4.2. Second Derivative and Graphs

Given $y = f(x)$, the derivative of the derivative is the ______________________________.

Notation 4.2.1. $f''(x) = f^{(2)}(x) = y'' = \dfrac{d^2y}{dx^2} = D^2 f(x)$

The n-th derivative: $f^{(n)}(x) = \dfrac{d^n y}{dx^n} = D^n f(x)$

Example 4.2.1. *Find the first and second derivatives of the function.*

$f(x) = (x^2 + 6)^9$

Applications

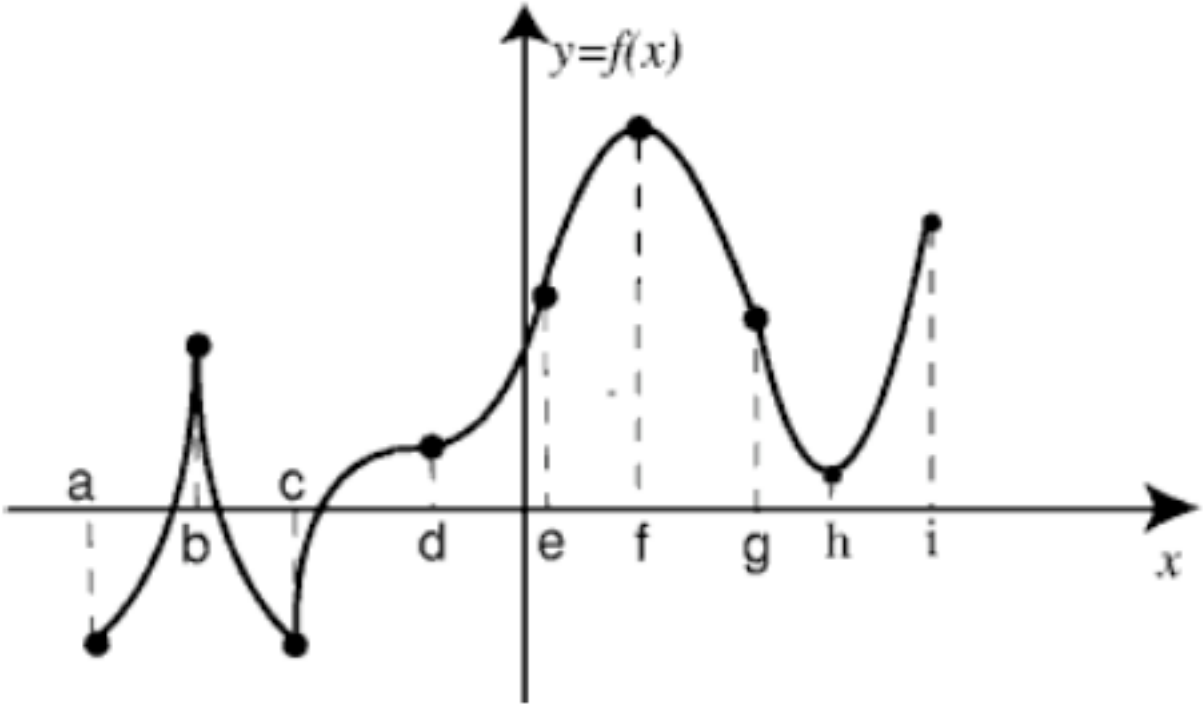
(1) Given the graph of $y = f(x)$
   - (a) $f'(x)$ provides the slope of the the line tangent to $y = f(x)$ at x
   - (b) $f''(x)$ provides the *rate of change* of the slope of the the line tangent to $y = f(x)$ at x.
   - (c) thus $f''(x)$ tells us if the FIRST DERIVATIVE, $f'(x)$, is increasing or decreasing
   - (d) so $f''(x)$ tells us if the tangent line is getting steeper or flatter.
   - (e) and so $f''(x)$ tells us if the ORIGINAL FUNCTION, $f(x)$, is concave up or concave down.

(2) If $f(t)$ give the position of a particle at time, t, then
   - (a) $f'(t)$ will provide the (instantaneous) ______________________ at time t and
   - (b) $f''(t)$ will provide the (instantaneous) ______________________________ at time t.
   - (c) $f'''(t)$ will provide the _____________ at time t.

Concavity

Theorem 4.2.1.
(1) If $f''(x) > 0$ for all x in an interval I, then f is concave up on I.
(2) If $f''(x) < 0$ for all x in an interval I, then f is concave down on I.
(3) If f changes concavity at $x = c$ and f is defined at $x = c$, then we say $(c, f(c))$ is a **inflection point**. *To find inflection points we find where the second derivative changes signs (and is in the domain of the original function).*

Example 4.2.2. *The graph given is the graph of $y = f(x)$*



(1) Find the intervals where the function is concave up and where concave down.

(2) Find the intervals where $f''(x) > 0$ and where $f''(x) < 0$

(3) Find the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing

(4) Find the intervals where $f'(x)$ is increasing and where $f'(x)$ is decreasing

(5) Find where the inflection points occur

(6) Find the local extrema of $f(x)$

(7) Find the local extrema of $f'(x)$

Finding Inflection Points

(1) Find the domain of f.
(2) Find all partition numbers p of $f''(x)$ (i.e. numbers where $f''(x) = 0$ or does not exist) such that $f(x)$ is continuous at $x = p$.
(3) Place all of these partition numbers AND values where f is undefined on a number line. These numbers will separate the number line into intervals.
(4) Determine the sign of f'' on each interval on the number line.
(5) If the sign chart of f'' changes signs at p (where f is defined at p), then $(p, f(p))$ is an inflection point of f. If the sign chart does not change signs at p, then there is no inflection point at $x = p$.

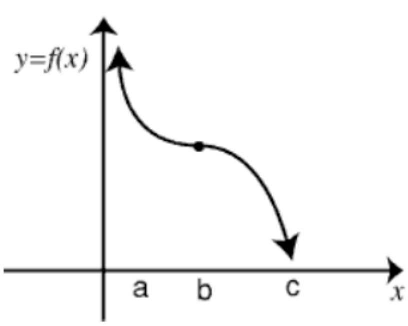
Example 4.2.3. *Find the inflection point(s) of* $f(x) = x^3 - 9x^2 + 24x - 10$.

Example 4.2.4. *Find the inflection point(s) of* $f(x) = \ln(x^2 - 2x + 5)$.

Example 4.2.5. *Select ALL the correct choices for* $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 3x + 4$

(1) the graph of $f(x)$ *has an inflection point at* $x = \frac{1}{4}$
(2) the graph of $f(x)$ *is concave downward on* $(-\infty, \frac{1}{4})$
(3) the graph of $f(x)$ *is concave downward on* $(\frac{1}{4}, \infty)$
(4) the graph of $f(x)$ *is increasing on* $(-1, \frac{3}{2})$
(5) the graph of $f(x)$ *is decreasing on* $(-\infty, -1) \cup (\frac{3}{2}, \infty)$
(6) the graph of $f(x)$ *has a local minimum at* $x = \frac{3}{2}$

Example 4.2.6. *The graph given is the graph of* $y = f(x)$. *Choose the correct statement for the graph.*



(1) $f'(x) > 0$ *on* (a, c); $f''(x) < 0$ *on* (a, b) *and* $f''(x) > 0$ *on* (b, c)
(2) $f'(x) > 0$ *on* (a, c); $f''(x) > 0$ *on* (a, b) *and* $f''(x) < 0$ *on* (b, c)
(3) $f'(x) < 0$ *on* (a, c); $f''(x) < 0$ *on* (a, b) *and* $f''(x) > 0$ *on* (b, c)
(4) $f'(x) < 0$ *on* (a, c); $f''(x) > 0$ *on* (a, b) *and* $f''(x) < 0$ *on* (b, c)