1   Continuity and I.V.T

Continuity

Definition (Intuitive idea used in algebra based on graphing). *A function $f(x)$ is continuous on the interval (a, b) if the graph of $y = f(x)$ can be drawn over the interval (a, b) without lifting your pencil.*

Definition (Precise Definition of Continuity).

1. *A function $f(x)$ is* **continuous** *at $x = a$ if the limit exits and $\lim_{x \to a} f(x) = f(a)$*

2. *A function $f(x)$ is* **continuous on the interval** *(a, b) if $f(x)$ is continuous at every value in (a, b).*

3. *A function $f(x)$ is* **Left continuous (or continuous from the left)** *at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$*

4. *A function $f(x)$, is* **right continuous (or continuous from the right)** *at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$*

5. *A function $f(x)$ is* **continuous on the interval** *$[a, b]$ if f is continuous at every value in (a, b), f is right continuous at a and f is left continuous at b.*

6. *A function $f(x)$ is* **discontinuous** *at $x = a$ if f is not continuous at $x = a$*

7. *Many discontinuities may be classified as either a* **removable**, **jump**, *or* **infinite** *discontinuity.*

Remark.

1. *All simple functions such as trig functions, exponential functions, polynomials are continuous in their domain.*

2. *Use what you know about functions from algebra.*

3. *Usually, one looks at the* domain *and finds* discontinuities *to determine where a function* **is** *continuous.*

Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous on the interval $[a, b]$. If N is a number between $f(a)$ and $f(b)$, then there is at least one number c in the interval (a, b) with $f(c) = N$.*

Example 1. *Show the equation $1 - 2x = \sin x$ has at least one real solution.*

Proof. Let $f(x) = 1 - 2x - \sin x$. Notice that $f(x)$ is a continuous function and that $f(0) = 1 > 0$ while $f(\pi) = 1 - 2\pi < 0$. The Intermediate Value Theorem guarantees there is a number, c between 0 and π such that $f(c) = 0$. Since $f(c) = 0$ we have $1 - 2c = \sin c$. Thus c is a real solution for $1 - 2x = \sin x$ showing this equation has at least one real solution. $\qquad\square$

Remark. *A useful result of the Intermediate Value Theorem is that a function may only change signs where it equals zero or at discontinuities. This is why the "sign chart" method of solving inequalities taught in algebra works.*