MAC 2313-0001 Print Name ____________________
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Test 2
10/06/2016 Signature ____________________

INSTRUCTIONS:

- Write answer in the space provided after the problems.
- Clearly show all work and circle/box answer.
- $\mathbb{R}^n$ denotes the n dimensional real space.
- Keep Calm and Enjoy Calculus!

1. 'Trick or Treat'

Determine whether the statement is true or false. If it is true, say so; if it is false, explain why or give an example that disproves the statement.

(1) (5pts) If a space curve $\vec{r}(t)$ has constant curvature $\kappa(t) = 0$, then it's a straight line.

True

(2) (5pts) $f_{xy} = f_{yx}$ for any differentiable function $z = f(x, y)$.

False [illegible]

(3) (5pts) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \dfrac{1}{2}$.

False [illegible]

(4) (5pts) I covered §13.4 about Kepler's Theorem on a Wednesday morning.

False [illegible]

2. 'Space Curve'

Let $\vec{r}(t)\colon \mathbb{R} \to \mathbb{R}^3$ be a differentiable function. The function gives a space curve C. Assume that the curve has tangent vector $\vec{r}\,'(t) = < -\sqrt{3}\sin t, \sin t, 2\cos t >$, and passes through the point $(\sqrt{3}, -1, 0)$ when $t = 0$.

(1) (8pts) Find the equation of C, i.e., the explicit expression of $\vec{r}(t) = < x(t), y(t), z(t) >$.

$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = < \sqrt{3}\cos t + C_1, -\cos t + C_2, 2\sin t + C_3 >$

since $\vec{r}(0) = <\sqrt{3}, -1, 0>$

$\Rightarrow <\sqrt{3} + C_1, -1 + C_2, 0 + C_3> = <\sqrt{3}, -1, 0> \Rightarrow C_1 = 0,\ C_2 = 0,\ C_3 = 0$

$\therefore \vec{r}(t) = <\sqrt{3}\cos t, -\cos t, 2\sin t>$

(2) (6pts) Re-parameterize curve C by its arc-length s.

$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = |\vec{r}\,'(t)|$

$= \sqrt{3\sin^2 t + \sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

$ds = 2\,dt$

$\Rightarrow \int ds = \int 2\,dt \Rightarrow s = 2t$

$t = \frac{s}{2}$

so C is reparameterization of

$<\sqrt{3}\cos\frac{s}{2}, -\cos\frac{s}{2}, 2\sin\frac{s}{2}>$

(3) (8pts) Find the curvature function $\kappa(t)$ of C.

$\vec{r}\,'(t) = <-\sqrt{3}\sin t, \sin t, 2\cos t>$

$\vec{r}\,''(t) = <-\sqrt{3}\cos t, \cos t, -2\sin t>$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} i & j & k \\ -\sqrt{3}\sin t & \sin t & 2\cos t \\ -\sqrt{3}\cos t & \cos t & -2\sin t \end{vmatrix} = <-2, -2\sqrt{3}, 0>$

$\Rightarrow \kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \frac{\sqrt{2^2 + (2\sqrt{3})^2}}{2^3} = \frac{4}{8} = \frac{1}{2}$

(4) (10pts) Find the unit tangent vector $\vec{\mathbf{T}}$, the normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$.

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\vec{r}'(t)}{2} = \left\langle -\frac{\sqrt{3}}{2}\sin t,\ \frac{1}{2}\sin t,\ \cos t \right\rangle$$

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = \frac{\left\langle -\frac{\sqrt{3}}{2}\cos t,\ \frac{1}{2}\cos t,\ -\sin t \right\rangle}{\sqrt{\frac{3}{4}\cos^2 t + \frac{1}{4}\cos^2 t + \sin^2 t}} = \left\langle -\frac{\sqrt{3}}{2}\cos t,\ \frac{1}{2}\cos t,\ -\sin t \right\rangle$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} i & j & k \\ -\frac{\sqrt{3}}{2}\sin t & \frac{1}{2}\sin t & \cos t \\ -\frac{\sqrt{3}}{2}\cos t & \frac{1}{2}\cos t & -\sin t \end{vmatrix} = \left\langle -\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ 0 \right\rangle$$

(5) (8pts) Find equation of the osculating plane **H** to the curve C at point $(0, 0, 2)$.

osculating plane [illegible] $\vec{B} = \left\langle -\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ 0 \right\rangle$

so equation is

$$-\frac{1}{2}(x-0) + -\frac{\sqrt{3}}{2}(y-0) + 0(z-2) = 0$$

$$\Rightarrow -\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 0$$

3. 'Functions'

3.1. **(20pts).** Let $w = x\sin(y-z)$ be a differentiable function of three variables.

(1) (6pts) Find the gradient vector ∇f.

$f_x = \sin(y-z)$

$f_y = x\cos(y-z)$

$f_z = -x\cos(y-z)$

$\nabla f = \langle \sin(y-z),\ x\cos(y-z),\ -x\cos(y-z) \rangle$

(2) (7pts) Consider the level surface S in $\mathbb{R}^3$ given by $w = 0$. Find the equation of the tangent plane to S at point $(3, 2017\pi, 2016\pi)$.

equation of tangent plane @ $(3, 2017\pi, 2016\pi)$ is

$f_x(x-3) + f_y(y-2017\pi) + f_z(z-2016\pi) = 0$

at $(3, 2017\pi, 2016\pi)$, $\nabla f = \langle 0, -3, 3 \rangle$

so equation is $-3(y-2017\pi) + 3(z-2016\pi) = 0$

(3) (7pts) Find the directional derivative $\nabla_u f$ of $w = f(x,y,z)$ along the direction $\vec{u} = \langle 1,1,1 \rangle$ at $(3, 2017\pi, 2016\pi)$.

$\nabla_u f = \nabla f \cdot \vec{u} =$ [illegible]

so $\nabla_u f \Big|_{(3,\ 2017\pi,\ 2016\pi)} = 0 + (-3) + 3 = 0$

3.2. **(10pts).** Find $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^3 z}{\partial x \partial y \partial x}$ if provided that $yz + x \ln y = z^2$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} \qquad f(x, y, z) = x \ln y + yz - z^2$$

$$f_x = \ln y \qquad f_z = y - 2z$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}} = \frac{-\ln y}{y - 2z} = \frac{\ln y}{2z - y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\ln y}{2z - y}\right) = \frac{\frac{1}{y}(2z - y) - \ln y(-1)}{(2z - y)^2}$$

$$= \frac{\frac{1}{y}(2z - y) + \ln y}{(2z - y)^2} = \frac{2z - y + y \ln y}{y(2z - y)^2}$$

$$\frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0.$$

3.3. **(10pts).** Prove that the following function is continuous **Everywhere**.

$$f(n) = \begin{cases} \frac{\sin(x^2-y^3)xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if}(x,y) = (0,0). \end{cases}$$

(Hint: Some trig functions have bounds.)

Pf. Let $f(x,y) = \begin{cases} \frac{\sin(x^2-y^3)\,xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

1) If $(x,y) \neq (0,0)$, [illegible] function [illegible] continuous everywhere in the domain

2) At $(0,0)$, we need to show that $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2-y^3)\,xy}{\sqrt{x^2+y^2}} = 0$.

To show that, notice that

① $-1 \leq \sin(\text{anything}) \leq 1$

② $-\frac{x^2+y^2}{2} \leq xy \leq \frac{x^2+y^2}{2}$

so

$$\frac{(-1)\left(\frac{1}{2}\right)(x^2+y^2)}{\sqrt{x^2+y^2}} \leq \frac{\sin(x^2-y^3)\,xy}{\sqrt{x^2+y^2}} \leq \frac{(1)\left(\frac{1}{2}\right)(x^2+y^2)}{\sqrt{x^2+y^2}}$$

$$-\frac{1}{2}\sqrt{x^2+y^2} \leq \frac{\sin(x^2-y^3)\,xy}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\sqrt{x^2+y^2}.$$

limit is 0 [illegible] limit is 0 [illegible]

Hence [illegible] $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2-y^3)\,xy}{\sqrt{x^2+y^2}} = 0$

Scratch Here

* Sol: Proof using ε-δ language.

For any $\varepsilon > 0$, let $\delta = 2\varepsilon$, then $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$

$|\sin(\text{anything})| \le 1$

since $\left| \dfrac{\sin(x^2+y^3)\,xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| \dfrac{\sin(x^2+y^2)\,xy}{\sqrt{x^2+y^2}} \right| \le \dfrac{|xy|}{\sqrt{x^2+y^2}}$

$\le \dfrac{1}{2} \dfrac{x^2+y^2}{\sqrt{x^2+y^2}}$

$= \dfrac{1}{2}\sqrt{x^2+y^2} < \dfrac{1}{2}\delta = \varepsilon$

so by definition

$$\lim_{(x,y)\to(0,0)} \frac{xy\sin(x^2+y^2)}{\sqrt{x^2+y^2}} = 0$$