# 1 Rolle's Theorem and the Mean Value Theorem

**Theorem 1.1** (Rolle's Theorem). *Let $f$ be a function satisfying the following properties:*

1. *$f$ is continuous on the interval $[a, b]$*
2. *$f$ is differentiable on the interval $(a, b)$*
3. *$f(a) = f(b)$*

*Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$*

**Theorem 1.2** (The Mean Value Theorem). *Let $f$ be a function satisfying the following properties:*

1. *$f$ is continuous on the interval $[a, b]$*
2. *$f$ is differentiable on the interval $(a, b)$*

*Then there is a number $c$ in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

*Equivalently, there is a number $c$ in $(a, b)$ such that*

$$(b - a)f'(c) = f(b) - f(a).$$

# 2 Examples

**Example 2.1.** *Use the Intermediate Value Theorem to show the equation $1 - 2x = \sin x$ has at least one real solution. Then use Rolle's Theorem to show it has no more than one solution.*

*Proof.* Let $f(x) = 1 - 2x - \sin x$. Notice that $f(x)$ is a continuous function and that $f(0) = 1 > 0$ while $f(\pi) = 1 - 2\pi < 0$. The Intermediate Value Theorem guarantees there is a number, $c$ between 0 and $\pi$ such that $f(c) = 0$. Since $f(c) = 0$ we have $1 - 2c = \sin c$. Thus $c$ is a real solution for $1 - 2x = \sin x$ showing this equation has at least one real solution.

Now suppose there are two zeros for $f$. That is, suppose $a$ and $b$ are two different real number with $f(a) = f(b) = 0$. Note that $f$ is both continuous and differentiable for all $x$ so by Rolle's Theorem there must be a real number $c$ between $a$ and $b$ with $f'(c) = 0$. However, $f'(x) = -2 - \cos x$ cannot equal zero since $-1 \leq \cos x \leq 1$ for all $x$. This creates a contradiction and so the original assumption, that there are two different real zeros for $f$, must be false. Thus there can only be one real solution for $1 - 2x = \sin x$. □

# 3 Very important results that use Rolle's Theorem or the Mean Value Theorem in the proof

**Theorem 3.1.** *Suppose $f$ is a function that is differentiable on the interval $(a, b)$. Then $f'(x) = 0$ for all $x$ in the interval $(a, b)$* **if and only if** *$f$ is a constant function on $(a, b)$.*

**Theorem 3.2.** *Suppose $f$ is a function that is differentiable on the interval $(a, b)$. Then $f'(x) > 0$ for all $x$ in the interval $(a, b)$, except possibly a finite number of points,* **if and only if** *$f$ is a strictly increasing function on $(a, b)$.*

**Theorem 3.3.** *Suppose $f$ is a function that is differentiable on the interval $(a, b)$. Then $f'(x) < 0$ for all $x$ in the interval $(a, b)$, except possibly a finite number of points,* **if and only if** *$f$ is a strictly decreasing function on $(a, b)$.*