MAC 2311-0003 Print Name ____________________
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Test 1
05/26/2016 Signature ____________________

INSTRUCTIONS:

- Clearly show all work where instructed.
- Write answer in space provided or circle/box answer.
- If a limit may be classified as ∞ or $-\infty$, say so (and which one).
- If a limit does not exist and cannot be classified as ∞ or $-\infty$, write DNE

Keep Calm and Do some Calculus !

1.(5pts). Let $f(x)$ be a function of x. Write down the definition of "$f(x)$ is continuous at $x = a$."

Ans. *The limit* $\lim_{x\to a} f(x)$ *exists and equals* $f(a)$. *Or*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

2.(10pts). Let

$$f(x) = \begin{cases} x^2 + 3x & \text{if } x \le -2 \\ x & \text{if } -2 < x < 3 \\ \sqrt{x+1} & \text{if } x \ge 3 \end{cases}$$

(1) $\lim_{x\to -3} f(x) = 0$
(2) $\lim_{x\to -2^+} f(x) = -2$
(3) $\lim_{x\to 3} f(x) = DNE$
(4) $f(x)$ is NOT continuous at $x = 3$.

3.(5pts). [Graph it] Sketch a possible graph for $f(x)$ if

$$\lim_{x\to -3^-} f(x) = 3; \quad \lim_{x\to -3^+} f(x) = -2; \quad f(-3) = 0; \quad \lim_{x\to\infty} f(x) = 0$$

Ans. *Omit here.*

4.(20pts). [Show me your limits]
Evaluate the following limits. **Simplify** answers as much as possible, show your work in the spaces provided, and circle or box your final answer.

(1) $\lim_{x\to\pi}(x^3+\cos(x))$

$$\begin{aligned}&\lim_{x\to\pi}(x^3+\cos(x))\\&=\lim_{x\to\pi}x^3+\lim_{x\to\pi}\cos(x)\\&=\pi^3+\cos(\pi)\\&=\pi^3-1\end{aligned}$$

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(2) $\lim_{x\to\infty}e^{2x-x^2}$

$$\begin{aligned}&\lim_{x\to\infty}e^{2x-x^2}\\&=e^{\left(\lim_{x\to\infty}2x-x^2\right)}\\&=e^{\left(\lim_{x\to\infty}-x^2\right)}\\&=e^{-\infty}\\&=0\end{aligned}$$

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(3) $\lim_{x\to0}\frac{\sin(3x)}{\tan(2x)}$

$$\begin{aligned}&\lim_{x\to0}\frac{\sin(3x)}{\tan(2x)}\\&=\lim_{x\to\infty}\frac{(\sin(3x))(3x)(2x)}{(3x)(2x)(\tan(3x))}\\&=\lim_{x\to\infty}\frac{\sin(3x)}{3x}\frac{3x}{2x}\frac{2x}{\tan(2x)}\\&=\lim_{x\to\infty}\frac{\sin(3x)}{3x}\lim_{x\to\infty}\frac{3x}{2x}\lim_{x\to\infty}\frac{2x}{\tan(2x)}\\&=1\cdot\frac{3}{2}\cdot1=\frac{3}{2}\end{aligned}$$

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(4) $\lim_{x\to-\infty}\frac{\sqrt{x^2-3}}{x+19}$

Since the limit here is $x\to-\infty$, we can think x to be negative, hence

$$x=-\sqrt{x^2}$$

and

$$\begin{aligned}
&\lim_{x\to-\infty}\frac{\sqrt{x^2-3}}{x+19}\\
&=\lim_{x\to-\infty}\frac{\frac{\sqrt{x^2-3}}{x}}{\frac{x+19}{x}}\\
&=\lim_{x\to-\infty}\frac{-\sqrt{\frac{x^2-3}{x^2}}}{\frac{x+19}{x}}\\
&=\lim_{x\to-\infty}\frac{-\sqrt{1-\frac{3}{x^2}}}{1+\frac{19}{x}}\\
&=-1
\end{aligned}$$

5.(10pts). Use the **limit definition** to find the 1st derivative function $f'(x)$ of $f(x)=\dfrac{1}{x+1}$. Show all your work clearly and circle your answer.

Ans. *The definition of $f'(x)$ is:*

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

Hence here we have:

$$\begin{aligned}
f'(x)&=\lim_{h\to0}\frac{\frac{1}{x+1+h}-\frac{1}{x+1}}{h}\\
&=\lim_{h\to0}\frac{\frac{(x+1)-(x+1+h)}{(x+1)(x+h+1)}}{h}\\
&=\lim_{h\to0}\frac{(x+1)-(x+1+h)}{h(x+1)(x+h+1)}\\
&=\lim_{h\to0}\frac{-h}{h(x+1)(x+h+1)}\\
&=\lim_{h\to0}\frac{-1}{(x+1)(x+h+1)}\\
&=\frac{-1}{(x+1)^2}
\end{aligned}$$

6.(15pts). The **Intermediate Value Theorem** is a very powerful theorem in Calculus. It guarantees the legality of the sign chart way, for examples.

(1) (5 pts) Fill out the Theorem.
Let $f(x)$ be a CONTINUOUS function on the closed interval $[a, b]$. If N is a number between $f(a)$ and $f(b)$, then:
There exists some number c belongs to (a, b) such that $f(c) = N$.

(2) (10 pts) Use the Theorem to prove that the equation $e^x = sin(\pi x) + 3x^2$ has a zero in $(0, 1)$.

Proof. Let $f(x) = e^x - sin(\pi x) + 3x^2$. This is a linear form of simple functions, hence it is continuous on the closed interval $[0, 1]$. Also one notices that

$$f(0) = e^0 - sin(\pi \cdot 0) + 3(0)^2 = 1 > 0$$

and

$$f(1) = e^1 - sin(\pi \cdot 1) + 3(1)^2 = e - 3 < 0$$

Hence one picks $N = 0$, and by the I.V.T there exists some $c \in (0, 1)$ such that $f(c) = 0$. □

7.(25pts). Find the 1st derivatives for the following functions. Show your work in the spaces provided, and circle or box your final answer.

(1) $f(x) = e^\pi - 2^x + 2\tan(x)$

$$\begin{aligned} f'(x) &= (e^\pi)' - (2^x)' + 2(\tan(x))' \\ &= 0 - (\ln 2)2^x + 2\sec^2(x) \\ &= -(\ln 2)2^x + 2\sec^2(x) \end{aligned}$$

(2) $f(x) = \sin(x)(\ln x)$

$$\begin{aligned} f'(x) &= (\sin(x)(\ln x))' \\ &= (\sin(x))'(\ln x) + (\sin(x))(\ln x)' \\ &= \cos(x)\ln x + \sin(x)\frac{1}{x} \\ &= \cos(x)\ln x + \frac{\sin(x)}{x} \end{aligned}$$

(3) $(f(x) = \dfrac{x^3 - x + 2}{x^2}$
We can rewrite $f(x)$ as:

$$f(x) = \frac{x^3}{x^2} - \frac{x}{x^2} + \frac{2}{x^2} = x - \frac{1}{x} + \frac{2}{x^2}$$

Hence

$$f'(x) = \left(x - \frac{1}{x} + \frac{2}{x^2}\right)'$$
$$= (x)' - (\frac{1}{x})' + (\frac{2}{x^2})'$$
$$= (x)' - (x^{-1})' + (2 \cdot x^{-2})'$$
$$= 1 - (-1)x^{-2} + 2(-2)x^{-3}$$
$$= 1 + \frac{1}{x^2} - \frac{4}{x^3}$$

(4) (8 pts)**Simplify your answer as much as possible**

$f(x) = \dfrac{x^2+1}{e^x}$

By Quotient Rule we have;

$$f'(x) = \frac{(x^2+1)'(e^x) - (x^2+1)(e^x)'}{(e^x)^2}$$
$$= \frac{(2x)(e^x) - (x^2+1)(e^x)}{(e^x)^2}$$
$$= \frac{(2x - (x^2+1))(e^x)}{(e^x)^2}$$
$$= \frac{2x - x^2 - 1}{(e^x)}$$
$$= \frac{-(x-1)^2}{(e^x)}$$

8.(10pts). Find the equation of the tangent line of the curve $y = x^3 - 3x + 7$ at $x = 1$.

Ans. *The curve is given by* $y = f(x) = x^3 - 3x + 7$, *hence at* $x = 1$, *the point is given by* $(1, f(1)) = (1, 1 - 3 + 7) = (1, 5)$.

In order to find the equation of a line, we also need the slope. The slope of the tangent line is given by 1*st derivative, ie,* $k = f'(1)$. *Here* $f'(x) = (x^3 - 3x + 7)' = 3x^2 - 3$, *hence* $k = f'(1) = 3 - 3 = 0$.

So the equation of the tangent line is:

$$y - 5 = (0) \cdot (x - 1)$$

Which simplifies as

$$y = 5$$