1. (11 points) Use the Cauchy-Riemann equations to determine where the following function is differentiable:

$$f(z) = e^y \cos(x) - ie^y \sin(x)$$

$u(x,y) = e^y \cos(x)$ $\quad$ $v(x,y) = -e^y \sin(x)$

$u_x = -e^y \sin(x)$ $\quad$ $v_x = -e^y \cos(x)$

$u_y = e^y \cos(x)$ $\quad$ $v_y = -e^y \sin(x)$

$u_x = v_y$

$u_y = -v_x$ $\longrightarrow$ $\boxed{\text{so differentiable everywhere}}$

2. (11 points) Use the definition of the derivative to prove that:

$$\frac{d}{dz} z^2 = 2z$$

(*Hint:* After some manipulation, you may determine the limit by "plugging in". You do not need to use an ϵ, δ argument.)

$$\frac{d}{dz} z^2 = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}$$

$$= 2z + \Delta z \longrightarrow \boxed{2z}$$

3. (11 points) Use the *polar* Cauchy-Riemann equations to determine where the following function is differentiable:

$$f(z) = \sqrt{r} e^{i\theta/2},$$

where, above, $\theta \in (0, 2\pi]$.

$u = \sqrt{r} \cos(\theta/2)$ $\quad$ $v = \sqrt{r} \sin(\theta/2)$

$r u_r = r \cdot \frac{1}{2} \frac{1}{\sqrt{r}} \cos(\theta/2)$ $\quad$ $r v_r = r \cdot \frac{1}{2} \frac{1}{\sqrt{r}} \sin(\theta/2)$

$= \frac{1}{2}\sqrt{r} \cos(\theta/2)$ $\quad$ $= \frac{1}{2}\sqrt{r} \sin(\theta/2)$

$u_\theta = -\frac{1}{2}\sqrt{r} \sin(\theta/2)$ $\quad$ $v_\theta = \frac{1}{2}\sqrt{r} \cos(\theta/2)$

$r u_r = v_\theta$ and $u_\theta = -r v_r$ so $\boxed{\text{f diff except at branch cut } \theta = 2\pi \text{ (i.e. pos. real axis)}}$

4. (11 points) Suppose that f is analytic everywhere in a domain D and that $f(z)$ is real-valued for all $z \in D$. Find f'

f real-valued $\Rightarrow$ $v = 0$ $\Rightarrow$ $v_x = v_y = 0$

f analytic $\Rightarrow$ $u_x = v_y = 0$

so $f' = u_x + i v_x$

$= 0 + i\,0 = \boxed{0}$

5. (11 points) Find the harmonic conjugate of the function:

$$u(x, y) = 2x(1 - y)$$

$v_y = u_x = 2(1-y)$

$v = \int 2(1-y)\,dy = 2y - y^2 + \phi(x)$

$\phi'(x) = v_x = -u_y = -(-2x)$

$\phi = \int 2x\,dx = x^2 + C$

$\boxed{v = 2y - y^2 + x^2 + C}$

6. (11 points) Show that for all $z \in \mathbb{C}$:

$$|e^{z^2}| \le e^{|z|^2}$$

$x^2 - y^2 \le x^2 + y^2$, e^t mon.

$|e^{z^2}| = |e^{x^2-y^2+2ixy}|$

$= |e^{x^2-y^2}| \cdot \underbrace{|e^{2ixy}|}_{=1} = |e^{x^2-y^2}| = e^{x^2-y^2} \le e^{x^2+y^2} = \boxed{e^{|z|^2}}$

7. (11 points) Use the chain rule to find $\frac{d}{dz}\,\mathrm{Log}(z)$.

$$z = e^{\mathrm{Log}\, z}$$

$$1 = \frac{d}{dz} z = \frac{d}{dz} e^{\mathrm{Log}\, z} = e^{\mathrm{Log}\, z} \frac{d}{dz} \mathrm{Log}\, z = z \frac{d}{dz} \mathrm{Log}\, z$$

$$\Rightarrow \boxed{\frac{d}{dz} \mathrm{Log}\, z = \frac{1}{z}}$$

8. (11 points) Find all values of $\log(1 + i)$

$$\log(1+i) = \ln(|1+i|) + i \arg(1+i)$$

$$= \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\boxed{= \frac{1}{2}\ln(2) + i\left(\frac{\pi}{4} + 2n\pi\right), \quad n \in \mathbb{Z}}$$

9. (12 points) Find all values of 2^i.

$$2^i = e^{i \log 2} = e^{i(\ln(2) + i \cdot 2n\pi)}$$

$$= \boxed{e^{-2n\pi}\left(\cos(\ln(2)) + i \sin(\ln(2))\right)}$$