MAC 2311-0003
Xiping Zhang
Test 3
07/14/2016

Print Name XP's Answer Key

Signature

INSTRUCTIONS:

- Clearly show all work where instructed.
- Write answer in space provided or circle/box answer.
- If a limit may be classified as ∞ or $-\infty$, say so (and which one).
- If a limit does not exist and cannot be classified as ∞ or $-\infty$, write DNE

KEEP CALM AND DO SOME CALCULUS !

1.(5pts). Show your work clearly and circle your answer. **Simplify** your answers.

Find the linearization $L(x)$ of $f(x) = e^{2x}$ at $a = 0$.

2 pts $\left\{ \begin{array}{l} f'(x) = 2 \cdot e^{2x} \\ f'(a) = f'(0) = 2 \cdot e^{2 \cdot 0} = 2 \end{array} \right.$

$\left. \begin{array}{l} f(0) = e^{2 \cdot 0} = 1 \\ L(x) = 1 + 2x \end{array} \right\}$ 1 pt

2pts $\{ L(x) = f(0) + f'(0) \cdot (x - 0)$

2.(10pts). Show your work clearly and circle your answer. Find the value(s) of x where the absolute maximum and the absolute minimum of $f(x) = x^3 - 3x + 19$ occur on the interval [-3, 0]. Show all work and put answer in spaces provided.

Absolute Maximum **at** $x = -1$

Absolute Minimum **at** $x = -3$

$\left. \begin{array}{l} f(x) = x^3 - 3x + 19 \\ \text{on } [-3, 0] \end{array} \right\}$ 2 pts

$\left. \begin{array}{l} f'(x) = 3x^2 - 3 = 3(x^2 - 1) \\ \text{critical pts: } 1, -1. \to \text{only } -1. \\ (\text{Since } 1 \text{ is not in domain } [-3, 0]) \end{array} \right\}$ 3pts

3pts $\left\{ \begin{array}{l} f(-3) = (-3)^3 - 3(-3) + 19 \\ \quad = -27 + 9 + 19 = 1 \\ f(-1) = (-1)^3 - 3(-1) + 19 = -1 + 3 + 19 = 21 \\ f(0) = 0^3 - 3(0) + 19 = 19. \end{array} \right.$

so $\underbrace{\text{Absolute max @ } -1.}_{1 \text{ pt}}$ $\underbrace{\text{Absolute min @ } -3.}_{1 \text{ pt}}$

3.(10pts). Show your work clearly and circle your answer. Simplify if possible. For the function $f(x) = 2x\sqrt{5-x}$, on what interval is $f(x)$ **decreasing**? Write your answer in interval notation.

$f(x) = 2x \cdot \sqrt{5-x}$

2pts: Domain of $f(x)$:

$5 - x \geq 0 \Rightarrow x \leq 5.$

$(-\infty, 5]$

$f'(x) = 2 \cdot \sqrt{5-x} + 2x \cdot \frac{1}{2} \cdot \frac{(-1)}{\sqrt{5-x}}$

$= \dfrac{10-3x}{\sqrt{5-x}}$

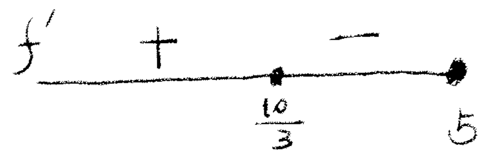
zeros: $\frac{10}{3}$.

DNE: 5

$\Rightarrow$ critical pts: $\frac{10}{3}$ & 5.

4 pts

sign chart



3pts

so $f(x)$ is decreasing on $\left(\frac{10}{3}, 5\right)$

1 pt

4.(10pts). Show your work clearly and circle your answer. Evaluate the following limit:

$$\lim_{x\to 0^+} x^{\tan x}$$

let $A = \lim_{x\to 0^+} x^{\tan x}$

$\ln A = \ln \lim_{x\to 0^+} x^{\tan x} = \lim_{x\to 0^+} \ln x^{\tan x}$ 2 pts

$\ln A = \lim_{x\to 0^+} \tan x \cdot \ln x$

$= \lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$, $\boxed{\frac{\infty}{\infty} \text{ type}}$

1pts

By l'hospital rule.

$\lim_{x\to 0^+} \dfrac{\ln x}{\cot x} = \lim_{x\to 0^+} \dfrac{(\ln x)'}{(\cot x)'}$

1pts

$= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\csc^2 x}$

$= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = \lim_{x\to 0^+} \dfrac{-\sin^2 x}{x}$, $\frac{0}{0}$ type

3pts

By l'hospital rule again

2pts: $\ln A = \lim_{x\to 0^+} \dfrac{\sin^2 x}{x} = \lim_{x\to 0^+} \dfrac{(\sin^2 x)'}{(x)'}$

$= \lim_{x\to 0^+} \dfrac{2\sin x \cos x}{1} = 0$

$\ln A = 0$

$\Rightarrow A = e^0 = 1$

1pt

5.(13pts). Use the Mean Value Theorem to finish the following proof:

Statement: $f(x) = \sin(x) - 2x + 1$ has at most one zero in $(0, \pi/2)$.

Proof. We will prove this statement by contradiction.

Step 1 Assume that there is more than 1 zero in the interval $(1, \pi/2)$, then there are at least 2 different zeros. Denote them by a and b.

Step 2 Since $f(x) = \sin(x) - 2x + 1$ is a linear form of simple functions, it is

continuous (2pts)

on the closed interval $[a, b]$, and is

differentiable (2pts)

on the open interval (a, b).

Hence by the Mean Value Theorem, there exists some number c in (a, b) such that

$$\overbrace{f'(c) = \frac{f(b) - f(a)}{b - a}}^{3pts} = \overbrace{\frac{0 - 0}{b - a} = 0}^{1 pt}$$ 4pts)

Step 3 However, by taking derivative we know that

$f'(x) =$ $\cos x - 2$ (2pts)

Step 4 Hence we have a contradiction because

$f'(x) = \cos x - 2$ is ALWAYS negative $\Rightarrow$ never zero (3pts)

Step 4 So the assumption is wrong, which proves that there is at most one zero in $(0, \pi/2)$.

□

6.(10pts). Show your work clearly and circle your answer. Let

$$f'(x) = \frac{2x^2 - 3x + 1}{x}$$

Find $f(x)$ if $f(-1) = 2016$.

$$3\text{pts}\left\{\begin{aligned} f'(x) &= \frac{2x^2-3x+1}{x} \\ &= 2x - 3 + \frac{1}{x}\end{aligned}\right.$$

"fake" quotient

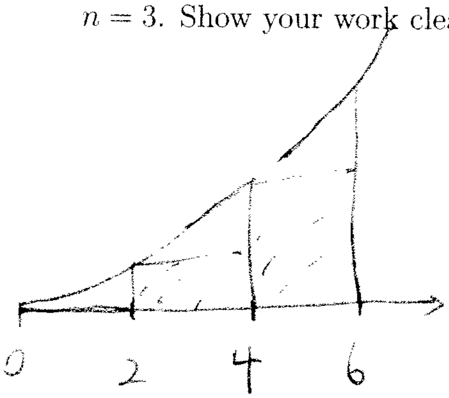
Hence the anti derivative of $f'(x)$ is $f(x)$

$$f(x) = \underbrace{2\cdot\frac{x^2}{2}}_{1\text{pt}} \underbrace{-3x}_{1\text{pt}} + \underbrace{\ln|x|}_{1\text{pt}} + \underbrace{C}_{1\text{pt}} = x^2 - 3x + \ln|x| + C$$

$$\left.\begin{aligned} &\text{Since } f(-1) = 2016 \Rightarrow (-1)^2 - 3(-1) + \ln(1) + C = 2016 \\ &\Rightarrow 1 + 3 + C = 2016 \Rightarrow C = 2012\end{aligned}\right\} 3\text{pts}$$

so $f(x) = x^2 - 3x + \ln|x| + 2012$

7.(10pts). Find the **left** Riemann sum of $f(x) = x^2$ on the closed interval $[0, 6]$ if $n = 3$. Show your work clearly and circle your answer.



$$\left.\begin{aligned} &n = 3, \text{ interval} = [0, 6] \\ &\text{so } \Delta x = \frac{6-0}{3} = 2 \\ &\text{section pts are } 0, 2, 4, 6\end{aligned}\right\} 3\text{pts}$$

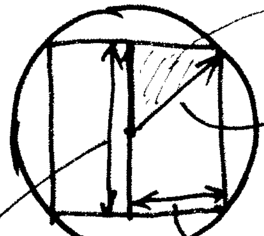
so the left Riemann sum is

$$L = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 \quad \} \ 4\text{pts}$$

$$\left.\begin{aligned} &= 0^2\cdot 2 + 2^2\cdot 2 + 4^2\cdot 2 \\ &= 0 + 8 + 32 \\ &= 40\end{aligned}\right\} 3\text{pts}$$

8.(12pts). A right cylinder is inscribed inside a sphere of radius $R = 5\sqrt{3}$ cm. Find out the largest possible volume of such a cylinder.

a cylinder inscribed inside a sphere looks like. (intersection view)



r $\frac{1}{2}h$ R

R = radius of the sphere $= 5\sqrt{3}$ cm

r = radius of the base circle of the cylinder

h = height of this cylinder

3 pts

so the volume of the cylinder $V = \pi r^2 h$.

From the graph, we know that the red shadow is a right triangle. Hence by theorem

$R^2 = r^2 + (\frac{1}{2}h)^2 \Rightarrow (5\sqrt{3})^2 = r^2 + \frac{1}{4}h^2 \Rightarrow r^2 = 75 - \frac{1}{4}h^2.$ 3pts

so $V(h) = \text{volume} = \pi \cdot (75 - \frac{1}{4}h^2) \cdot h = 75\pi h - \frac{\pi}{4}h^3$, $\cancel{h>0}$. 1 pt.

since the cylinder is inscribed inside, $0 < h < 2R \Rightarrow 0 < h < 10\sqrt{3} \approx 17$.

$V'(h) = 75\pi - 3 \cdot \frac{\pi}{4}h^2 \Rightarrow$ critical pt @: $75\pi - \frac{3\pi}{4}h^2 = 0$ 3pts

$h^2 = \frac{4}{3\pi} \cdot 75\pi = 100$

$h = \pm 10 \Rightarrow h = 10$

sign chart $V'(h)$ $+$ $-$

0 10 $5\sqrt{3}$

so @ $h = 10$, $V(h)$ achieves maximum. 2pts

$V = V(10) = 75\pi \cdot 10 - \frac{\pi}{4}(10)^3 = 750\pi - 250\pi = 500\pi$

9.(20pts). Given the following function, its derivative, and its second derivative, answer the remaining questions. If there are none, say so. If there is more than one, list all. Write intervals in interval notation.

$$f(x) = e^{1-x^2} \quad f'(x) = -2xe^{1-x^2} \quad f''(x) = (2x^2-1)e^{1-x^2}$$

(1) (a)(4pts) Asymptotes

(i) Vertical Asymptote(s)

None

(ii) Horizontal Asymptote(s)

$\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = 0$

so $y=0$ is the horizontal asymptote.

(2) (b)(2pts) interval(s) f is increasing

sign chart f' $+$ $-$; 0

$(-\infty, 0)$

(3) (c)(2pts) interval(s) f is decreasing

$(0, +\infty)$

(4) (d)(2pts) interval(s) f is concave up

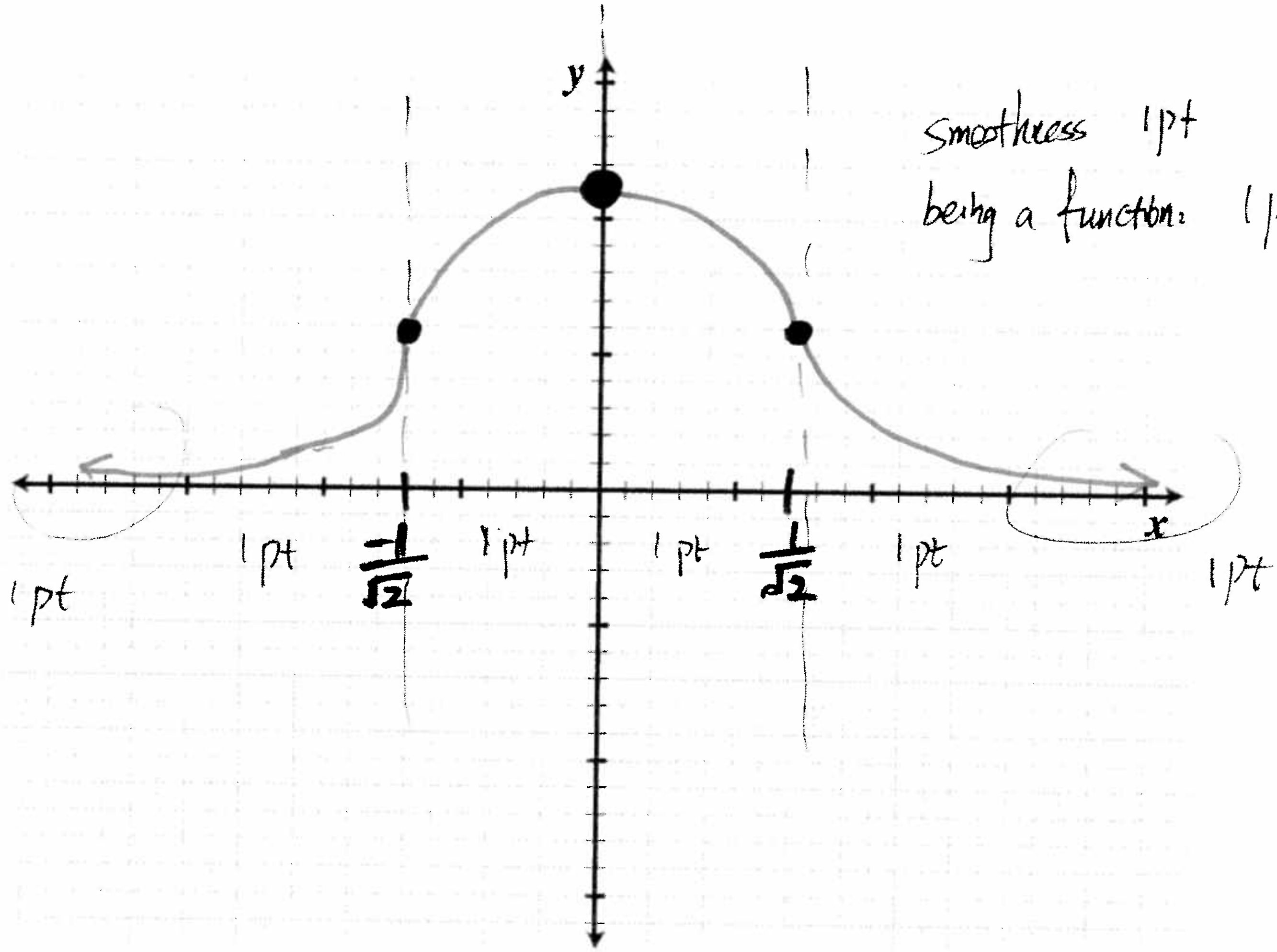
sign chart f'' $+$ $-$ $+$; $-\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$

$(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, +\infty)$

(5) (e)(2pts) interval(s) f is concave down

$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

(6) (f)(8pts) Sketch the graph of f on the next page, illustrating the above information. The graph need not be to scale. Label the critical points and inflection points.



smoothness 1pt

being a function: 1pt

sign chart of f' & f''

f'	+	+	−	−
f''	+	−	−	+
	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	

shape.

$f(0) = e^{1-0^2} = e$

$f(\frac{1}{\sqrt{2}}) = f(\frac{-1}{\sqrt{2}}) = e^{\frac{1}{2}}$

Scratch Here