Limits and Basic Laws

Definitions of Left/Right Limits

Let $y = f(x)$ be a function of x. a be a real number.

Definition (Left Limit). *We say that the Left Limit of f(x) at a equals A, if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $a - x > \delta$. We denote it by*

$$\lim_{x \to a^-} = A$$

.

Definition (Right Limit). *We say that the Right Limit of f(x) at a equals A, if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $x - a > \delta$. We denote it by*

$$\lim_{x \to a^+} = A$$

.

Remark. *Generally the left limt doesn't necessarily equal the right limit. If they do, we have a limit.*

Definitions of Limits

Definition (Limit). *We say that the Limit of f(x) at a existes and equals A if the left limit equals the right limit*

$$\lim_{x \to a^-} = \lim_{x \to a^+} = A$$

and we denote it by

$$\lim_{x \to a} = A.$$

Definition (Equivalent Definition of Limit). **[Will Be Tested]**
We say that

$$\lim_{x \to a} = A$$

if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $|x - a| > \delta$.

Remark. *Generally the limit of $f(x)$ at a, if exists, does NOT necessarily equal the value $f(a)$. See the examples we did in class.*

Basic Laws for Limit

Theorem (”Common Sense” Limit Laws)**.** *If c is a constant and all limits involved exist (are real numbers), then*

1. $$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

2. $$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

3. $$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$$

4. $$\lim_{x\to a}[f(x)g(x)] = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$$

5. $$\lim_{x\to a}[f(x)/g(x)] = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x))$$

6. $$\lim_{x\to a}[f(x)]^n = (\lim_{x\to a} f(x))^n$$

7. *If f is a continuous function at a, then*
$$\lim_{x\to a} f(x) = f(a)$$
.

Remark. *Here all the limits can be replaced by left/right limits, and the theorem keeps.*

Remark. *The line 7 is actually the definition of continuous, which will be seen next class.*

Some Theorems

Theorem. *If $f(x) = g(x)$ for all x in an open interval containing a (except possibly at a), then*
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$
.

Remark. *This theorem plays an important role in computing $\frac{0}{0}$ limits for rational functions.*

Theorem (The Sandwich Theorem)**.** *If f, g, and h are functions such that*

$$f(x) \leq g(x) \leq h(x)$$

for all x around a (possibly except at a), then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)$$

Here a can be ∞ or $-\infty$.

Remark. *This theorem is particularly useful when $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = A$, which indicates that $\lim_{x \to a} g(x) = A$.*