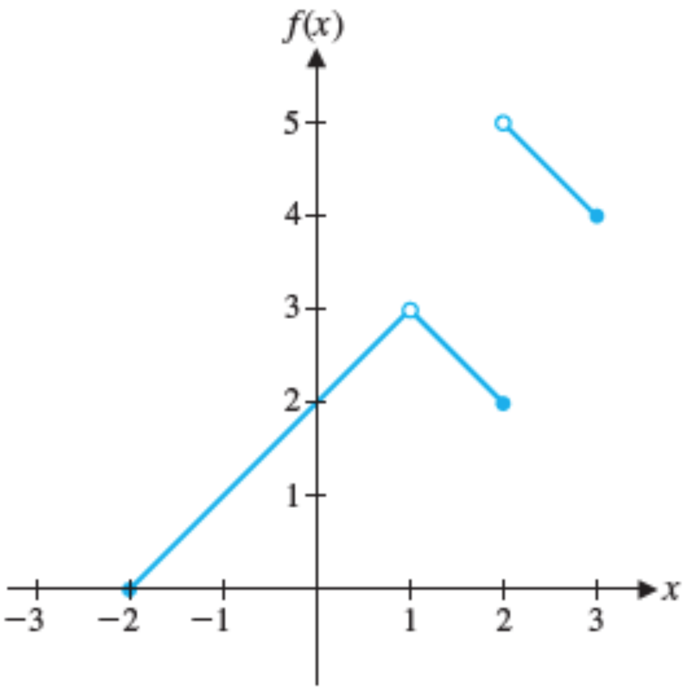
Definition 2.1.1 (Intuitive Definition). *The* **limit of** $f(x)$, **as** x **approaches** a, **equals** L *means that as* x *gets arbitrarily close to the value* a *(but not actually equal to* a*), the value of* $f(x)$ *gets close to the value* L*. This is also written*

$$\lim_{x \to a} f(x) = L$$

Remark 2.1.1. *Note, the limit has nothing to do with the* y*-value at* $x = a$*, but rather the behavior of the graph as we approach* $x = a$ *from both sides of* $x = a$*.*

Example 2.1.1. *Given the graph of* $y = f(x)$ *below*



(1) *Find* $f(a)$ *for* $a = -2, 0, 1, 2$.

(2) *Find* $\lim_{x \to a} f(x)$ *for* $a = -2, 0, 1, 2$.

Left and Right Limits

Definition 2.1.2. *The* **limit of** $f(x)$**, as** x **approaches** a **from the left, equals** L *means that as x gets arbitrarily close to the value a AND $x < a$, the value of $f(x)$ gets close to the value L. This is also written*

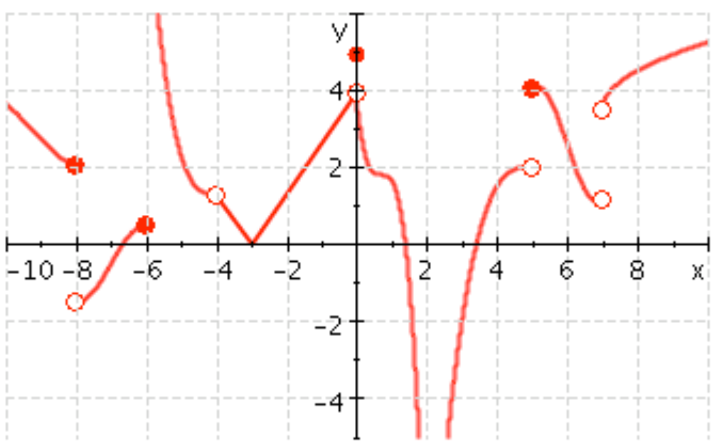
$$\lim_{x \to a^-} f(x)$$

Definition 2.1.3. *The* **limit of** $f(x)$**, as** x **approaches** a **from the right, equals** L *means that as x gets arbitrarily close to the value a AND $x > a$, the value of $f(x)$ gets close to the value L. This is also written*

$$\lim_{x \to a^+} f(x)$$

Theorem 2.1.1. $\lim_{x \to a} f(x) = L$ *if and only if*

Example 2.1.2. *Given the graph of $y = f(x)$ below*



(1) *Find* $\lim_{x \to a^-} f(x)$ *for* $a = -6, -4, -2, 5, 7$.

(2) *Find* $\lim_{x \to a^+} f(x)$ *for* $a = -6, -4, -2, 5, 7$.

(3) *Find* $f(a)$ *for* $a = -6, -4, -2, 5, 7$.

Theorem 2.1.2 (Limit Laws)**.** *If c is a constant and all limits involved exist (are real numbers), then*

(1) $\lim_{x\to a}[f(x) + g(x)] =$

(2) $\lim_{x\to a}[f(x) - g(x)] =$

(3) $\lim_{x\to a}[cf(x)] =$

(4) $\lim_{x\to a}[f(x)g(x)] =$

(5) $\lim_{x\to a}[f(x)/g(x)] =$

(6) $\lim_{x\to a}[f(x)]^n =$

(7) If f is a function that you know from previous experience is "continuous" (for example, polynomials) at $x = a$, $\lim_{x\to a} f(x) =$

Example 2.1.3. *Evaluate* $\lim_{x\to -1}(3x + 5)$

Example 2.1.4. *Evaluate* $\lim_{x\to 8} -5$

Example 2.1.5. *Find* $\lim_{x\to 0} 5x(x^2 + 3)$

Example 2.1.6. *Find* $\lim_{x\to -1/4} \dfrac{16x^2+1}{2-8x}$

Example 2.1.7. *Find* $\lim_{x\to 5} \sqrt[4]{3(47-4x)}$

Example 2.1.8. *Find* $\lim_{x\to -2} \dfrac{g(x)-2f(x)}{3g(x)}$, *if* $\lim_{x\to -2} f(x) = 4$ *and* $\lim_{x\to -2} g(x) = -1$

What if $f(x)$ is not continuous at $x = a$?

Example 2.1.9. $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$

Definition 2.1.4. $\lim_{x\to a} f(x) = \frac{0}{0}$ *is an* **indeterminant form**

Remark 2.1.2. $\lim_{x\to a} f(x) = \frac{n}{0}$ *where* $n \neq 0$ *is NOT an indeterminant form. The steps to solving each of these limits will be different.*

Example 2.1.10. $\lim_{x\to 2} \dfrac{x^2+4}{x-2}$

Steps to Finding Limits of (non-piecewise) Functions:

(1) Always plug in the value $x = a$ in the function first!
(2) If step 1 gives you a real number, you have found the limit!
(3) If step 1 gives you the indeterminant form $\frac{0}{0}$, then $f(x)$ is a rational function and you have to simplify the function by factoring the polynomial in the numerator and denominator of $f(x)$ and canceling the common factor $(x - a)$.
(4) If step 1 gives you the non-indeterminant form $\frac{n}{0}$, $n \neq 0$, then the limit does not exist (we will work further on this case in section 2.2)

Example 2.1.11. *Find* $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 - x - 6}$

Example 2.1.12. *Find* $\lim_{x \to 4} \dfrac{x - 4}{x^2 - 2x - 8}$

Definition 2.1.5. *The limit of a* **difference quotient** *is*

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Example 2.1.13. *Find* $\lim_{h\to 0} \dfrac{[3(x+h)^2-(x+h)]-[3x^2-x]}{h}$

Example 2.1.14. *If* $f(x)=4x-5$*, find the following limit of the different quotient:*

$$\lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$

Limits of Piecewise Functions

Example 2.1.15. *f is given by*

$$f(x) = \begin{cases} -1 & \text{if } x < -2 \\ 2x + 3 & \text{if } -2 \le x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

(1) *Evaluate* $\lim_{x \to 0} f(x)$

(2) *Evaluate* $\lim_{x \to 1^-} f(x)$

(3) *Evaluate* $\lim_{x \to 1^+} f(x)$

(4) *Evaluate* $\lim_{x \to 1} f(x)$

(5) *Evaluate* $f(1)$

(6) *Evaluate* $\lim_{x \to -2} f(x)$

Example 2.1.16. *Find* $\lim_{x \to 2^+} \frac{|2-x|}{2-x}$

Example 2.1.17. *Find* $\lim_{x \to 2^-} \frac{|2-x|}{2-x}$

Example 2.1.18. *Find* $\lim_{x \to 2} \frac{|2-x|}{2-x}$