

4.1. CONTINUITY AND GRAPHS

Definition 4.1.1 (Intuitive idea used in algebra based on graphing). *A function, f , is continuous on the interval (a, b) if the graph of $y = f(x)$ can be drawn over the interval (a, b) without lifting your pencil.*

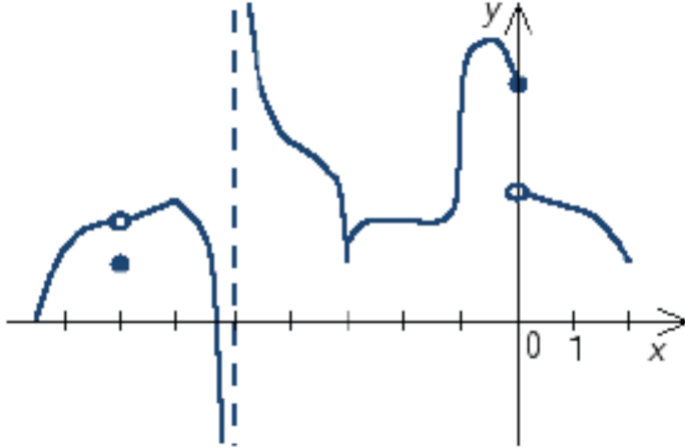
Definition 4.1.2 (The carefully thought-out calculus version based on limits).

- (1) A function, f , is **continuous** at $x = a$ if _____
- (2) A function, f , is **continuous on the interval** (a, b) if f is continuous at every value in (a, b) .
- (3) A function, f , is **left continuous (or continuous from the left)** at $x = a$
if _____
- (4) A function, f , is **right continuous (or continuous from the right)** at
 $x = a$ if _____
- (5) A function, f , is **continuous on the interval** $[a, b]$ if f is continuous at every value in (a, b) , f is right continuous at a and f is left continuous at b .
- (6) A function, f , is **discontinuous** at $x = a$ if _____

Remarks 4.1.1.

- (1) Use what you know about functions from algebra.
- (2) If a is NOT in the domain of f then f is CANNOT be continuous at that value.
- (3) Usually, one looks for the domain and discontinuities to determine where a function is continuous.

Example 4.1.1. Determine the discontinuities and the intervals on which f is continuous.



Example 4.1.2. Determine the discontinuities and the intervals on which f is continuous.

$$f(x) = 5x^4 + 2x^2 - \sqrt[3]{x}$$

Example 4.1.3. Where is $f(x) = \frac{x+1}{x^2-x-2}$ continuous?

- (1) None of these
- (2) $(-\infty, \infty)$
- (3) $(-\infty, 2) \cup (2, \infty)$
- (4) $(-\infty, -1) \cup (-1, \infty)$
- (5) $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

Example 4.1.4. Where is $f(x) = \sqrt{-2x-6}$ continuous?

- (1) $(-\infty, -3]$
- (2) $[-6, \infty)$
- (3) $[0, \infty)$
- (4) $[-3, \infty)$
- (5) $(-\infty, \infty)$

Example 4.1.5. Where is $f(x) = \frac{1}{\sqrt[3]{-2x-6}}$ discontinuous?

- (1) For all values of x
- (2) For no values of x
- (3) $x = -3$ only
- (4) $x \geq -3$ only
- (5) $x < -3$ only
- (6) $x = 3$ only

Example 4.1.6. Determine the discontinuities and the intervals on which f is continuous.

$$f(x) = \begin{cases} -1 & \text{if } -10 < x < -2 \\ 2x + 3 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Example 4.1.7. Where is $f(x)$ continuous if

$$f(x) = \begin{cases} -3x^3 & \text{if } x < -1 \\ -\sqrt{x+10} & \text{if } x = -1 \\ 2-x & \text{if } x > -1 \end{cases}$$

- (1) Nowhere
- (2) at $x = -1$ only
- (3) For all x except $x = -1$
- (4) $(-\infty, \infty)$
- (5) None of these

Example 4.1.8. Select all functions that are continuous for all values of x .

- (1) None of these
- (2) $f(x) = 5x^2$
- (3) $f(x) = \frac{x+1}{x^2+1}$
- (4) $f(x) = 4 - 3x - 6x^2$
- (5) $f(x) = \sqrt[3]{x+2}$

Infinite Limits and Vertical Asymptotes

Definition 4.1.3. If $f(x)$ fails to exist as x approaches a from the left because the values of $f(x)$ are becoming very large positive numbers (or very large negative numbers) then we say

$$\lim_{x \rightarrow a^-} f(x) = \infty \text{ or } (-\infty)$$

If the above behavior happens when x approaches a from the right then we say

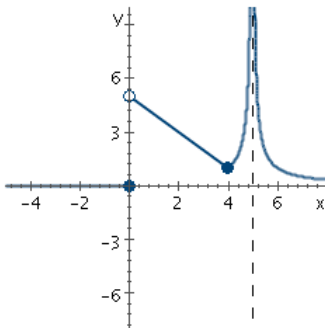
$$\lim_{x \rightarrow a^+} f(x) = \infty \text{ or } (-\infty)$$

If both one-sided limits exhibit same behavior then we say

$$\lim_{x \rightarrow a} f(x) = \infty \text{ or } (-\infty)$$

Moreover, in these cases the graph $y = f(x)$ has a vertical asymptote at $x = a$.

Example 4.1.9. Find the infinite limits for the following function.



Steps for Determining Infinite Limits Given Equations

- (1) Simplify the equation completely by factoring
- (2) Determine where the denominator is 0
- (3) For each value found in last step, plug in numbers very close to the left and right of each value to determine sign (positive or negative). This tells you if left-/right- handed limits are positive or negative infinity.

Example 4.1.10. Find the limits $\lim_{x \rightarrow 0^+} \frac{1}{x}$ and $\lim_{x \rightarrow 0^-} \frac{1}{x}$

Example 4.1.11. $\lim_{x \rightarrow 4^-} \frac{3}{x - 4}$

Example 4.1.12. $\lim_{x \rightarrow 4^+} \frac{3}{x - 4}$

Example 4.1.13. $\lim_{x \rightarrow -1^-} \frac{-5}{x + 1}$

Example 4.1.14. $\lim_{x \rightarrow -1^+} \frac{-5}{x + 1}$

Example 4.1.15. Find the vertical asymptotes of $f(x) = \frac{-5}{x+1}$

Example 4.1.16. $\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 3}{x^2 - x - 2}$

Example 4.1.17. $\lim_{x \rightarrow -1^+} \frac{x^2 - 2x - 3}{x^2 - x - 2}$

Example 4.1.18. Find the vertical asymptotes of $f(x) = \frac{x^2 - 2x - 3}{x^2 - x - 2}$

Solving Inequalities

Theorem 4.1.1. *If a function is continuous on the interval (a, b) and does not equal zero on the interval, then the function must be the same sign on the entire interval.*

Steps as taught in College Algebra

Step 1. Add or subtract so that all terms of the inequality are on one side and 0 is on the other side. You need to get the equation in one of the following forms:

$$\begin{array}{ll} f(x) > 0 & f(x) \geq 0 \\ f(x) < 0 & f(x) \leq 0 \end{array}$$

Step 2. Simplify the expression $f(x)$. Combine and factor.

Step 3. Find where $f(x)$ is equal to 0 or undefined (i.e. where is the numerator and where is the denominator equal to zero). We will call these values the **Partitioning Values**.

Step 4. Put these numbers on a number line. The Partition values will separate (partition) the number line into one or more intervals.

Step 5. Make a chart by making each interval in Step 4 a column and each factor in $f(x)$ a row. Note you may combine more than one factor in the rows, but every factor should be accounted for.

Step 6. Select a test number and determine if the factor in each row is positive or negative in each interval.

Step 7. Add a row for $f(x)$ at the bottom of your table and determine the sign of $f(x)$ in each interval (column) by counting the negatives.

Step 8. Answer the question based on which of the inequalities

$$\begin{array}{ll} f(x) > 0 & f(x) \geq 0 \\ f(x) < 0 & f(x) \leq 0 \end{array}$$

you are to solve and the signs given in the last row of your table.

Example 4.1.19. Solve $-2x(x+4)^2(x^2+9)(4-x)^3 < 0$

Example 4.1.20. Solve $\frac{4x-1}{(x+2)(x+3)} \geq 0$

Example 4.1.21. Solve $\frac{1}{3-x} \geq 2$

Homework: 4.1 p. 237 # 11, 13, 15, 17, 25, 29, 35, 37, 47, 49, 53, 57 work e-grade practice at least 2 times.