,dz\,dA$$

$$= \iint_D \sqrt{(x^2+y^2)}\,(4-x^2-y^2)\,dA$$

$$= \iint_{\tilde D} r(4-r^2)\,r\,d\tilde A$$

$$= \int_0^{2\pi}\int_0^2 r^2(4-r^2)\,dr\,d\theta$$

$$= \cancel{2\pi\left(r^3-\frac{r^5}{6}\right)\Big|_0^2}$$

$$= \cancel{2\pi\left(16-\frac{64}{6}\right)}$$

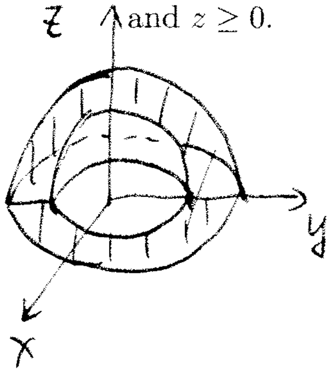
$$= \cancel{2\pi\left(16-\frac{32}{3}\right) = \frac{32\pi}{3}}$$

$$= 2\pi\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)\Big|_0^2$$

$$= 2\pi\left(\frac{32}{3}-\frac{32}{5}\right)$$

$$= \frac{128\pi}{15}$$

2.2. **(10 pts).** Find the triple integral over the solid E enclosed by $1 \le x^2+y^2+z^2 \le 4$ and $z \ge 0$.



$$\iiint_E z dV$$

so the solid E is part of a ball

$\longrightarrow$ spherical coordinates

$$\begin{cases} x = r\sin\alpha\cos\beta \\ y = r\sin\alpha\sin\beta \\ z = r\cos\alpha \end{cases} \qquad \begin{aligned} r&: 1 \longrightarrow 2 \\ \alpha&: 0 \longrightarrow \tfrac{\pi}{2} \\ \beta&: 0 \longrightarrow 2\pi \end{aligned}$$

so $\iiint_E z dV = \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (r\cos\alpha)(r^2\sin\alpha) \, d\beta \, d\alpha \, dr$

$= \int_1^2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^3 \sin\alpha\cos\alpha \, d\beta \, d\alpha \, dr$

$= 2\pi \int_1^2 r^3 dr \int_0^{\frac{\pi}{2}} \sin\alpha\cos\alpha \, d\alpha$

$= 2\pi \left(\frac{r^4}{4}\Big|_1^2\right) \left(-\frac{1}{4}\cos 2\alpha \Big|_0^{\frac{\pi}{2}}\right)$

$= 2\pi \cdot \frac{15}{4} \cdot \frac{1}{2} = \frac{15\pi}{4}$

2.3. **(10 pts).** Compute the following integral by changing variables.

$$\iint_R y^2 dA.$$

Here R is enclosed by $1 \le xy \le 2$ and $1 \le xy^2 \le 2$

Set $\begin{cases} u = xy. \\ v = xy^2. \end{cases}$

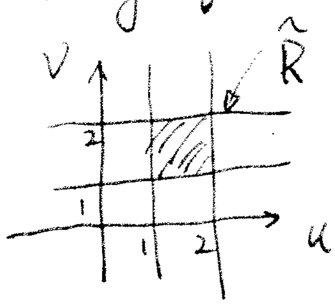
$\Rightarrow \begin{cases} x = \frac{u^2}{v}. \\ y = \frac{v}{u}. \end{cases}$ so the Jacobian is

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$= \frac{2}{v} - \frac{1}{v} = \frac{1}{v}.$$

And the range of u, v

are :



so $\iint_R y^2 dA$

$$= \iint_{\tilde{R}} \left(\frac{v}{u}\right)^2 \cdot \left(\frac{1}{v}\right) d\tilde{A} = \int_1^2 \int_1^2 \frac{v}{u^2} \, du \, dv$$

$$= \int_1^2 v \, dv \int_1^2 \frac{1}{u^2} du$$

$$= \left(\frac{1}{2} v^2 \Big|_1^2\right) \left(\frac{-1}{u} \Big|_1^2\right)$$

$$= \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

3. Vector Calculus

3.1. **(10 pts).** Let C be a piece of space helix given by $\vec{r}(t) =< \cos t, \sin t, t >$, $0 \leq t \leq \pi/4$. Let $\vec{F} =< x, y, xy >$ be a smooth vector field. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

$$\int_C \vec{F}\, d\vec{r} = \int_C x\,dx + y\,dy + xy\,dz$$

since $\vec{r}(t) = < \cos t, \sin t, t >$, $t: 0 \longrightarrow \frac{\pi}{4}$

$$\begin{cases} dx = -\sin t\, dt \\ dy = \cos t\, dt \\ dz = dt \end{cases} \Rightarrow \int_C \vec{F}\, d\vec{r} = \int_0^{\frac{\pi}{4}} \cos t(-\sin t) + (\sin t \cos t) + (\cos t)(\sin t)\; dt$$

$$= \int_0^{\frac{\pi}{4}} \sin t \cos t\, dt$$

$$= -\frac{1}{4} \cos 2t \Big|_0^{\frac{\pi}{4}} = \frac{1}{4} \cos 2t \Big|_{\frac{\pi}{4}}^{0} = \frac{1}{4}.$$

3.2. **(20 pts).** A vector field $\vec{F}$ is given by $< yz, xz, xy + 2z >$.

(1) (5 pts) Show that $\vec{F}$ is a conservative vector field.

* $\vec{F}$ is defined on $\mathbb{R}^3 \Rightarrow \vec{F}$ is conservative iff $\text{curl}\,\vec{F} = 0$.

* $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+2z \end{vmatrix} = \langle x - x, y - y, z - z \rangle$

$= \langle 0, 0, 0 \rangle$

* so $\vec{F}$ is conservative.

(2) (10 pts) Find a differentiable function $f(x, y, z)$ such that $\nabla f = \vec{F}$.

* $f_x = yz \Rightarrow f(x.y.z) = xyz + g(y.z)$.

* so $\begin{cases} f_y = xz + g_y = xz \\ f_z = xy + g_z = xy + 2z \end{cases} \Rightarrow \begin{cases} g_y = 0 \\ g_z = 2z \end{cases} \Rightarrow g(y.z) = z^2 + C$

so $f(x.y.z) = xyz + z^2 + C$.

(3) (5 pts) Find $\oint_C \vec{F} \cdot d\vec{r}$. Here C is a smooth curve from $(0, 0, -1)$ to $(0, 0, 1)$.

By the fundamental theorem of calculus.

$\int_C \vec{F}\,d\vec{r} = f(0.0.1) - f(0,0,-1)$

$= 1 - 1 = 0$

3.3. **(10 pts).** For any differentiable functions $f(x, y, z)$ and $g(x, y, z)$ with continuous second prtial derivatives, compute and **simplify** $div(\nabla f \times \nabla g)$.

$$\nabla f = \langle f_x \quad f_y \quad f_z \rangle$$

$$\nabla g = \langle g_x \quad g_y \quad g_z \rangle$$

$$\nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \langle f_y g_z - g_y f_z \, , \, g_x f_z - g_z f_x \, , \, f_x g_y - f_y g_x \rangle$$

so $$div(\nabla f \times \nabla g) = (f_y g_z - g_y f_z)_x + (g_x f_z - g_z f_x)_y + (f_x g_y - f_y g_x)_z$$

$$= f_{yx} g_z + f_y g_{zx} - g_{yx} f_z - g_y f_{zx}$$

$$+ g_{xy} f_z + g_x f_{zy} - g_{zy} f_x - g_z f_{xy}$$

$$+ f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz}$$

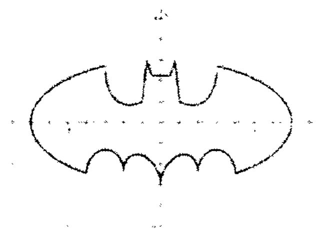
$$= 0$$

The above vanishing is due to the fact that f, g have continuous 2nd derivatives, hence the partial derivatives are independent of order.

3.4. **(10 pts).** Find the following line integral

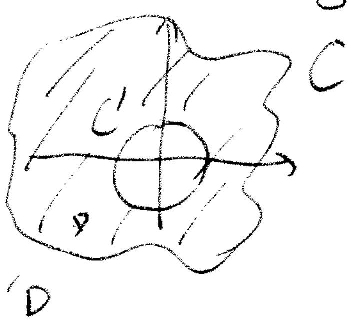
$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Here C is the closed cirve given by the figure:



[Hint: 'Poled' Green's Theorem.]

* Consider the region enclosed by the curve C with a hole in the middle



so $\vec{F}$ is defined & differentiable on D, Green's Theom

$$\Rightarrow \iint_D \text{curl}\vec{F}\cdot d\vec{A} = \oint_C \vec{F}dr - \oint_{C'} \vec{F}dr.$$

since $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \Rightarrow \text{curl}\vec{F} = Q_x - P_y = 0$

so $\oint_C \vec{F}d\vec{r} - \oint_{C'} \vec{F}d\vec{r} = 0 \Rightarrow \oint_C \vec{F}d\vec{r} = \oint_{C'} \vec{F}dr$

so we can use any closed curve C' to replace C.

Pick C' to be the unit circle $x^2+y^2=1 \Rightarrow \begin{cases} x = \cos\alpha \\ y = \sin\alpha \end{cases}$ $\alpha: 0 \to 2\pi$

$$\text{so } \oint_{C'} \vec{F}d\vec{r} = \int_0^{2\pi} \frac{-\sin\alpha}{1}(-\sin\alpha) + \frac{\cos\alpha}{1}(\cos\alpha)\, d\alpha$$

$$= \int_0^{2\pi} (\sin^2\alpha + \cos^2\alpha)\, d\alpha = 2\pi$$