# Derivatives

## Recall Tangents and Velocity

1. **Slopes**

   (a) The slope of the secant line of $y = f(x)$ through $(a, f(a))$ and $(b, f(b))$ is
   $$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

   (b) The Slope of the tangent line of $y = f(x)$ through $(a, f(a))$ is
   $$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

2. **Velocity**

   (a) The **average velocity** of a particle with position at time $t$ given by $s(t)$ over the time interval $[a, b]$ is
   $$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}$$

   (b) The **instantaneous velocity** of a particle with position at time $t$ given by $s(t)$ at time $t = a$ is
   $$v(a) = \lim_{t \to a} \frac{v(t) - v(a)}{t - a} = \lim_{h \to 0} \frac{v(a + h) - v(a)}{h}$$

**Remark.** *Notice that the forms of the above are the same and this is just two of many applications using this form. So mathematicians take the form and generalize.*

## Rate of Change

1. The **Average Rate of Change** of $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is
   $$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

2. The **Instantaneous Rate of Change** or the **Derivative** of $y = f(x)$ at $(a, f(a))$. is
   $$f'(a) = \frac{dy}{dx}\Big|_{x=a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

## Definition of a Derivative

**Definition.** *For the following we let $y = f(x)$ be a* **continuous** *function, and $a$* **lives inside the domain** *of $f(x)$.*

1. *The* **first derivative of $f$ with respect to $x$** *at $\underline{\ x = a\ }$ is*

$$f'(a) = y'(a) = \frac{df}{dx}\Big|_{x=a} = \frac{dy}{dx}\Big|_{x=a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2. *The* **first derivative of $f$ with respect to $x$** *is*

$$f'(x) = y' = \frac{df}{dx}(x) = \frac{dy}{dx} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3. *The second derivative of $f$ with resect to $x$ is the derivative of $f'$ with respect to $x$. It's denoted by $f''(x) = f^{(2)}(x) = y'' = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{df^2}{dx^2}(x)$*

**Remark.** *The first derivative $f'(x)$ can be treated as the function of slopes of the tangents of $f(x)$. In other words, the value of $f'(x)$ at $a$, or $f'(a)$, is the slope of the tangent line to $y = f(x)$ at $x = a$.*

All of the following concepts are found using the derivative, so basically that's why **everyone needs to learn Calculus !**

1. the slope of a tangent line,
2. velocity of a particle using the position,
3. the acceleration of a particle using velocity,
4. instantaneous rate of change of a quantity
5. marginal cost using a cost function
6. marginal revenue using a revenue function

## Differentiable Function

**Definition.** *Let $f(x)$ be a function.*

1. *The function $f$ is* **differentiable** *at $x = a$ if*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

*exists as a finite real number.*

2. *The function $f$ is* **left differentiable or differentiable from the left** *at $x = a$ if the following limit is a real number:*

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

3. *The function $f$ is* **right differentiable or differentiable from the right** *at $x = a$ if the following limit is a real number:*

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

**Remark.** *A function is* **NOT** *differentiable at $x = a$ if one of the following happens.*

1. *$f(x)$ is NOT continuous at $a$.*
2. *$f(x)$ is continuous at $a$, but*

   $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   *doesn't exist.*
3. *$f(x)$ is continuous at $a$, but*

   $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   *exists as infinity.*
   *Which graphically correspond to* **discontinuity ,"corner" or vertical tangent** *respectively.*

**Remark.** *A differentible function must be continuous, but continuous function are NOT necessarily differentiable. cf.Weierstrass Function*