PRACTICE FOR FINAL EXAM

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1. All the Previous Tests

2. Vector Operations and Equations of planes and surfaces

2.1. Let $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$, $\vec{c} = \vec{j} - 5\vec{k}$.

(1) Find $\vec{a} \times (\vec{b} + \vec{c})$.
(2) Find $\vec{a} \cdot \vec{b}$.
(3) Area of the parallelogram generated by $\vec{a}$ and $\vec{b}$. What does it mean if the area is 0 ?
(4) Volume of the parallelpiped generated by $\vec{a}$, $\vec{b}$ and $\vec{c}$. What does it mean if the volume is 0 ?
(5) Angel between $\vec{a}$ and $\vec{b}$.
(6) Find the projection of $\vec{a}$ onto $\vec{b}$.

2.2. Find the equations of following lines or planes.

(1) Angle between 2 lines.
(2) Angle between 2 planes.
(3) The line that passes through 2 points.
(4) The plane that passes through 3 points.
(5) The line that passes through a given point and orthogonal to a given plane.
(6) The plane that is orthogonal to a given line and contains a point.
(7) The line cut by 2 planes.
(8) The point cut by 3 planes.

2.3. Find the following distances.

(1) Distance between a point to a line.
(2) The distance between two lines. **Here you need to determine what is the relatuion of these two lines, do they intersect, or are they parallel, or are they skew lines?**

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(3) Distance between a point and a plane.
(4) Distance between a point to a general surface. (See Max/Min problrms)

3. Vector Functions and Space Curves

3.1. Let $\vec{r}(t) =< \ln(x^2 - x - 2), \frac{x^2-4}{x-2}, 2xe^{x^2} >$ be a vector function.

(1) Find the domain of $\vec{r}(t)$
(2) Find the 1st derivetive $\vec{r}'(t)$.
(3) Find the following limits: $\lim_{t\to 2} \vec{r}(t)$ and $\lim_{t\to 1} \vec{r}(t)$.
(4) Find $\int \vec{r}'(t)dt$

3.2. Let $\vec{r}(t) =< 2\cos t, \sin t, t >$ be a space curve.

(1) Find the equation of the tangent line at $(2, 0, 0)$.
(2) Find the arc length of the curve from $(2, 0, 0)$ to $(-2, 0, \pi)$.
(3) Find the frenet frames, i.e., the unite tangent vector $\vec{\mathbf{T}}$, the normal vector $\vec{\mathbf{N}}$, and the binomal vector $\vec{\mathbf{B}}$.
(4) Find the curvature function $\kappa(t)$. At which points does the curve reaches the maximal/minal curvature ?
(5) Find the equations of the osculating plane and the normal plane of the curve.

4. Function of Several Variables

4.1. Use the $\epsilon - \delta$ language to prove that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

4.2. Show that

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2} = 0$$

doesn't exist

4.3. Maximal and Minimal Problem. Find the Absolute Max and Absolute Min of the following function $f(x, y)$ with the domain D.

$$f(x, y) = x^2 + y^2 - 4x - 2y$$

and D is the closed disk $x^2 + y^2 = 20$.

4.4. Implicit Differantion. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ if

$$x^2 + y^2 + z^2 = z.$$

Then find the equation of the tangent plane to the surface at point $(0, 0, 1)$.

5. Multiple Integrals

5.1. Translatation between Riemann sum and integral.

5.2. Sketch, REMEMBER, and PARAMETRIZE the following surfaces.

(1) $x + y + z = 1$, $z = 1 + x$
(2) $x^2 + y^2 + z^2 = 1$, $2x^2 + 3y^2 + z^2 = 1$.
(3) $x^2 + y^2 + z^2 = 2x$.
(4) $z = 1 - x^2 - y^2$, $z = x^2 + y^2$, $z = \sqrt{x^2 + y^2}$.
(5) $x^2 + y^2 = 1$, $z = x^2$, $x^2 - y^2 = z$.
(6) THe 'Icercam': $x^2 + y^2 + z^2 = z$ intersect with $z = \sqrt{x^2 + y^2}$.

5.3. Sketch the Regions and Compute the following integrals.

(1) $\iint_R xy dA$, $R = \{1 \le y \le 1; y^2 \le x \le y + 2\}$.
(2) $\iint_R xy dA$, R is the region in the first quadrant bounded by $x = y^2$ and $x = 8 - y$.
(3) $\iint_R (x^2 + y^2)^{3/2} dA$, R is the region in the first quadrant bounded by $y = 0$, $y = \sqrt{3}x$, and the circle $x^2 + y^2 = 1$.
(4) $\iint_R (x^2+y^2)^{3/2} dA$, R is the region in the upper half plane bounded by $x^2+y^2 = 1$ and $x^2 + y^2 = 4$.
(5) Find the areas of the abouve regions.

5.4. Compute the following integral by suitable change of coordinates.

(1) $\iint_R y dA$, R is bounded by $y \ge 0$, $y^2 = 4 - 4x$, $y^2 = 4 + 4x$. [Hint: $x = u^2 - v^2$, $y = 2uv$.]
(2) $\iint_R e^{\frac{x+y}{x-y}} dA$, R is the trapezodial region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, $(0, -1)$.
(3) $\iint_R \cos \frac{x+y}{x-y} dA$, R is the trapezodial region with vertices $(1, 0)$, $(2, 0)$, $(0, 2)$, $(0, 1)$.
(4) $\iint_R (x + y)e^{x^2-y^2} dA$, R is the rectangle enclosed by $x - y = 0$, $x - y = 2$, $x + y = 0$, $x + y = 3$.

5.5. Sketch the Regions and Compute the following integrals.

(1) $\iiint_E xydV$, E is the solid tetrahedron cut by plane $3x+y+z=1$ with coordinate lines.
(2) $\iiint_E xydV$, E is bounded by $x=1-y^2-y^2$ and $x=0$.
(3) $\iiint_E zdV$, E is bounded by $y=0$, $z=0$, $x+y=2$, and the cylinder $x^2+y^2=1$.
(4) $\iiint_E z^3\sqrt{x^2+y^2+z^2}dV$, E is the solid bounded by $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ and $z\geq 0$.
(5) Find the volumes of the above solids.

6. Line and Surface Integrals

6.1. Evaluate the following line integrals.

(1) $\int_C xds$, C is the arc of $y=x^2$ from $(0,0)$ to $(1,1)$.
(2) $\int_C ydx+(x+y^2)dy$, C is the the ellipse $4x^2+9y^2=36$ with **clockwise** orientation.
(3) $\int_C y^3dx+x^2dy$, C is the arc of $x=1-y^2$ from $(0,-1)$ to $(0,1)$.
(4) $\int_C \vec{F}\cdot d\vec{r}$, here $\vec{F}=<e^z, xz, (x+y)>$, C is given by $\vec{r}(t)=<t^2,t^3,-t>$, $0\leq t\leq 1$.

6.2. Let $\vec{F}=<e^y, xe^y+e^z, ye^z>$ be a vector field.

(1) Show that $\vec{F}$ is a conservative vector field.
(2) Find such a function $f(x,y,z)$ such that $\nabla f=\vec{F}$.
(3) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ for C given by **any curve** from $(0,2,0)$ to $(4,0,3)$.
(4) Use the Fundamental Theorem of Calculus to verify your result.
(5) Find the arc length of the above curves.

6.3. Verify thet Green's Theorem is true for the following line integral.

$$\oint_C xy^2dx-x^2ydy$$

Here C consists of the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$, and the line segment from $(1,1)$ to $(-1,1)$.

6.4. Find the following line integral.

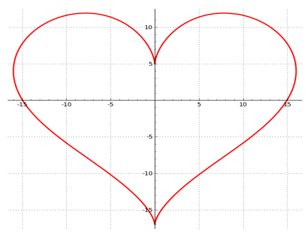
$$\int_C \sqrt{1++x^3}dx + 2xydy.$$

Here C is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

6.5. Find the following line integral.

$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Here C is the closed curve given by the figure:



[Hint: 'Poled' Green's Theorem.]

6.6. Evaluate the following Surface integrals.

(1) $\iint_S zdS$, S is the part of $z = x^2 + y^2$ the lies under $z = 4$.
(2) $\iint_S (x^2z + y^2z)dS$, S is the part of $4 + x + y - z = 0$ the lies inside $x^2 + y^2 = 1$.
(3) $\iint_S \vec{F} \cdot d\vec{S}$, here $\vec{F} =< xz, -2y, 3x >$, and S is the sphere $x^2 + y^2 + z^2 = 4$ with **inside orientation**.
(4) $\iint_S \vec{F} \cdot d\vec{S}$, here $\vec{F} =< x^2, xy, z >$, and S is the part of the paraboloid $z = x^2 + y^2$ below $z = 1$ with **upward orientation**.

7. IMPORTANT

During the test I won't tell you whether you should use Green's Theorem or Stoke's Theorem or Divergence Theorem or other ways to compute the surface integrals/line integrals, you need to decide by yourself.

8. Stoke's Theorem and the Divergence Theorem

8.1. Verify that the Stoke's Theorem is true for the vector field $\vec{F} =< x^2, y^2, z^2 >$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy plane. S is oriented **downward**.

8.2. Use Stoke's Theorem to compute $\iint_S curl\vec{F}d\vec{S}$, where $\vec{F} =< x^2yz, yz^2, z^3e^{xy} >$, S is the part of $x^2 + y^2 + z^2 = 5$ that lies above $z = 1$.

8.3. Use Stoke's Theorem to compute $\int_C \vec{F}d\vec{r}$. $\vec{F} =< xy, yz, zx >$, and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented **clockwise**.

8.4. Use the Divergence Theorem to calculate $\iint_S \vec{F}d\vec{S}$. Here $\vec{F} =< x^3, y^3, z^3 >$, and S is the boundary surface of the solid E bounded by $x^2+y^2 = 1$, $z = x+y$ and $z = 0$.

8.5. Use the Divergence Theorem to calculate $\iint_S \vec{F}d\vec{S}$. Here $\vec{F} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$, and S is the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

8.6. Use the Divergence Theorem to calculate $\iint_S \vec{F}d\vec{S}$. Here $\vec{F} =< x^2y, 1/3y^3 + \tan z^{1997}, xy^2 >$, and S is the upper half semi-shpere $z = \sqrt{1 - y^2 - z^2}$.