1. (10 points) Find $z^2 + 3z + 4$ when $z = 1 + 2i$

$(1+2i)^2 = 1 - 4 + 4i = -3+4i$

$(1+2i)^2 + 3(1+2i) + 4 = -\cancel{3} + 4i + \cancel{3} + 6i + 4$

$\boxed{= 4 + 10i}$

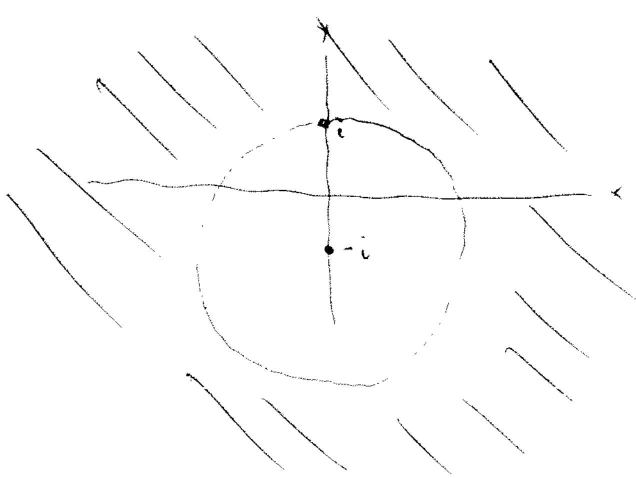
2. (10 points) Reduce the following quantity to a purely imaginary number.

$$\frac{-20+10i}{3-4i} + 4$$

$= \dfrac{-20+10i}{3-4i} \dfrac{(3+4i)}{(3+4i)} + 4 = \dfrac{-60 - 40 - 80i + 30i}{25} + 4$

$= -4 - 2i + 4 = \boxed{-2i}$

3. (10 points) Sketch the set of points determined by the inequality $|z + i| \geq 2$



$|z - (-i)| \geq 2$

"distance from $(-i) \geq 2$"

closed exterior of circle radius 2 centered at $-i$

4. (10 points) Write down the rectangular coordinates for $(1+\sqrt{3}i)^7$ (hint: use exponential form to calculate).

$$1+\sqrt{3}i = 2\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = 2e^{i\pi/3}$$

$$(1+\sqrt{3}i)^7 = 2^7 e^{i7\pi/3} = 2^7 e^{i\pi/3} = 128\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)$$

$$= \boxed{64+64\sqrt{3}i}$$

5. (10 points) Find all roots: $(1+i)^{1/3}$ (you may leave your answer in exponential form)

$$1+i = \sqrt{2}\left(\tfrac{1}{\sqrt{2}}+\tfrac{i}{\sqrt{2}}\right) = \sqrt{2}e^{i\pi/4}$$

$$(1+i)^{1/3} = (\sqrt{2})^{1/3}\left(e^{i\pi/4}\right)^{1/3}$$

$$= \boxed{\sqrt[6]{2}\, e^{i\pi/12}}$$

$$= \sqrt[6]{2}\, e^{i(\pi/12+2\pi/3)} = \boxed{\sqrt[6]{2}\, e^{i3\pi/4}}$$

$$= \sqrt[6]{2}\, e^{i(\pi/12+4\pi/3)} = \boxed{\sqrt[6]{2}\, e^{i17\pi/12}}$$

6. (10 points) Write the function $f(z)=z^3$ in the form $f(z)=u(x,y)+iv(x,y)$

$$(x+iy)^2 = x^2-y^2+2ixy$$

$$(x+iy)^3 = (x+iy)(x^2-y^2+2ixy)$$

$$= x^3-xy^2-2xy^2+i(yx^2-y^3+2x^2y)$$

$$= \boxed{x^3-3xy^2+i(3x^2y-y^3)}$$

$$u(x,y) = x^3-3xy^2$$

$$v(x,y) = 3x^2y-y^3$$

7. (10 points) Write the function $f(z) = z + \frac{1}{z}$ in the form $f(z) = u(r,\theta) + iv(r,\theta)$

$= re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$= r(\cos(\theta) + i\sin(\theta)) + \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$

$= r(\cos(\theta) + i\sin(\theta)) + \frac{1}{r}(\cos(\theta) - i\sin(\theta))$

$= \boxed{\cos(\theta)\left(r + \frac{1}{r}\right) + i\sin(\theta)\left(r - \frac{1}{r}\right)}$

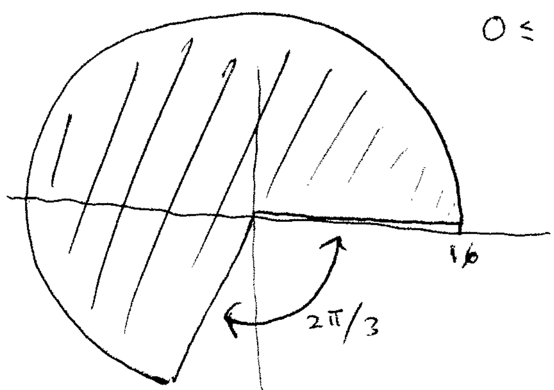
$u(r,\theta) = \cos(\theta)\left(r + \frac{1}{r}\right)$

$v(r,\theta) = \sin(\theta)\left(r - \frac{1}{r}\right)$

8. (10 points) Sketch the region onto which the sector $r \le 2$, $0 \le \theta \le \pi/3$ is mapped by the transformation $f(z) = z^4$

$\longrightarrow$ $r \le 2^4 = 16$

$0 \le \theta \le 4\pi/3$



9. (10 points) Use the definition of the limit to prove that $\lim_{z \to i} iz = -1$

Need $|iz - (-1)| < \varepsilon$

$|iz - i \cdot i| < \varepsilon$

$|i(z - i)| < \varepsilon$

$=1 \rightarrow |i||z - i| < \varepsilon$

$|z - i| < \varepsilon$

So, take $\delta = \varepsilon$. If $|z - i| < \delta$ then $|iz - (-1)| < \varepsilon$.

10. (10 points) Find

$$\lim_{z \to \infty} \frac{2z + i}{z + 1}$$

and justify your answer.

$$\lim_{z \to \infty} f(z) = \lim_{z \to 0} f\left(\frac{1}{z}\right)$$

$$\lim_{z \to \infty} \frac{2z+i}{z+1} = \lim_{z \to 0} \frac{2 \cdot \frac{1}{z} + i}{\frac{1}{z} + 1}$$

$$= \lim_{z \to 0} \frac{\frac{2 + iz}{z}}{\frac{1 + z}{z}}$$

$$= \lim_{z \to 0} \frac{2 + iz}{1 + z} = \boxed{2}$$