1. (12 points) Calculate the following integral:

$$\int_1^2 \left(\frac{1}{t} - 2i\right)^2 \, dt$$

$$= \int_1^2 \frac{1}{t^2} - \frac{4i}{t} - 4 \, dt$$

$$= -\frac{1}{t} - 4i\ln(t) - 4t \Big|_{t=1}^{t=2}$$

$$= -\left(\tfrac{1}{2} - 1\right) - 4i\ln(2) - 4(2-1)$$

$$= \boxed{-\frac{7}{2} - 4i\ln(2)}$$

2. (11 points) Let C be the positively oriented circle of radius 2 centered at 0. Calculate by parametrizing the contour:

$$\int_C \frac{\bar{z}+4}{z} \, dz$$

$z(t) = 2e^{it}$ $\quad 0 \le t \le 2\pi$

$z'(t) = 2ie^{it}$

$= iz(t)$

$$= \int_0^{2\pi} \frac{\overline{2e^{it}}+4}{z(t)} \, i z(t) \, dt$$

$$= i\int_0^{2\pi} 2e^{-it} + 4 \, dt = 2i\int_0^{2\pi} e^{-it} + 2 \, dt = 2i\left(\frac{1}{-i}e^{-it} + 2t \Big|_{t=0}^{t=2\pi}\right)$$

$$= \boxed{} \, 8\pi i$$

3. (11 points) Let C be the line starting at 0 and ending at $1+i$. Calculate by parametrizing the contour:

$$\int_C z + \bar{z} \, dz$$

$z(t) = t + it$ $\quad 0 \le t \le 1$

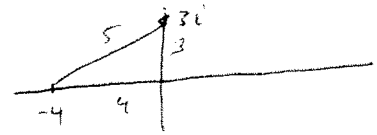
$z'(t) = 1 + i$

$$= \int_0^1 2\,\mathrm{Re}(t+it)(1+i) \, dt$$

$$= (2+2i)\int_0^1 t \, dt = (2+2i)\frac{1}{2} = \boxed{1+i}$$

4. (11 points) Suppose C is the positively oriented triangle with vertices 0, $3i$, -4. Show that:

$$\left|\int_C e^z - \bar{z}\, dz\right| \leq 60.$$



on C:

$$|e^z - \bar{z}| \leq |e^z| + |z|$$

$$= e^x + |z| \leq 1 + |z| \leq 1 + 4 = 5$$

$\uparrow$ $x \leq 0$ on C

arc length$(C) = 3 + 4 + 5 = 12$

$M = 5$

$$\left| \int_C e^z - \bar{z}\, dz \right| \leq M \text{ arc length}(C) = 5 \cdot 12 = 60$$

5. (11 points) Suppose C is any contour starting at $2i$ and ending at $3i$. Use an antiderivative to calculate:

$$\int_C (z-i)^3\, dz$$

$$\frac{d}{dz} \frac{1}{4}(z-i)^4 = (z-i)^3$$

$$= \frac{1}{4}(z-i)^4 \Big|_{z=2i}^{z=3i} = \frac{1}{4}\left((2i)^4 - i^4\right)$$

$$= \frac{1}{4}(16 - 1) = \boxed{\frac{15}{4}}$$

6. (11 points) Suppose that C is any contour that starts at 0 and ends at $i/2$. Use an antiderivative to calculate:

$$\int_C e^{\pi z}\, dz$$

$$\frac{d}{dz} \frac{1}{\pi} e^{\pi z} = e^{\pi z}$$

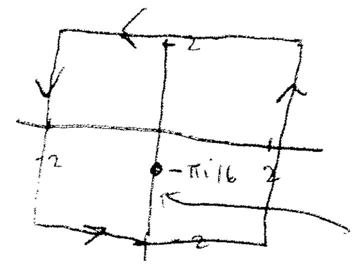
$$= \frac{1}{\pi} e^{\pi z} \Big|_{z=0}^{z=i/2}$$

$$= \frac{1}{\pi}(i - 1) = \frac{-1}{\pi} + \frac{i}{\pi}$$

7. (11 points) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$ Evaluate:

$$\int_C \frac{e^{-z}}{z + (\pi i/6)}\, dz$$

$\pi/6 \approx \frac{1}{2}$



$$= \int_C \frac{e^{-z}}{z - (-\pi i/6)}\, dz = 2\pi i\, e^{-z}\Big|_{z=-\pi i/6}$$

$$= 2\pi i\, e^{\pi i/6}$$

$$= 2\pi i\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

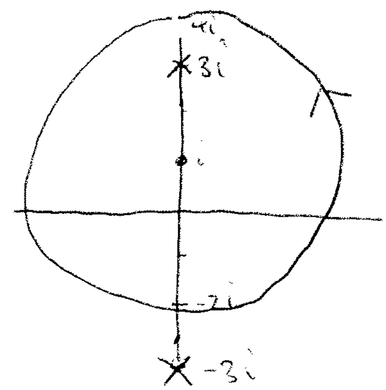
$$= \boxed{-\pi + \sqrt{3}\,\pi i}$$

8. (11 points) Let C be the positively oriented circle of radius 3 centered at i. Find

$$\int_C \frac{1}{(z^2+9)^2}\, dz$$

$z^2 + 9 = z^2 - (-9) = (z - 3i)(z + 3i)$

analytic on Int(C)



$$= \int_C \frac{\frac{1}{(z+3i)^2}}{(z-3i)^2}\, dz$$

$(6i)^3 = 2^3 \cdot 27 \cdot (-i)$

$$= \frac{2\pi i}{1!}\,\frac{d}{dz}\,\frac{1}{(z+3i)^2}\Big|_{z=3i}$$

$$= 2\pi i \cdot (-2)\,\frac{1}{(z+3i)^3}\Big|_{z=3i} = \frac{-2^2\pi \cancel{i}}{2^{\cancel{3}}\cdot 27\cdot \cancel{-i}} = \boxed{\frac{\pi}{54}}$$

9. (11 points) Let C be any positively oriented simple closed contour. Show that $g(w) = 6\pi i w$ for w inside C where:

$$g(w) = \int_C \frac{z^3 + 2z}{(z-w)^3}\, dz. = \frac{2\pi i}{2!}\left(\frac{d}{dz}\right)^2 (z^3 + 2z)\Big|_{z=w}$$

$\frac{d}{dz}(z^3 + 2z) = 3z^2 + 2$

$\frac{d}{dz}\, 3z^2 + 2 = 6z$

$$= \frac{2\pi i}{2!}\, 6w = \boxed{6\pi i w}$$