MAC 2313-0001
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Test 3
11/03/2016

Print Name Key

Signature

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $[a, b] \times [c, d]$ denotes the rectangle $\{a \leq x \leq b; c \leq y \leq d\}$ in xy plane.
- Keep Calm and Enjoy the Computations!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) $\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx$ for any continuous function $f(x,y)$.

True

(2) (5pts) A saddle point could be an absolute maximum.

False

(3) (5pts) Polar transform maps a disk with center at origin in xy plane to a rectangle in $r\theta$ plane.

True

(4) (5pts) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$ if $R_1 \cup R_2 = R$ and R_1 doesn't intersect R_2.

True

2. 'Extrema'

2.1. **(15pts).** Let $f(x, y) = x^2 + y^2 - 4x - 2y$ be a differentiable function defined on the closed disk D: $x^2 + y^2 \leq 20$. Find the **Absolute maximum** and the **Absolute minimum** of $f(x, y)$ over D.

1) $\nabla f = \langle 2x-4, 2y-2 \rangle = 0$ 5 pts

$\Rightarrow x=2 \quad y=1$

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow D=4 \Rightarrow$ local min $f(2.1) = -5.$

2) on the boundary $x^2+y^2=20. \Rightarrow \varphi(x.y) = x^2+y^2-20$

$L(x.y.\lambda) = x^2-4x+y^2-2y + \lambda(x^2+y^2-20)$

$\nabla L = \langle 2x-4+2\lambda x, 2y-2+2\lambda y, x^2+y^2-20 \rangle$ 5 pts

$\nabla L = \vec{0} \Rightarrow \begin{cases} 2(1+\lambda)x = 4 \\ 2(1+\lambda)y = 2 \\ x^2+y^2=20 \end{cases} \Rightarrow \begin{cases} (1+\lambda)x = 2 \\ (1+\lambda)y = 1 \\ x^2+y^2=20 \end{cases} \Rightarrow \begin{cases} x = 2y \\ x^2+y^2=20 \end{cases}$

$\Rightarrow y=\pm 2. \quad x=\pm 4$ or $\begin{cases} y=2 \\ x=4 \end{cases} \times \begin{cases} y=-2 \\ x=-4 \end{cases}$

@ (4.2) $f(4.2) = 4^2-4\times 4 + 2^2-2\times 2 = 0$

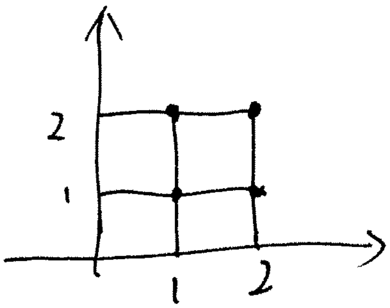
@ $(-4,-2)$. $f(-4,-2) = 4^2+4\times 4 + 2^2+2\times 2 = 40$ 5 pts

so Absolute Max $= 40$ @ $(-4,-2)$

Absolute Min $= -5$ @ (2.1)

3. 'Riemann Sum'

3.1. **(10pts).** Estimate the volume of the solid that lies over the square $R : [0, 2] \times [0, 2]$ and below the surface $z = xy$ via **Riemann Sum**. Here we divide R into **four** equal squares and use the **upper-right** corner as sample points.



$$\Delta x = \frac{2-0}{2} = 1$$

$$\Delta y = \frac{2-0}{2} = 1$$

$$\Delta x \Delta y = \Delta A = 1 \times 1$$

$$\begin{aligned} R = & f(1,1) \times 1 \\ & + f(2,1) \times 1 \\ & + f(1,2) \times 1 \\ & + f(2,2) \times 1 \\ = & 1 + 2 + 2 + 4 \\ = & 9 \end{aligned}$$

3.2. **(15pts).** Express the following Riemann Sum over the rectangle $[0, \pi] \times [0, 1]$ as a double integral, and then compute it.

$$\lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{\sin(x_i^*)}{1+(y_i^*)^2} \Delta x_i \Delta y_j$$

(1) (5 pts) [Express as a Double Integral and sketch the integration region]

$$\int_0^{\pi} \int_0^1 \left(\frac{\sin x}{1+y^2} \right) \, dy \, dx$$

(2) (10 pts)[Compute the above integral]

$f(x,y) = \dfrac{\sin x}{1+y^2} = (\sin x) \cdot \dfrac{1}{1+y^2}$ is separated.

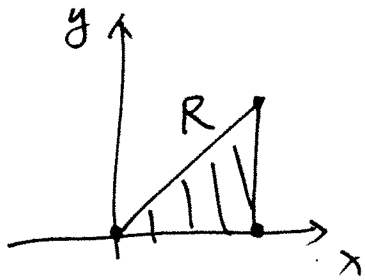
so $\displaystyle\int_0^{\pi} \int_0^1 \frac{\sin x}{1+y^2} \, dy \, dx = \int_0^{\pi} \sin x \, dx \cdot \int_0^1 \frac{1}{1+y^2} \, dy$

$= \left(-\cos x \Big|_0^{\pi} \right) \cdot \left(\arctan y \Big|_0^1 \right)$

$= 2 \cdot \dfrac{\pi}{4} = \dfrac{\pi}{2}$

4. Integrals and Applications

4.1. **(10pts).** Find the area of the surface $z = x^2 + 2y$ that lies above the triangle T in xy plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.



$R: \ 0 \le x \le 1$, $0 \le y \le x$ ← type I

Area Unit $= \sqrt{1 + f_x^2 + f_y^2}$.

$f_x = 2x$. $f_y = 2$.

so area $= A = \iint_R \sqrt{1 + (2x)^2 + 2^2}\, dA$

$= \int_0^1 \int_0^x \sqrt{5+4x^2}\, dy\, dx = \int_0^1 x\sqrt{5+4x^2}\, dx$

$u = 5 + 4x^2 \Rightarrow \frac{du}{dx} = 8x \Rightarrow dx = \frac{du}{8x}$

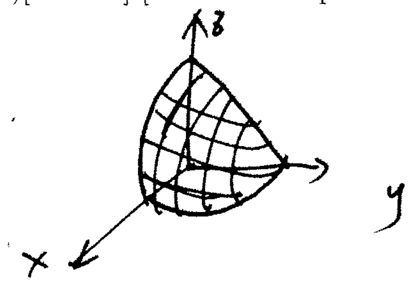
so $A = \int_5^9 \frac{\sqrt{u}}{8}\, du = \frac{1}{8} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} \Big|_5^9$

$= \frac{1}{12}\left(9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = \frac{1}{12}\left(27 - (\sqrt{5})^3\right)$

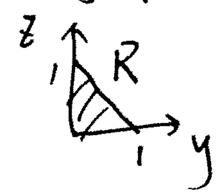
$= \frac{1}{12}\left(27 - \sqrt{125}\right)$

4.2. **(20 pts).** Use **integral method** to find the Volume of the 'sphere tetrahedron' in $\mathbb{R}^3$ formed by surface $x^2 + y + z = 1$, $x = 0$, $y = 0$ and $z = 0$.

(1) (5 pts)[**Bonus**] [Sketch the 'sphere tetrahedron']



project to yz plane



surface is

$x^2 = 1 - y - z$

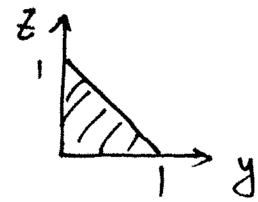
$x = \sqrt{1-y-z}$

since $x > 0$.

(2) (5 pts) Express the volume as a double integral, and sketch the integration region.

$$V = \iint_R \left(\sqrt{1-y-z} - 0\right) dA.$$

R:



(3) (10 pts) Compute the above integral.

$$V = \int_0^1 \int_0^{1-y} \sqrt{1-y-z}\, dz\, dy \qquad u = 1-y-z \ \underline{\text{sub}}.$$

$$= \int_0^1 \frac{2}{3}(1-y)^{\frac{3}{2}}\, dy \qquad u = 1-y \ \underline{\text{sub}}$$

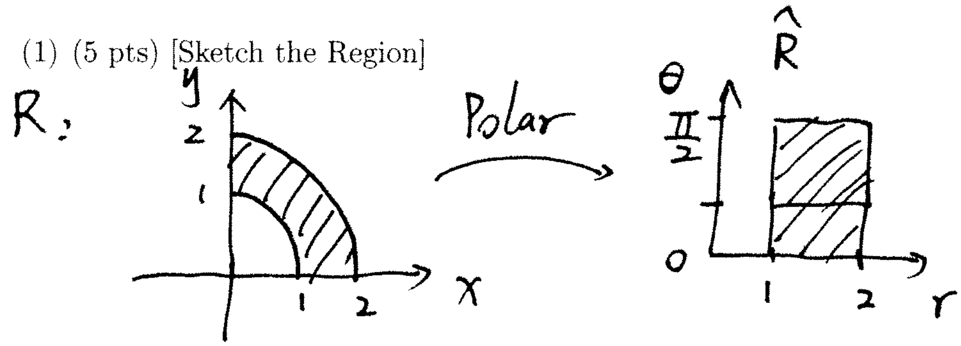
$$= \frac{2}{3} \cdot \frac{2}{5} u^{\frac{5}{2}} \Big|_0^1 = \frac{4}{15}.$$

4.3. **(15 pts).** Sketch the region and evaluate

$$\iint_R xy\,dA$$

where R is the region in xy plane enclosed by $\{x \geq 0\}$, $\{y \geq 0\}$ and $\{1 \leq x^2+y^2 \leq 4\}$.

(1) (5 pts) [Sketch the Region]



(2) (10 pts)[Compute the above integral]

$$\iint_R xy\,dA = \int_1^2 \int_0^{\frac{\pi}{2}} (r\cos\theta)(r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_1^2 \int_0^{\frac{\pi}{2}} r^3 \cos\theta\sin\theta\,d\theta\,dr$$

$$= \left(\int_1^2 r^3\,dr\right)\left(\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta\right) = \left(\frac{r^4}{4}\Big|_1^2\right)\cdot\left(-\frac{1}{2}\cos 2\theta\Big|_0^{\frac{\pi}{2}}\right)$$

$$= \left(4-\frac{1}{4}\right)(1) = \frac{15}{4}$$