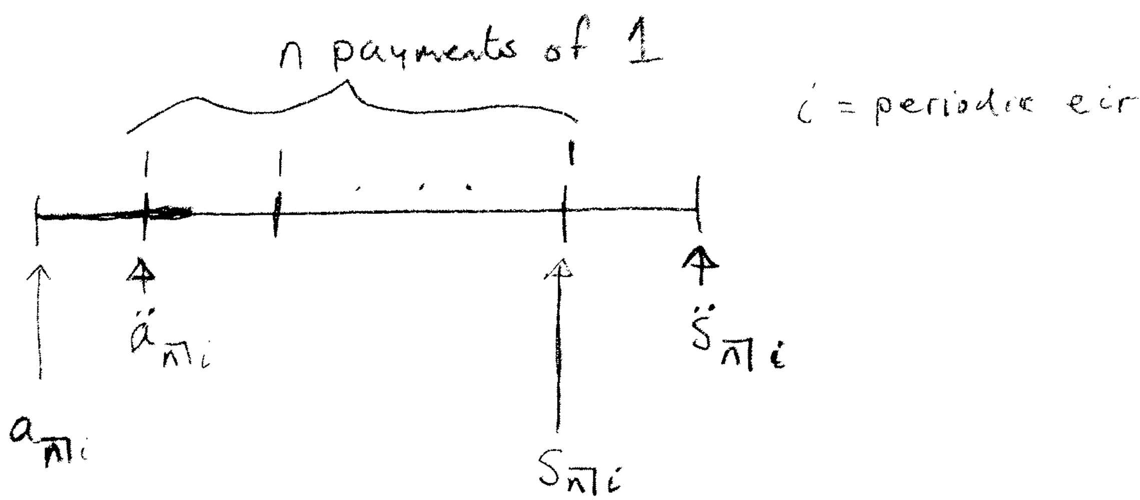
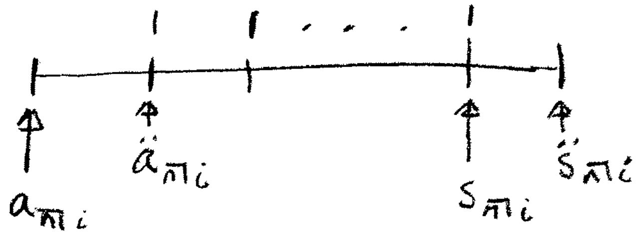
Module 2

Section 1: Annuities – Definition, Terminology, and Notation

An **annuity** is a sequence of periodic payments. The annuities on Exam FM/2 are called **certain annuities** because we assume the payments are certain to be paid, as opposed to the annuities on Exam MLC in which the payments will depend on the occurrence of some event. The **annuity start date** is at the beginning of the first period and the **annuity end date** is at the end of the last period. An **ordinary annuity** is an annuity in which the payments are thought of as taking place at the end of each period. An ordinary annuity is also called an **annuity immediate**. An **annuity due** is an annuity in which payments are thought of as taking place at the beginning of each period. All financial calculations have a **valuation date**. The **value** of an annuity at a given valuation date is the single sum value at the valuation date in which one is indifferent to receiving instead of receiving the periodic payments. The **present value** of an annuity is the value of the annuity at the annuity start date, and the **future** (or **accumulated**) **value** of an annuity is the value of the annuity at the annuity end date. If the valuation date is between the annuity start and end dates, then the value is called a **(current) value**. If the valuation date is before the annuity start date, then the annuity is called a **deferred annuity**. An annuity is called **level** if the payments are equal. We first focus on level annuities, and just as with accumulation functions, the development of formulas is based on a "per dollar of payment" basis. So we start with a level annuity with payments of 1, which I'll call a **basic level annuity**. Of course, the value of an annuity will depend on the interest rate. Unless (on the rare occasion) told otherwise, we determine values of annuities using periodic effective *interest* rates (eir's). In the timeline below, we introduce the notation for the value of a basic level annuity at each of the valuation dates shown.



Summary:



$a_{\overline{n}|i}$ is read "a, angle n, at *i*". It represents the value, one payment period before the first payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of an (basic level) annuity immediate. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$\ddot{a}_{\overline{n}|i}$ is read "a, double dot, angle n, at *i*". It represents the value, immediately before the first payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of an (basic level) annuity due. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$s_{\overline{n}|i}$ is read "s, angle n, at *i*". It represents the value, immediately after the last payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the accumulated (or future) value of an (basic level) annuity immediate. If there is no ambiguity, then the "*i* " in the notation is often omitted.

$\ddot{s}_{\overline{n}|i}$ is read "s, double dot, angle n, at *i*". It represents the value, one payment period after the last payment, of *n* periodic payments of 1, using a periodic eir of *i*. It is referred to as the accumulated (or future) value of an (basic level) annuity due. If there is no ambiguity, then the "*i* " in the notation is often omitted.

Some Elementary Relationships:

$a_{\overline{n}|} = \ddot{a}_{\overline{n}|} \cdot v$	(or equivalently)	$\ddot{a}_{\overline{n}|} = a_{\overline{n}|} \cdot (1+i)$
$s_{\overline{n}|} = \ddot{s}_{\overline{n}|} \cdot v$	(or equivalently)	$\ddot{s}_{\overline{n}|} = s_{\overline{n}|} \cdot (1+i)$
$a_{\overline{n}|} = s_{\overline{n}|} \cdot v^n$	(or equivalently)	$s_{\overline{n}|} = a_{\overline{n}|} \cdot (1+i)^n$
$\ddot{a}_{\overline{n}|} = \ddot{s}_{\overline{n}|} \cdot v^n$	(or equivalently)	$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|} \cdot (1+i)^n$

A **perpetuity** is an annuity in which the payments continue forever.

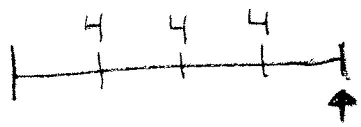
$a_{\overline{\infty}|i}$ represents the value, one payment period before the first payment, of an infinite number of periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of a (basic level) perpetuity immediate. Likewise $\ddot{a}_{\overline{\infty}|i}$ represents the value, immediately before the first payment, of an infinite number of periodic payments of 1, using a periodic eir of *i*. It is referred to as the present value of a (basic level) perpetuity due.

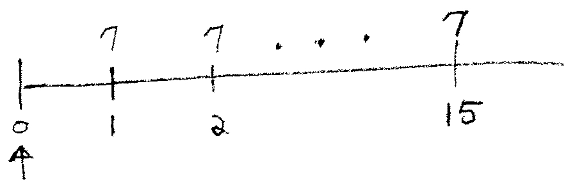
Note that $a_{\overline{\infty}|} = \ddot{a}_{\overline{\infty}|} \cdot v$ (or equivalently) $\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} \cdot (1+i)$

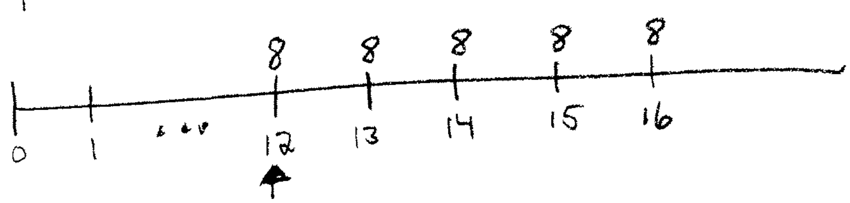
Module 2 Section 1 Problems:

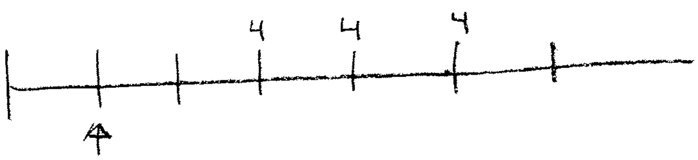
For Numbers 1-10, give an expression for the value, at the given valuation date, of the cash flow shown. The valuation date is marked with a vertical arrow. You cannot determine a numeric value for the expression until you are given an interest rate to use. We'll do that in the next section.

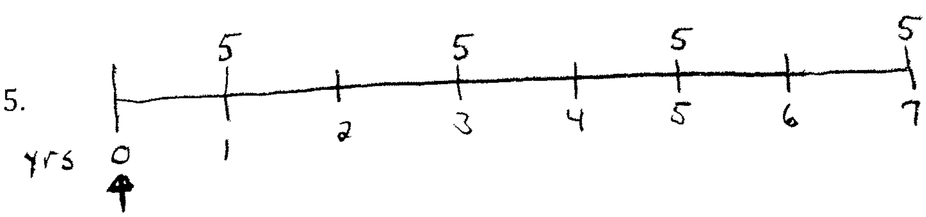
Note: You will not see these types of problems on Exam FM/2, but it is essential that you know how to do these problems to continue.

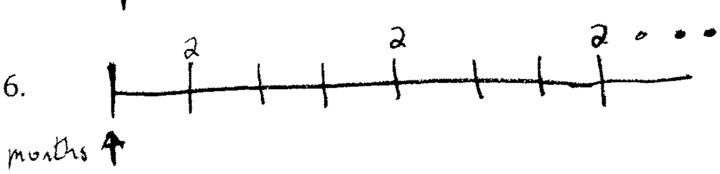
1. 

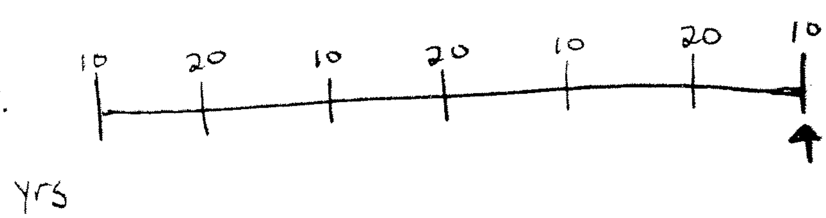
2. 

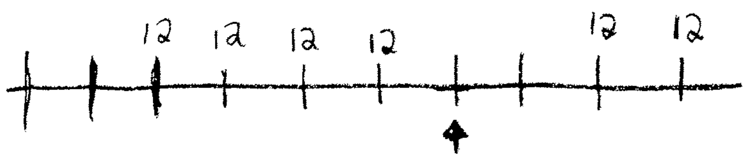
3. 

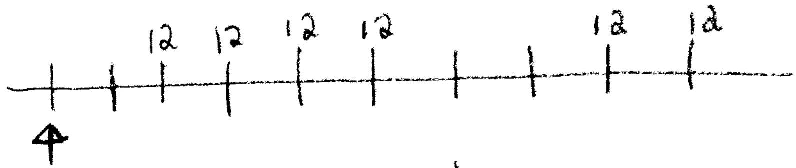
4. 

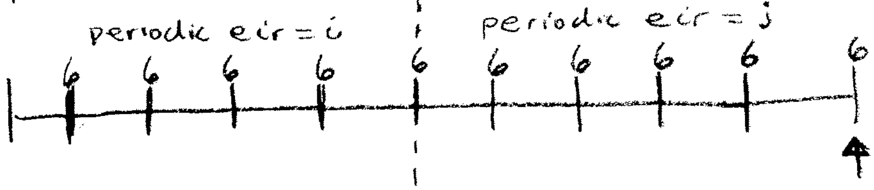
5. 

6. 

7. 

8. 

9. 

10. 

For Numbers 11-14, draw the timeline that corresponds to the given expression.

11. $3s_{\overline{6}|i} + 4\ddot{s}_{\overline{2}|j}$ where i is an annual effective interest rate and j is a biannual effective interest rate.

12. $7a_{\overline{7}|} - 7\ddot{a}_{\overline{3}|}v^3$

13. $5a_{\overline{6}|} + 100v^6$

14. $10(1+i)^5 + 3s_{\overline{5}|i}$

Solutions to Module 2 Section 1 Problems:

Please read the entire solutions. There are many correct, equivalent, answer choices for each of the problems. Listed are the ones that are more intuitive to me.

1. Since the valuation date is one period after the last of 3 payments of 4, then the simplest expression for the value of the payments is $4\ddot{s}_{\overline{3}|}$. Another correct expression is $4s_{\overline{3}|}(1+i)$. Yet other correct expressions are $4\ddot{a}_{\overline{3}|}(1+i)^3$ and $4a_{\overline{3}|}(1+i)^4$. Make sure you understand why these are all equivalent expressions for the value of the annuity.

2. Since the valuation date is one period before the first of 15 payments of 7, then the simplest expression for the value of the payments is $7a_{\overline{15}|}$. Another correct expression is $7\ddot{a}_{\overline{15}|}v$.

3. The simplest expression for the value of the payments is $8\ddot{a}_{\overline{5}|}$. Notice that the time values on the timeline are irrelevant. What matters is that the valuation date is at the time of the first payment.

4. If we think of this annuity as an annuity immediate, then the valuation date is one payment period before the annuity start date. In this case we would call the annuity a 1 period deferred, 3 payment annuity immediate with payments of 4. The value is $4a_{\overline{3}|}v$, and this expression is denoted by $4 \cdot {}_{1|}a_{\overline{3}|}$.

   If we think of this annuity as an annuity due, then the valuation date is two payment periods before the annuity start date. In this case we call the annuity a 2 period deferred, 3 payment annuity due with payments of 4. The value is $4\ddot{a}_{\overline{3}|}v^2$, and this expression is denoted by $4 \cdot {}_{2|}\ddot{a}_{\overline{3}|}$.

5. Note that the payments are made every two years, or biannually, but the valuation date is one year before the first payment. We can't change the payments and so the period set by the payments is biannual.

   If we think of this annuity as an annuity due, then the valuation date is one-half period before the annuity start date. Then the value is $5\ddot{a}_{\overline{4}|j} \cdot v_j^{\frac{1}{2}}$ where j is the baeir (biannual eir). Note that if i is the equivalent aeir, then $1+j=(1+i)^2$ or equivalently, $v_j^{\frac{1}{2}} = v_i$ and so we could write the value as $5\ddot{a}_{\overline{4}|j} \cdot v_i$. We could also think of this annuity as an annuity immediate, in which case the valuation date is one-half period *after* the annuity start date. In this case we get that the value is $5a_{\overline{4}|j} \cdot (1+j)^{\frac{1}{2}} = 5a_{\overline{4}|j} \cdot (1+i)$.

6. Note that the payments are made every three months, or quarterly, but the valuation date is one month before the first payment. We can't change the payments and so the period set by the payments is quarters.

If we think of this perpetuity as a perpetuity due, then the valuation date is 1/3 of a period before the perpetuity start date. Then the value is $2\ddot{a}_{\overline{\infty}|j} \cdot v_j^{\frac{1}{3}}$ where j is the qeir. Note that if i is the equivalent meir, then $1 + j = (1 + i)^3$ or equivalently, $v_j^{\frac{1}{3}} = v_i$ and so we could write the value as $2\ddot{a}_{\overline{\infty}|j} \cdot v_i$.

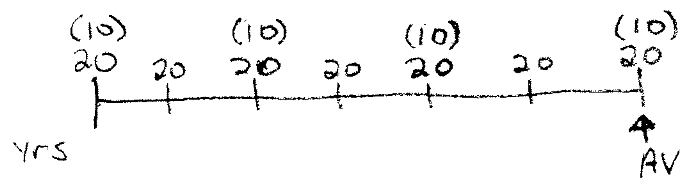
We could also think of this perpetuity as a perpetuity immediate. Since a perpetuity (or annuity) immediate has a start date one payment period before the first payment, the valuation date in this question is two months (or 2/3 of a quarter) *after* the annuity start date. Here we get that the value is $2a_{\overline{\infty}|j} \cdot (1 + j)^{\frac{2}{3}}$ or equivalently $2a_{\overline{\infty}|j} \cdot (1 + i)^2$.

In summary, all of the following are equal and represent the value of the perpetuity on the given valuation date:

$$2\ddot{a}_{\overline{\infty}|j} \cdot v_j^{\frac{1}{3}} \quad = \quad 2\ddot{a}_{\overline{\infty}|j} \cdot v_i \quad = \quad 2a_{\overline{\infty}|j} \cdot (1 + j)^{\frac{2}{3}} \quad = \quad 2a_{\overline{\infty}|j} \cdot (1 + i)^2$$

7. At first, students are inclined to view this as two different biannual payment annuities, one with level payments of 10 and another with level payments of 20. There's nothing wrong with this, but before we get to an expression for the value of the payments at the valuation date with this thought process, let's look other ways to value the payments. This first method is my preferred method for this problem.

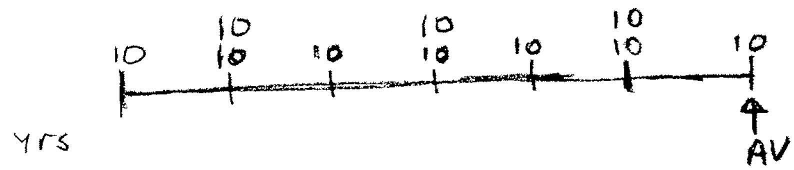
Rewrite each of the payments of 10 as 10 = 20 – 10. In our timelines, parenthesis around a number means the number is negative. Our timeline becomes



With this thought process, the value of the given payments at the valuation date is the accumulated value of an annuity immediate with 7 annual payments of 20 minus the accumulated value of an annuity immediate with 4 biannual payments of 10. We get $AV = 20s_{\overline{7}|i} - 10s_{\overline{4}|j}$ where i = the aeir and j = the baeir.

Note that this is a very useful thought process that can be applied to many exam type problems. Alternative methods for solving this problem follow. Keep in mind that what works best for one person may not work best for you. There are many correct answers.

Alternatively, we could rewrite each of the payments of 20 as 20 = 10 + 10. We get



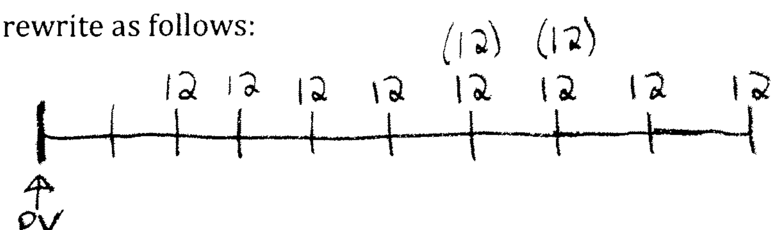
Then we get $AV = 10s_{\overline{7}|i} + 10s_{\overline{3}|j}(1+i)$ where i and j are the aier and baeir as above. The first term in this expression for the AV cannot be written any simpler, but there are other equivalent ways to write the second term. For example, instead of $10s_{\overline{3}|j}(1+i)$ we could have used $10s_{\overline{3}|j}(1+j)^{\frac{1}{2}}$ or we could have even used the annuity due notation $10\ddot{s}_{\overline{3}|j}v_i$ or $10\ddot{s}_{\overline{3}|j}v_j^{\frac{1}{2}}$. Unless told otherwise, use whatever works for you; just get it correct.

Finally, if we leave the payments alone, as indicated in the first paragraph of this solution, and view this as two different biannual payment annuities, one with level payments of 10 and another with level payments of 20, then we get $AV = 10s_{\overline{4}|j} + 20s_{\overline{3}|j}(1+i)$. As in the previous paragraph, there are many different equivalent ways we could write the second term of this expression.

It is EXTREMELY IMPORTANT to recognize the difference between expressions like $10s_{\overline{7}|i}$ and $10s_{\overline{7}|j}$ where the only difference is in the interest rate. If i is an aeir and j is a baeir, then $10s_{\overline{7}|i}$ represents the value just after the last of 7 *annual* payments of 10, whereas $10s_{\overline{7}|j}$ represents the value just after the last of 7 *biannual* payments of 10. This last expression can be thought of as the accumulated value of a 14 year annuity immediate with payments of 10 at the end of every other year.

8. $V = 12\ddot{s}_{\overline{4}|} + 12\ddot{a}_{\overline{2}|}v^2$. Note that since there is no ambiguity with interest rates, the interest rate is omitted. It is implied to be the periodic eir where the period matches the period for the payment. Also, it's simplest to use annuity due (double dot) notation for the first term. For the second term, we could have used annuity immediate notation (no double dots) but since we used double dots in the first term, it makes more sense to use double dot notation in the second term as well.

9. Let's rewrite as follows:



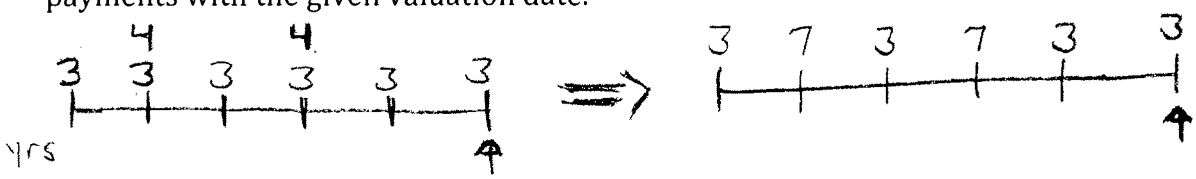
Then $PV = 12 \cdot {}_{1|}a_{\overline{8}|} - 12 \cdot {}_{5|}a_{\overline{2}|} = 12a_{\overline{8}|}v - 12a_{\overline{2}|}v^5$

Note that the timeline in this problem is the same as the timeline in the previous problem, with a different valuation date. Since the valuation date for this problem is 6 periods before the valuation date in the previous problem, we could get an expression for the value of the annuity at the valuation date in this problem by taking the expression for the value in the previous problem and discounting it for 6 periods (by multiplying by v^6). Namely, we have $PV = (12\ddot{s}_{\overline{4}|} + 12\ddot{a}_{\overline{2}|}v^2)v^6$ also.

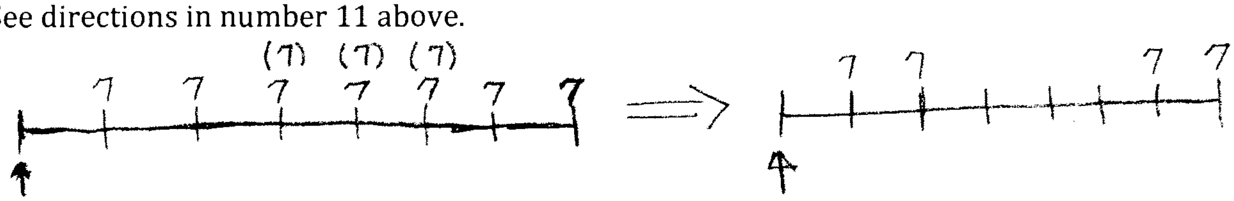
10. We've got to use an intermediate valuation date at the time in which the periodic eir changes, which is represented by the dotted line in the timeline. Be careful not to "cross over" this intermediate valuation date. So we will need two terms to represent the accumulated value of this annuity; one term for the set of payments before the intermediate valuation date and one term for the set of payments after the intermediate valuation date. The payment at the intermediate valuation date (at time $t = 5$) can be grouped with either the payments coming before it or with the payments coming after it. If we group the time $t = 5$ payment with the ones coming before it, then we get $AV = 6s_{\overline{5}|i}(1+j)^5 + 6s_{\overline{5}|j}$. This is my preferred answer.

If we group the time $t = 5$ payment with the ones coming after it, then we get $AV = 6\ddot{s}_{\overline{4}|i}(1+j)^5 + 6s_{\overline{6}|j}$. I would not consider this a preferred answer since the expression has both annuity due (double dot) notation and ordinary annuity (immediate) notation. Note that the first term, $6\ddot{s}_{\overline{4}|i}(1+j)^5$, can also be written as $6s_{\overline{4}|i}(1+i)(1+j)^5$, and so we can avoid the double dot notation, but the resulting expression is more complicated than the preferred answer above.

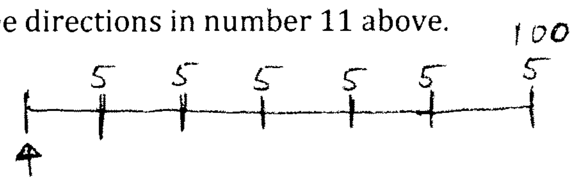
11. Start with one of the terms and draw the timeline for that term. This sets the valuation date for the timeline. Then there is only one way to draw in the other payments with the given valuation date.



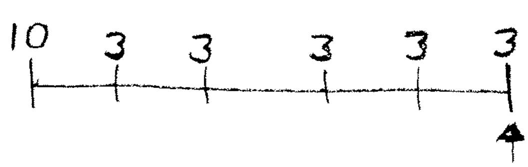
12. See directions in number 11 above.



13. See directions in number 11 above.



14. See directions in number 11 above.



Section 2: Basic Annuity Formulas

Key Formulas for Valuing Geometric Sums and Series:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

$$1 + r + r^2 + \cdots = \frac{1}{1 - r}$$

VEP means to *value each payment at the valuation date.* The number of terms in the VEP expression will equal the number of payments. The closed rule formulas (CRF's) follow from the VEP expressions using the above key formulas for valuing geometric sums and series.

Value Each Payment (VEP) and Closed Rule Formulas (CRF) for Basic Level Annuities

$$a_{\overline{n}|} \overset{VEP}{=} v + v^2 + \cdots + v^n \overset{CRF}{=} \frac{1 - v^n}{i}$$

$$\ddot{a}_{\overline{n}|} \overset{VEP}{=} 1 + v + v^2 + \cdots + v^{n-1} \overset{CRF}{=} \frac{1 - v^n}{d}$$

$$s_{\overline{n}|} \overset{VEP}{=} 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1} \overset{CRF}{=} \frac{(1 + i)^n - 1}{i}$$

$$\ddot{s}_{\overline{n}|} \overset{VEP}{=} (1 + i) + (1 + i)^2 + \cdots + (1 + i)^n \overset{CRF}{=} \frac{(1 + i)^n - 1}{d}$$

$$a_{\overline{\infty}|} \overset{VEP}{=} v + v^2 + \cdots \overset{CRF}{=} \frac{1}{i}$$

$$\ddot{a}_{\overline{\infty}|} \overset{VEP}{=} 1 + v + v^2 + \cdots \overset{CRF}{=} \frac{1}{d}$$

Module 2 Section 2 Problems:

For Numbers 1-10, first draw the timeline (include the valuation date) for each given cash flow. Then check your work by comparing your timeline to the to the timeline given in the corresponding Numbers 1-10 of the previous section. Then compute the numeric value of the annuity at the valuation date, using the given interest assumption.

1. Determine the accumulated value of a 4-year annuity due with annual payments of 3 using an annual effective interest rate of 5%.

2. Determine the present value of a 15-month annuity immediate with monthly payments of 7 using a monthly effective interest rate of 0.6%

3. An annuity pays 8 at the end of each year for 5 years, starting at the end of the 12th year. Determine the value of the annuity immediately before the first payment using an annual effective interest rate of 7%.

4. An annuity pays 3 semiannual payments of 4. Determine the present value of the annuity 1 year before the first payment using a nominal interest rate of 10% compounded semiannually.

5. An annuity consists of 4 biannual payments of 5. Determine the present value of the annuity 1 year before the first payment using an annual effective interest rate of 3%.

6. Determine the present value, one month before the first payment, of a perpetuity consisting of quarterly payments of 2, using an interest rate of 8% compounded quarterly.

7. A 7-year annuity due with annual payments has a first payment of 10, a second payment of 20, a third payment of 10, a fourth payment of 20, etc. Determine the accumulated value of the annuity immediately after the last payment, using an annual effective interest rate of 6%.

8. An annuity with monthly payments has 4 payment of 12 followed by two months without payments followed by two more months with payments of 12. Determine the value of the annuity one month after the 4th payment of 12, using a nominal rate of 12% compounded monthly.

9. For the same annuity as in Number 8, determine the present value of the annuity two months before the first payment using $i^{(12)} = 9\%$.

10. Determine the accumulated value of a 10-year annuity immediate with annual payments of 6, where interest is credited using 3% per annum for the first 5 years and 4% per annum thereafter.

11. Determine the present value of a 10-year annuity immediate with monthly payments of 30 using an annual effective interest rate of 6%.

12. Determine the present value of a perpetuity due with quarterly payments of 60 using a nominal interest rate of 12.18% compounded semiannually.

Solutions to Module 2 Section 2 Problems:

1. $AV = 4\ddot{s}_{\overline{3}|.05} \overset{VEP}{=} 4(1.05) + 4(1.05)^2 + 4(1.05)^3 = 4(1.05 + (1.05)^2 + (1.05)^3) =$ 13.2405. Since this annuity only has 3 payments, it's easy to use the VEP formula. Using the CRF (just to show consistency), and recalling that $d = \frac{i}{1+i}$ we get $AV = 4\ddot{s}_{\overline{3}|.05} \overset{CRF}{=} 4 \cdot \frac{1.05^3 - 1}{.05/1.05} = 4 \cdot \frac{1.05^3-1}{.05}(1.05) = 13.2405$ as expected.

Now let's use the Time Value of Money (TVM) buttons on the TI BA II Plus (Professional or Regular version). It is EXTREMELY IMPORTANT to become proficient with this required calculator. You should get to the point that you're using TVM for almost all numerical calculations of annuity values. Of course you must also know VEP's and CRF's because some exam problems may use them.

There appears to be 5 TVM buttons on your calculator (the third row from the top); namely, $\boxed{\text{N}}$, $\boxed{\text{I/Y}}$, $\boxed{\text{PV}}$, $\boxed{\text{PMT}}$, and $\boxed{\text{FV}}$. These buttons are self-explanatory, except for maybe the $\boxed{\text{I/Y}}$ button. For this button, we will *always* use the periodic eir, entered as a percent. You should think of there being one more input into each TVM calculation, that being whether the annuity in question is an annuity due or an annuity immediate. If the annuity is an annuity due then we are thinking of the payments as being made at the beginning of each period. To let the calculator know this, we must change the calculator setting to the "Begin" BGN mode. In order to toggle between the BGN and END modes, first type $\boxed{\text{2nd}}\boxed{\text{PMT}}$. Then, we toggle between the BGN and END modes by repeatedly typing $\boxed{\text{2nd}}\boxed{\text{ENTER}}$. When we're in the BGN mode, we will see "BGN" on the display screen. Once you have the correct mode, then exit by typing $\boxed{\text{2nd}}\boxed{\text{CPT}}$.

Students often get confused about when to use +/- signs when using the TVM buttons. We will do this first example in detail. We start with $AV = 4\ddot{s}_{\overline{3}|.05}$ and then, like we would do if we were solving a quadratic equation, we rewrite the equation so that one side is 0. For example, we could rewrite as $AV - 4\ddot{s}_{\overline{3}|.05} = 0$. We will be computing the AV (it is the unknown in the equation) and the payment will be input as a negative (there's a negative sign in front to the payment). First make sure we're in BGN mode, and then we type: 3 $\boxed{\text{N}}$ 5 $\boxed{\text{I/Y}}$ 0 $\boxed{\text{PV}}$ 4 $\boxed{+/-}$ $\boxed{\text{PMT}}\boxed{\text{CPT}}\boxed{\text{FV}}$. The display reads 13.2405 as expected. Notice the AV is displayed as a positive value, consistent with how we have rewritten the equation above.

Finally, there are a couple of remarks that need to be made before we move on. First, suppose we had rewritten the original equation as $4\ddot{s}_{\overline{3}|.05} - AV = 0$. Then we would input the payment of 4 as a positive. Type the rest in as above and notice that when we compute the AV it is displayed as a negative, which should be expected because of the negative in front of AV in the rewritten equation. Finally, note that we have three payments of 4. If there were an additional payment at the same time as the first payment of 4, then the amount of that payment would be input as a present value, *PV*. Since there is no such payment, we use *PV* = 0 for this problem.

2. $PV = 7a_{\overline{15}|0.006}$ Use TVM: $\boxed{\text{END}}$ 15 $\boxed{\text{N}}$ 0.6 $\boxed{\text{I/Y}}$ 7 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ (Result: 100.1268367)

   It is unreasonable to write a VEP expression for this present value since there are 15 payments. Using the CRF, we get
   $7a_{\overline{15}|0.006} = 7\frac{1-v_{0.006}^{15}}{.006} = 7\frac{1-(1.006)^{-15}}{.006} = 100.1268367$ as expected.

3. Be careful. The temptation is to use annuity immediate notation since the problem states that payments are at the end of each year. However, it is the *valuation date* that determines whether to use annuity immediate or annuity due notation.

   Draw the timeline. You should get exactly the timeline in Number 3 of the Module 2 Section 1 Problems. Since the valuation date is immediately before the first payment, the simplest expression for the value of the payments is $8\ddot{a}_{\overline{5}|}$. Evaluating at an aeir of 7% gives $8\ddot{a}_{\overline{5}|.07} \overset{TVM}{=} 35.09769005$. Note that since the symbol we're using is an *a* (actually an $\ddot{a}$), then we're computing a *PV*. The details of the TVM calculation are: $\boxed{\text{BGN}}$ 5 $\boxed{\text{N}}$ 7 $\boxed{\text{I/Y}}$ 8 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.

   One last comment: using annuity immediate notation, the symbol $8a_{\overline{5}|.07}$ represents the value one period (in this case, year) before the first payment. We would need to accumulate this value one year (multiply by 1.07) in order to an equivalent value at the given valuation date. As an exercise, show that $8a_{\overline{5}|.07}(1.07) = 35.09769005$.

4. Since there are only 3 payments, we could use VEP, but for more practice let's use TVM. Notice that the interest rate is given as $i^{(2)} = 0.1$. Since the payments are semiannual, the first thing to do is determine the semiannual effective interest rate. We have $i = \frac{0.1}{2} = 0.05$ = seir. Also notice the valuation date is 2 periods before the first payment. Therefore we have a deferred annuity. Thinking of this annuity as a 1 period deferred annuity immediate, $PV = 4 \cdot {}_{1|}a_{\overline{3}|.05} = 4a_{\overline{3}|}v \overset{TVM}{=} 10.37427821$. A shortcut to this last TVM calculation is to think of $4v$ (=4/1.05) as the payment. Then we type $\boxed{\text{END}}$ 3 $\boxed{\text{N}}$ 5 $\boxed{\text{I/Y}}$ 4 $\boxed{\div}$ 1.05 $\boxed{=}$ $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.

   If, instead, we think of this annuity as a 2 period deferred annuity due, then we get $PV = 4 \cdot {}_{2|}\ddot{a}_{\overline{3}|.05} = 4\ddot{a}_{\overline{3}|}v^2 \overset{TVM}{=} 10.37427821$, as expected. Similar to the shortcut above, we can think of the payment as $4v^2 = \frac{4}{1.05^2}$. Note that with TVM calculations, the order of the inputs is not important. For example, for this calculation we could type 1.05 $\boxed{Y^X}$ 2 $\boxed{+/-}$ $\boxed{=}$ $\boxed{\text{X}}$ 4 $\boxed{=}$ $\boxed{+/-}$ $\boxed{\text{PMT}}$ 5 $\boxed{\text{I/Y}}$ 3 $\boxed{\text{N}}$ 0 $\boxed{\text{FV}}$ $\boxed{\text{BGN}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.

5. Since the aeir is 0.03, then the baeir is $j = 1.03^2 - 1 = 0.0609$. Using annuity due notation we get $PV = 5\ddot{a}_{\overline{4}|.0609} \cdot v_{.03} = 17.80857871$. We could have used annuity immediate notation, in which case we get $PV = 5a_{\overline{4}|.0609} \cdot (1.03) = 17.80857871$.

6. We are given $i^{(4)} = 0.08$, and so the qeir is $i = \frac{.08}{4} = .02$. Therefore the quarterly accumulation factor is $af = 1.02$, the quarterly discount factor is $qdf = v = \frac{1}{1.02}$, and the quarterly effective discount rate is $d = \frac{.02}{1.02}$.
Since the valuation date is one month (one-third of a quarter) before the first payment, using perpetuity due notation, $PV = 2\ddot{a}_{\overline{\infty}|.02} \cdot v_{.02}^{\frac{1}{3}} = \frac{2}{d} v^{\frac{1}{3}} = 101.3289279$.
Note that using perpetuity immediate notation, the present value of a perpetuity immediate is one period (in this case, quarter, or three months) before the first payment, which is two months before the valuation date. Therefore we need to accumulate the present value of the perpetuity immediate for two months, or 2/3 of a quarter, in order to get to the valuation date. We get
$PV = 2a_{\overline{\infty}|.02} \cdot (1.02)^{\frac{2}{3}} = \frac{2}{i} \cdot (1.02)^{\frac{2}{3}} = 101.3289279$.

7. Be careful. The temptation is to use annuity due notation, but the valuation date is immediately after the last payment. See the timeline and solution from the last section. For the preferred solution we have $AV = 20s_{\overline{7}|i} - 10s_{\overline{4}|j}$ where $i = .06$ is the aeir, which implies the baeir is $j = 1.06^2 - 1 = .1236$. Thus $AV = 119.8307923$. You should check that the other expressions given in the solution to Number 7 in the previous section give the same value of 119.8307923.

This is a calculation in which it is useful to use the "store" button. One way to proceed is as follows: $\boxed{\text{END}}$ 7 $\boxed{\text{N}}$ 6 $\boxed{\text{I/Y}}$ 20 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$ $\boxed{\text{STO}}$ 1 (this stores the numeric value of the first term in $\boxed{\text{STO}}$ 1). You may use 0, 1, ... , 9 as "store" places. Now proceed with determining the numeric value of the second term and then subtract it from the first term: 4 $\boxed{\text{N}}$ 12.36 $\boxed{\text{I/Y}}$ 10 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$ $\boxed{+/-}$ $\boxed{+}$ $\boxed{\text{RCL}}$ 1 $\boxed{=}$. Result: $V = 119.8307923$.

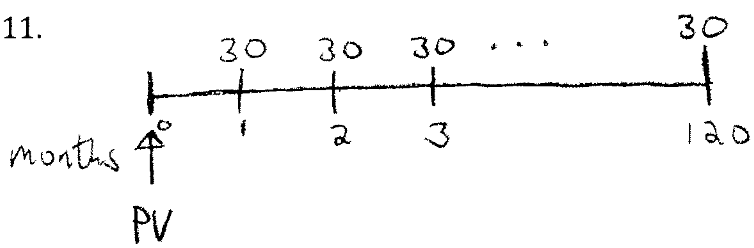
8. Using annuity due notations, we get $V = 12\ddot{s}_{\overline{4}|} + 12\ddot{a}_{\overline{2}|}v^2$ where the understood interest rate is the meir, $i = \frac{.12}{12} = .01$. Therefore $V = 72.62269449$.
This is another calculation in which it is useful to use the "store" button. One way to proceed is as follows: $\boxed{\text{BGN}}$ 4 $\boxed{\text{N}}$ 1 $\boxed{\text{I/Y}}$ 12 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 0 $\boxed{\text{PV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$ $\boxed{\text{STO}}$ 1. Now proceed with determining the numeric value of the second term and then add it to the first term. Note that we are already in "BGN" mode, and since the periodic interest rate is not changed there is no need to restore it. We also already have 12 in as the payment. So one way to proceed is as follows:
2 $\boxed{\text{N}}$ 0 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$ $\boxed{\div}$ 1.01 $\boxed{X^2}$ $\boxed{=}$ $\boxed{+}$ $\boxed{\text{RCL}}$ 1 $\boxed{=}$. Result: $V = 72.62269449$.

9. Using ordinary annuity notation, $PV = 12 \cdot {}_{1|}a_{\overline{8}|} - 12 \cdot {}_{5|}a_{\overline{2}|} = 12a_{\overline{8}|}v - 12a_{\overline{2}|}v^5$, where the understood interest rate is the meir, $i = \frac{.09}{12} = .0075$. Therefore, we get $PV = 69.2858683$.

10. As discussed in the previous section, $AV = 6s_{\overline{5}|.03}(1.04)^5 + 6s_{\overline{5}|.04}$ is the preferred symbolic answer. Therefore, $AV = 71.25418831$.

A shortcut to computing the numeric value of this expression is obtained by recognizing that we can think of the factor $6s_{\overline{5}|.03}$ in the first term of the expression as the "present value" for the TVM calculation for the second 5 year period. In order to input the correct +/- signs, recall that we rewrite the equation with a 0 on one side. For example, $AV - PV(1.04)^5 - PMTs_{\overline{5}|.04} = 0$. Written this way, we see that we would input $6s_{\overline{5}|.03}$ as a negative PV and 6 as a negative payment and compute the AV, which will be displayed as a positive amount. The keystrokes are:
$\boxed{\text{END}}$ 5 $\boxed{\text{N}}$ 3 $\boxed{\text{I/Y}}$ 0 $\boxed{\text{PV}}$ 6 $\boxed{+/-}$ $\boxed{\text{PMT}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$ (this gets us the value of $6s_{\overline{5}|.03}$ which we need to change the sign an input as the PV). Proceed as follows:
$\boxed{+/-}$ $\boxed{\text{PV}}$ 4 $\boxed{\text{I/Y}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$

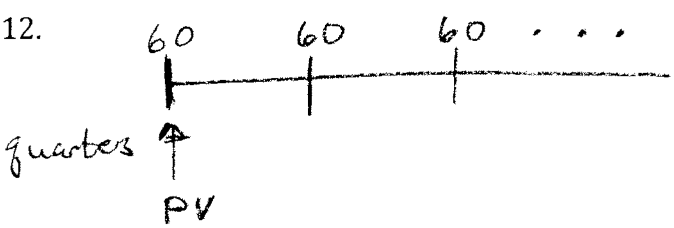
11.



aeir = .06

Note that payments are made monthly but interest is compounded annually. So we convert the interest rate from the given aier to its equivalent meir, getting an meir of $i = 1.06^{\frac{1}{12}} - 1$. Then we get $PV = 30a_{\overline{120}|i} = 2721.729649$.

12.



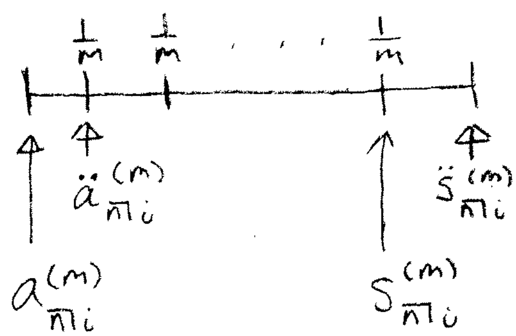
$i^{(2)} = .1218$

Note that payments are made quarterly but interest is compounded semiannually. So we convert the interest rate from the given nominal rate to its equivalent qeir. We are given $i^{(2)} = .1218$ which implies the seir is 0.0609. Therefore, the qeir is $j = 1.0609^{0.5} - 1 = .03$, and $PV = 60\ddot{a}_{\overline{\infty}|.03} = \frac{60}{d} = \frac{60}{j}(1+j) = \frac{60}{.03}(1.03) = 2060$.

Section 3: Basic upper m Notation

We have already seen "upper m" notation in the context of nominal interest rates. In the context of annuities, we use upper m notation when the annuity payments are more frequent than interest is compounded. The last two problems in the previous problem set illustrate such problems. We can see from the solutions of these problems that we can generally determine annuity values for such annuities by converting the given interest rate to its equivalent periodic effective interest rate, where the period matches the payment period. When possible, this is the approach most students take in such problems. However, you may see (a very few) problems on your actuarial exam that make use of the following upper m annuity notation.

Without loss of generality, we assume an n year annuity with payments made m times per year, for a total of nm payments, and we assume we're given an annual effective interest rate i. Just as with regular annuities, the term "basic" will represent the case that the annual payment is 1. Since there are m payments per year, then each payment is $1/m$. The timeline for "m^{thly} annuities" is:



period $= \left(\frac{1}{m}\right)^{th}$ of a year

$(n \cdot m)$ payments of $\frac{1}{m}$

i = aeir

n = # of years

Formulas for "m^{thly} annuities":

$$a^{(m)}_{\overline{n|}i} \overset{VEP}{=} \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{n} \overset{CRF}{=} \frac{1-v^n}{i^{(m)}}$$

$$\ddot{a}^{(m)}_{\overline{n|}i} \overset{VEP}{=} \frac{1}{m} + \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{\frac{nm-1}{m}} \overset{CRF}{=} \frac{1-v^n}{d^{(m)}}$$

$$s^{(m)}_{\overline{n|}i} \overset{VEP}{=} \frac{1}{m} + \frac{1}{m}(1+i)^{\frac{1}{m}} + \frac{1}{m}(1+i)^{\frac{2}{m}} + \cdots + \frac{1}{m}(1+i)^{\frac{nm-1}{m}} \overset{CRF}{=} \frac{(1+i)^n-1}{i^{(m)}}$$

$$\ddot{s}^{(m)}_{\overline{n|}i} \overset{VEP}{=} \frac{1}{m}(1+i)^{\frac{1}{m}} + \frac{1}{m}(1+i)^{\frac{2}{m}} + \cdots + \frac{1}{m}(1+i)^{n} \overset{CRF}{=} \frac{(1+i)^n-1}{d^{(m)}}$$

$$a^{(m)}_{\overline{\infty|}i} \overset{VEP}{=} \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots \overset{CRF}{=} \frac{1}{i^{(m)}}$$

$$\ddot{a}^{(m)}_{\overline{\infty|}i} \overset{VEP}{=} \frac{1}{m} + \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots \overset{CRF}{=} \frac{1}{d^{(m)}}$$

Module 2 Section 3 Problems:

1. Determine the present value of a 10-year annuity immediate with monthly payments of 30 using an annual effective interest rate of 6%. (This is Number 11 of the previous problem set.)

2. Determine the accumulated value of a 10-year annuity immediate with monthly payments of 30 using an annual effective interest rate of 6%.

3. Determine the accumulated value of a 15-year annuity due with quarterly payments of 20 using an annual effective interest rate of 3%.

4. Determine the present value of a perpetuity immediate with semiannual payments of 60 using an annual effective interest rate of 6.09%.

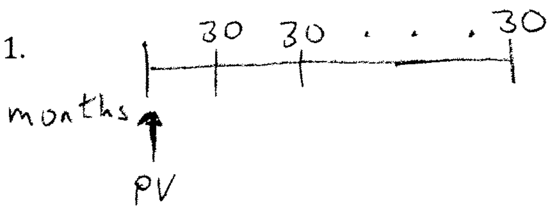
5. Determine the present value of a perpetuity due with semiannual payments of 60 using an annual effective interest rate of 6.09%.

6. Determine the present value of a perpetuity due with quarterly payments of 60 using a nominal interest rate of 12.18% compounded semiannually. (This is Number 12 of the previous problem set.)

Solutions to Module 2 Section 3 Problems:

The most common approach to all the problems in this section is to convert the given interest rate to the appropriate periodic eir and proceed without upper m notation. We use upper m notation in the solutions below to show the methods are equivalent.

1.

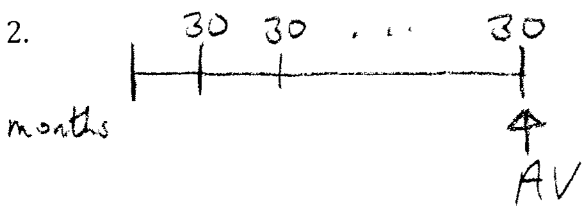


aeir = .06
10 years ⟹ 120 payments

We have $PV = 360a^{(12)}_{\overline{10}|.06} = 360 \cdot \frac{1-v^{10}_{.06}}{i^{(12)}}$ Note that the coefficient of the annuity symbol is the annual payment 12(30)=360. Also we cannot use TVM to determine a numeric value; instead we just use regular calculator keys. Finally, we can see that this expression is equal to the expression in the solution to Number 11 in Section 2 by recognizing that $i^{(12)} = 12i$ where $i = 1.06^{\frac{1}{12}} - 1$ is the equivalent meir, and $v^{10}_{.06} = v^{120}_i$. Therefore

$$PV = 360a^{(12)}_{\overline{10}|.06} = 360 \cdot \frac{1-v^{10}_{.06}}{i^{(12)}} = 360 \cdot \frac{1-v^{120}_i}{12i} = 30 \cdot \frac{1-v^{120}_i}{i} = 30a_{\overline{120}|i} = 2721.729649.$$

2.

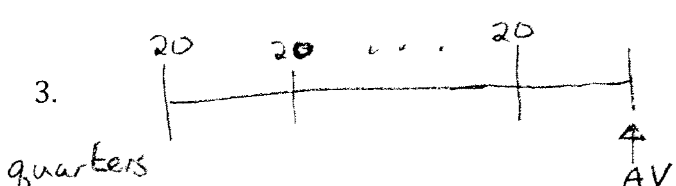


aeir = .06
10 years ⟹ 120 payments

Using upper m notation, we have $AV = 360s^{(12)}_{\overline{10}|.06} = 360 \cdot \frac{1.06^{10}-1}{i^{(12)}} = 4874.203273$ where, as above, we have $i = 1.06^{\frac{1}{12}} - 1$ is the equivalent meir to a 6% aeir, and $i^{(12)} = 12i$. Note that we could have just used $AV = 30s_{\overline{120}|i} = 4874.203273$.

Finally, notice that the annuity in this problem is exactly the same annuity as in the previous problem in which we found its present value. Therefore, we can determine the accumulated value from the present value in the previous problem by using the fact that $AV = PV \cdot (1+i)^n \Rightarrow AV = 2721.729649 \cdot 1.06^{10} = 4874.203273$.

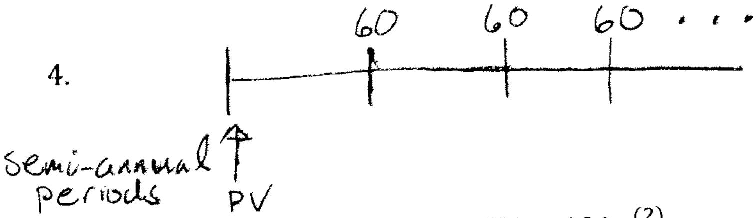
3.



aeir = .03
15 years

Using upper m notation, $AV = 80\ddot{s}^{(4)}_{\overline{15}|.03} = 80 \cdot \frac{1.03^{15}-1}{d^{(4)}}$. Note that $d^{(4)} = 4d$, where $d = 1 - 1.03^{\frac{-1}{4}}$ is the equivalent qedr to a 3% aeir. Thus, $AV = 1515.708403$.

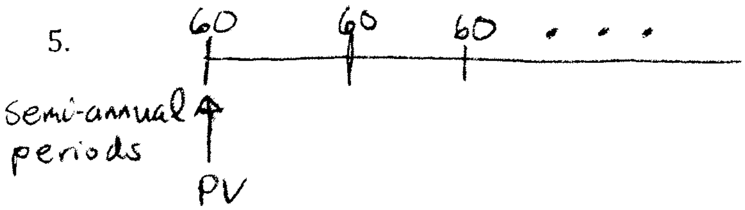
4.



$aeir = .0609$

Using upper m notation, $PV = 120a^{(2)}_{\overline{\infty}|.0609} = \frac{120}{i^{(2)}}$. Note that $i^{(2)} = 2i$, where $i = 1.0609^{\frac{1}{2}} - 1 = .03$ is the seir. Thus, $PV = \frac{120}{2(.03)} = \frac{60}{.03} = 2000$. Note that we could have just used $PV = 60a_{\overline{\infty}|i} = \frac{60}{i} = \frac{60}{.03} = 2000$.

5.



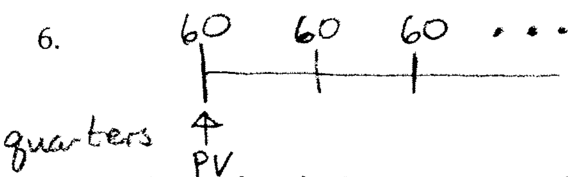
$aeir = .0609$

Using upper m notation, $PV = 120\ddot{a}^{(2)}_{\overline{\infty}|.0609} = \frac{120}{d^{(2)}}$. Note that $d^{(2)} = 2d$, where d is sedr that's equivalent to the given aeir of 6.09%. Therefore $d = 1 - 1.0609^{\frac{-1}{2}} = \frac{.03}{1.03}$ Note that in the solution to the last problem, we saw that the equivalent seir to the aeir of 6.09% is $i = .03$, and so this illustrates the fact that $d = \frac{i}{1+i}$. Continuing, we get $PV = \frac{120}{2(\frac{.03}{1.03})} = \frac{60}{.03}(1.03) = 2060$.

Without using upper m notation, $PV = 60\ddot{a}_{\overline{\infty}|i} = \frac{60}{d} = \frac{60}{\left(\frac{.03}{1.03}\right)} = \frac{60}{.03}(1.03) = 2060$.

Finally, notice that we can think of this perpetuity due as follows: separate the first payment of 60, and then the rest of the payments can be grouped together to form the perpetuity immediate from the last problem. We saw that the *PV* of the perpetuity immediate from the last problem was 2000, and since the first payment of 60 is made at the valuation date, the value of this payment of 60 at the valuation date is 60. Therefore the *PV* of the perpetuity due is 60 + 2000 = 2060.

6.



$i^{(2)} = .1218$

Notice that the interest conversion period in this problem is semiannual, not annual, but the same principles apply; namely, payments are more frequent than interest is compounded. Since there are two payment periods (quarters) per interest conversion period (semiannual period), then we use upper 2 notation. The *i* in the annuity symbol in this problem is the seir, and so $i = \frac{i^{(2)}}{2} = \frac{.1218}{2} = .0609$. We then get $PV = 120\ddot{a}^{(2)}_{\overline{\infty}|.0609}$. At this point we recognize that, mathematically, this problem is exactly the same as the last problem. *PV* = 2060 as in the last problem.

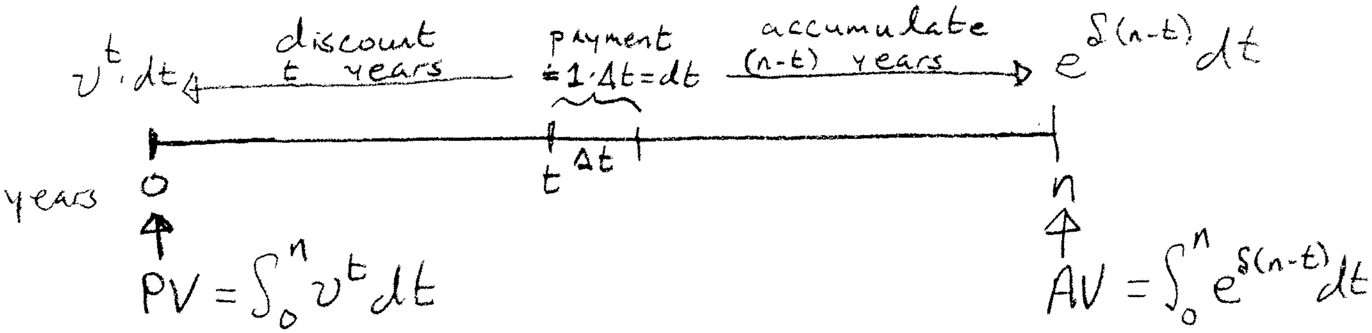
Section 4: Basic Level Continuous Annuities

By continuous annuity, we mean an annuity in which the payments are made continuously throughout the payment period. Of course this is not practical, since payments cannot be made continuously, but there is theoretical value to these annuities. The payment period will always be in years, and the "basic level" part of the title of this section means the total amount paid per year is 1. In this context of continuous annuities, we say the payment rate is 1 per year.

The idea is to start with the m^{thly} annuities from the last section, those for which payments of $1/m$ are made m times per year, and then take the limit as m goes to infinity. The result from this limiting process is called a continuous annuity.

A timeline and formulas to value continuous annuities follow. The VEP formulas can be related back to the VEP formulas from the last section. For example, the VEP formula for the present value of a basic level continuous annuity uses the fact that the VEP's for $\ddot{a}_{\overline{n}|i}^{(m)}$ and $a_{\overline{n}|i}^{(m)}$ are upper and lower Riemann sums for $\int_0^n v^t\,dt$, respectively. We derive the CRF's by integrating in the VEP formulas. It is common to be given a constant force of interest, δ, in continuous annuity problems.

Timeline: (payment rate = 1 per year)



$$\bar{a}_{\overline{n}|} \overset{VEP}{=} \int_0^n v^t\,dt \overset{CRF}{=} \frac{1-v^n}{\delta}$$

$$\bar{s}_{\overline{n}|} \overset{VEP}{=} \int_0^n e^{\delta(n-t)}\,dt \overset{CRF}{=} \frac{e^{\delta n}-1}{\delta}$$

Note that, similar to regular annuities, $\bar{s}_{\overline{n}|} = \bar{a}_{\overline{n}|} \cdot (1+i)^n = \bar{a}_{\overline{n}|} \cdot e^{\delta n}$.

Module 2 Section 4 Problems:

1. Determine the present value of a 10-year annuity with payments made continuously at a rate of 1 per year, using an annual effective interest rate of 6%.

2. Determine the accumulated value of a 20-year annuity with payments made continuously at a rate of 5 per year, using a force of interest equal to 4%.

3. An n-year annuity with payments made continuously at a rate of K per year has a present value of 10000 using an interest rate of 5% per annum. Using the same interest rate, the accumulated value of the annuity is 11025. Determine K.

Solutions to Module 2 Section 4 Problems:

1. $PV = \bar{a}_{\overline{10|}} = \frac{1-v^{10}}{\delta}$. We're given i = .06 aeir, and so $v^{10} = 1.06^{-10}$ and $\delta = \ln(1.06)$. Therefore, $PV = \bar{a}_{\overline{10|}} = \frac{1-1.06^{-10}}{\ln(1.06)} = 7.578745463$.

2. $AV = 5\bar{s}_{\overline{20|}} = 5\frac{e^{20\delta}-1}{\delta}$. $\delta = 0.04$ and so $AV = 5\bar{s}_{\overline{20|}} = 5\frac{e^{20(.04)}-1}{.04} = 155.1926161$

3. $AV = PV \cdot (1+i)^n \implies 11025 = 10000 \cdot 1.05^n \implies n = 2$. Therefore, using the fact that PV = 10000 (we could use AV = 11025) then $10000 = K\bar{a}_{\overline{2|}.05} = K\frac{1-v_{.05}^{2}}{\ln(1.05)}$ and so K = 5247.917658.

Section 5: Geometric Annuities

A sequence of terms forms a geometric progression if there is a "common ratio" between consecutive terms of the sequence. This means that given any term in the sequence, we can get the next term by multiplying by the common ratio.

A **geometric annuity** is an annuity for which the payments form a geometric progression. There are no special actuarial annuity symbols for the values of geometric annuities. When valuing geometric annuities, since the payments form a geometric progression, then the VEP expression for the value of the annuity will be a geometric sum with common ratio $r > 0$. We can determine the value of a geometric annuity using the following 3-step process.

Step 1: VEP
Step 2: Factor out the first term
Step 3: Recognize basic level VEP expressions and use TVM

After we factor out the first term of the VEP expression, as stated in Step 2, the second factor in the resulting expression will be a geometric sum that looks like $1 + r + r^2 + \cdots$ where the number of terms of the sum equals the number of payments of the annuity. For Step 3, we recognize the following facts:

1. If $r < 1$, think of $r = v = \frac{1}{1+i}$ and the sum is $1 + v + v^2 + \cdots \overset{VEP}{=} \ddot{a}_{\overline{n}|i}$ where n is the number of payments and $i = \frac{1}{r} - 1$.
2. If $r > 1$, think of $r = 1 + i$ and the sum is $1 + (1+i) + (1+i)^2 + \cdots \overset{VEP}{=} s_{\overline{n}|i}$ where n is the number of payments and $i = r - 1$.

Summarizing, for Step 3, we recognize that $1 + r + r^2 + \cdots = \begin{cases} \ddot{a}_{\overline{n}|\left(\frac{1}{r}-1\right)} & \text{if } r < 1 \\ s_{\overline{n}|(r-1)} & \text{if } r > 1 \end{cases}$

Geometric Annuity Immediate: In the special case of determining the present value of a geometric annuity immediate with periodic payments of $K, K(1+j), \cdots$, we can use the following formula, where i is the periodic effective interest rate:

$$PV = K\frac{1 - \left(\frac{1+j}{1+i}\right)^n}{i - j}$$

Geometric Perpetuity: If the annuity is a perpetuity, then the geometric sum representing its present value is actually a convergent geometric series with common ratio $r < 1$. We determine the present value by using the fact that the geometric series converges to the ratio of the first term to $(1 - r)$; that is,

$$PV = \frac{\text{First Term}}{1 - r}$$

Module 2 Section 5 Problems:

1. Determine the value of the sum $1 + 1.02 + 1.02^2 + \cdots + 1.02^{19}$

2. Determine the value of the sum $1 + 0.97 + 0.97^2 + \cdots + 0.97^{23}$

3. Determine the value of the sum $1.04 + 1.04^2 + \cdots + 1.04^{14}$

4. Determine the value of the sum $0.95 + 0.95^2 + \cdots + 0.95^{36}$

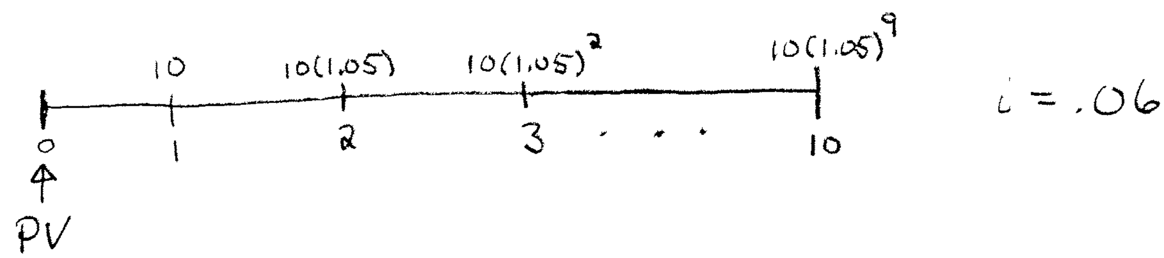
5. Determine the value of the sum $1 + \left(\frac{1.03}{1.05}\right) + \left(\frac{1.03}{1.05}\right)^2 + \cdots + \left(\frac{1.03}{1.05}\right)^{59}$

6. Determine the value of the sum $\left(\frac{1.0506}{1.03}\right) + \left(\frac{1.506}{1.03}\right)^2 + \cdots + \left(\frac{1.0505}{1.03}\right)^{20}$

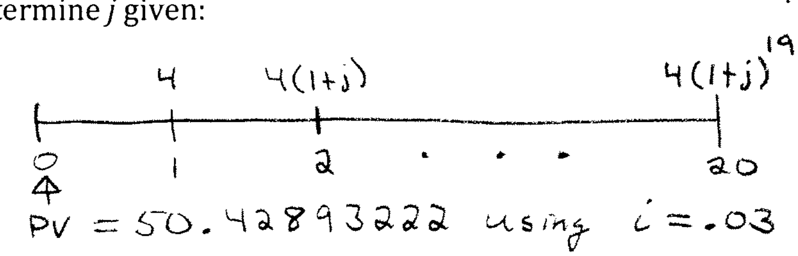
7. Determine the value of the series $1.07(0.98) + 1.07(0.98)^2 + 1.07(0.98)^3 \cdots$

For Numbers 8-15, determine the value, at the given valuation date and using the given periodic effective interest rate, of the cash flow shown.

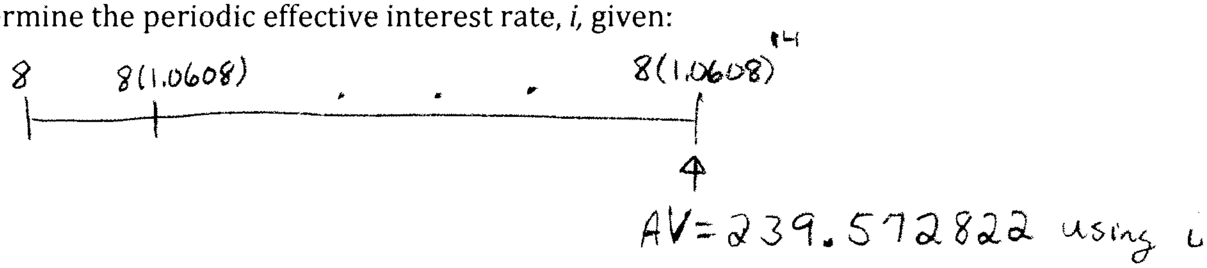
8.



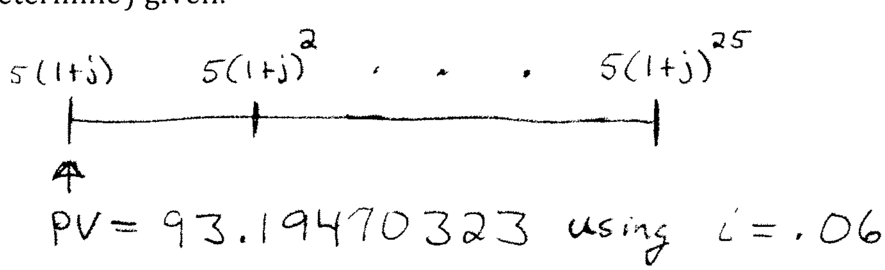
9. Determine j given:



10. Determine the periodic effective interest rate, i, given:



11. Determine j given:



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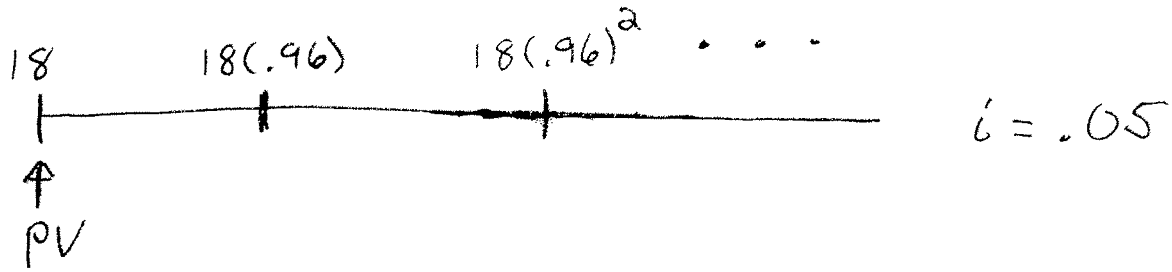
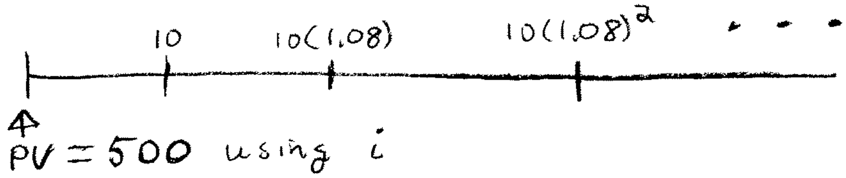
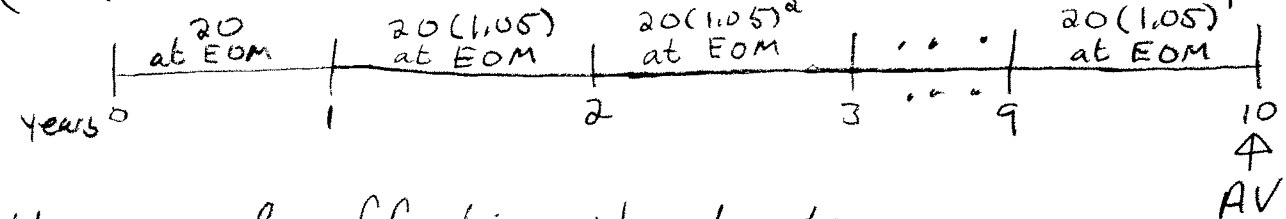


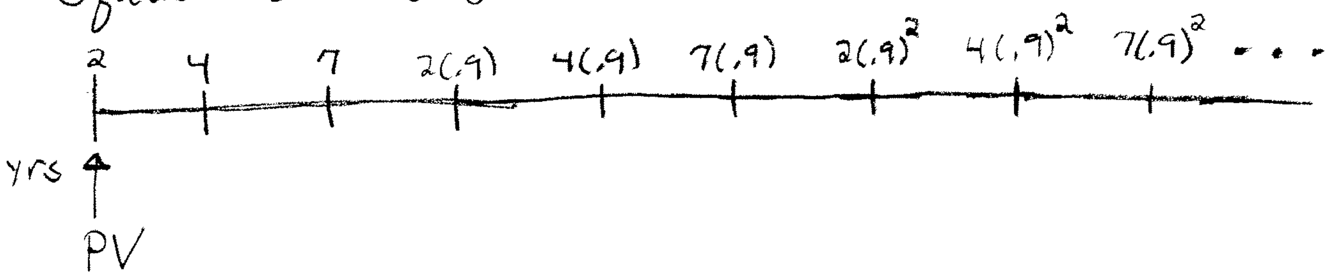
13. Determine the periodic effective interest rate *i*, given:



14. (EOM - End of Month) Use annual effective interest rate = 6%



15. Use annual effective interest rate equal to 2.5%



Solutions to Module 2 Section 5 Problems:

1. Since $r = 1.02 > 1$, we think of the sum as $1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1}$, which is the VEP expression for $s_{\overline{n}|i}$. In this case, $i = r - 1 = 0.02$ and $n = 20$ (there are 20 terms in the sum). Therefore the sum is $s_{\overline{20}|.02} \overset{TVM}{=} 24.2973698$.

   Alternatively, we can always value geometric sums using the formula
   $$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$
   In this case, with $r = 1.02$ and $n = 20$, the expression on the right is $\frac{1-1.02^{20}}{1-1.02} = \frac{1-1.02^{20}}{-0.02}$ which reduces to $\frac{1.02^{20}-1}{0.02}$; the CRF for $s_{\overline{20}|.02}$

2. Since $r = 0.97 < 1$, we think of the sum as $1 + v + v^2 + \cdots + v^{n-1}$, which is the VEP expression for $\ddot{a}_{\overline{n}|i}$. In this case, $i = \frac{1}{r} - 1 = \frac{1}{0.97} - 1$ and $n = 24$ (there are 24 terms in the sum). Therefore the sum is $\ddot{a}_{\overline{24}|i} \overset{TVM}{=} 17.2860926$.

   Alternatively, we can use the formula $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$
   In this case, with $r = 0.97 = v$ and $n = 24$, the expression on the right is $\frac{1-0.97^{24}}{1-0.97} = \frac{1-v^{24}}{1-v}$. Since $1 - v = d$, this reduces to the CRF for $\ddot{a}_{\overline{24}|i}$ as above.

3. Since $r = 1.04 > 1$, we think of the sum as $(1+i) + (1+i)^2 + \cdots + (1+i)^n$, which is the VEP expression for $\ddot{s}_{\overline{n}|i}$. In this case, $i = 0.04$ and $n = 14$ (there are 14 terms in the sum). Therefore the sum is $\ddot{s}_{\overline{14}|.04} = 19.02358764$.

4. Since $r = 0.95 < 1$, we think of the sum as $v + v^2 + \cdots + v^n$, which is the VEP expression for $a_{\overline{n}|i}$. In this case, $i = \frac{1}{r} - 1 = \frac{1}{0.95} - 1$ and $n = 36$ (there are 36 terms in the sum). Therefore the sum is $a_{\overline{36}|i} \overset{TVM}{=} 16.00219492$.

5. Since $r = \frac{1.03}{1.05} < 1$, this is like #2. We think of the sum as $1 + v + v^2 + \cdots + v^{n-1}$, which is the VEP expression for $\ddot{a}_{\overline{n}|i}$. In this case, $i = \frac{1}{r} - 1 = \frac{1.05}{1.03} - 1$ and $n = 60$ (there are 60 terms in the sum). Therefore the sum is $\ddot{a}_{\overline{60}|i} \overset{TVM}{=} 35.94097196$.

6. Since $r = \frac{1.0506}{1.03} = 1.02 > 1$, this is like #3. We think of the sum as $(1+i) + (1+i)^2 + \cdots + (1+i)^n$, which is the VEP expression for $\ddot{s}_{\overline{n}|i}$. In this case, $i = r - 1 = 0.02$ and $n = 20$ (there are 20 terms in the sum). Therefore the sum is $\ddot{s}_{\overline{20}|.02} \overset{TVM}{=} 24.78331719$.

Note that after recognizing that r just reduces to 1.02, then the expression we're valuing in this problem is just 1.02 times the expression we're valuing in Problem 1. You should check that the numeric answer to this problem is just 1.02 times the answer to Problem 1, and this is also capture symbolically by recognizing that $\ddot{s}_{\overline{n}|i} = s_{\overline{n}|i} \cdot (1+i)$.

7. As illustrated in the previous problems, the geometric sums you'll see on Exam FM/2 can be valued using TVM. The geometric series you'll see on Exam FM/2 are even easier to value than geometric sums. Letting Σ denote the sum, we use the formula:

$$\Sigma = \frac{First\ Term}{1-r}$$

For the series in this problem, we have $First\ Term = 1.07(0.98)$ and $r = 0.98$. So

$$\Sigma = 1.07(0.98) + 1.07(0.98)^2 + 1.07(0.98)^3 \cdots = \frac{1.07(0.98)}{1-0.98} = \frac{1.07(0.98)}{0.02} = 52.43$$

8. Using the 3-step process, we first determine the VEP expression for the PV, getting $PV = 10v + 10(1.05)v^2 + \cdots + 10(1.05)^9 v^{10} = \frac{10}{1.06} + \frac{10(1.05)}{1.06^2} + \cdots + \frac{10(1.05)^9}{1.06^{10}}$. Next, factor out the first term, getting $PV = \frac{10}{1.06}\left[1 + \frac{1.05}{1.06} + \cdots + \left(\frac{1.05}{1.06}\right)^9\right]$. Finally, the second factor is a geometric sum with $r = \frac{1.05}{1.06} < 1$, and so we recognize it as $\ddot{a}_{\overline{n}|i}$ with $n = 10$ and $i = \frac{1}{r} - 1 = \frac{1.06}{1.05} - 1$. Using TVM, we have $PV = \frac{10}{1.06}\ddot{a}_{\overline{10}|i} = 90.43374826$.

Note what happens if we reverse the order of the VEP expression. Then we get $PV = 10(1.05)^9 v^{10} + 10(1.05)^8 v^9 + \cdots + 10v = \frac{10(1.05)^9}{1.06^{10}} + \frac{10(1.05)^8}{1.06^9} + \cdots + \frac{10}{1.06}$. Next, factor out the first term, getting $PV = \frac{10(1.05)^9}{(1.06)^{10}}\left[1 + \frac{1.06}{1.05} + \cdots + \left(\frac{1.06}{1.05}\right)^9\right]$. Finally, the second factor is a geometric sum with $r = \frac{1.06}{1.05} > 1$. We recognize it as $s_{\overline{n}|i}$ with $n = 10$ and $i = r - 1 = \frac{1.06}{1.05} - 1$. Using TVM, we get $PV = \frac{10(1.05)^9}{(1.06)^{10}} s_{\overline{10}|i} = 90.43374826$ as above. This illustrates the fact that the order in which we write the VEP expression is irrelevant. Also note that in this last expression, we are calculating the present value of the annuity using an "*s*" symbol. That is, we are computing *FV* using TVM, but the resulting value is the *PV* of the annuity. This is a phenomenon that sometimes occurs using this process to value geometric annuities.

Alternatively, we can recognize this calculation as the present value of a geometric annuity immediate and use the formula for such. In this case we get,

$$PV = 10 \cdot \frac{1 - \left(\frac{1.05}{1.06}\right)^{10}}{.06 - .05} = 90.43374826$$

9. Most geometric annuity problems will be straight forward like the previous problem, but occasionally you will see problems where you must find a rate, either an interest rate or a rate of increase/decrease of the payments. The next several problems illustrate these types of problems.

Using the 3-step process, we first determine the VEP expression for the present value, getting $PV = 4v + 4(1+j)v^2 + \cdots (20 \text{ terms}) = \frac{4}{1.03} + \frac{4(1+j)}{1.03} + \cdots (20 \text{ terms})$. Note that all we need are the first two terms of the expression and the number of terms in the expression. Then factor out the first term, and using the numeric value of the *PV* given, we get $50.42893222 = \frac{4}{1.03}\left[1 + \left(\frac{1+j}{1.03}\right) + \cdots (20 \text{ terms})\right]$. Now we use the fact that the geometric sum in brackets can be valued using either an "$\ddot{a}$" or an "s", depending on whether its common ratio $r = \frac{1+j}{1.03}$ is less than 1 or greater than 1, respectively. Since the number of terms in the sum is 20, if its common ratio is greater than 1, then the sum in the brackets would be greater than 20. In turn, after multiplying by $\frac{4}{1.03}$ we would have a product greater than 50.42893222. We conclude that the common ratio $r = \frac{1+j}{1.03} < 1$, and so the sum in the brackets is $\ddot{a}_{\overline{n}|i}$ with $n = 20$ and $i = \frac{1}{r} - 1 = \frac{1.03}{1+j} - 1$. Using TVM and solving for I/Y, we get $i = 5.102040816\%$, and so $j = -0.02$. (That is, each payment is 2% less than its previous payment.)

Now consider what happens if we reverse the terms in the VEP expression for the present value. We would get $PV = 4(1+j)^{19}v_{.03}^{20} + 4(1+j)^{18}v_{.03}^{19} + \cdots (20 \text{ terms})$. After factoring out the first term and using the numeric value of the *PV* given, we get $50.42893222 = \frac{4(1+j)^{19}}{1.03^{20}}\left[1 + \left(\frac{1.03}{1+j}\right) + \cdots (20 \text{ terms})\right]$. At this point, we are stuck. We know that the sum in the brackets is either an "$\ddot{a}$" or an "s", but because the factor in front of the brackets also has the unknown j, then we will not be able to use TVM to complete the calculation. The only way to proceed is to use a "guess and check" technique (keep in mind this is a multiple choice exam and there will be 5 answer choices). This is an example of a problem in which the order in which we write the sum in the VEP expression matters. If, when working a problem, you run into a situation such as this, try reversing the order in the VEP expression. If all else fails, use guess and check!!

Finally, as in the last problem, we can recognize this calculation as the present value of a geometric annuity immediate and use the formula for such. In this case we get,

$$50.42893222 = 4 \cdot \frac{1 - \left(\frac{1+j}{1.03}\right)^{20}}{.03 - j}$$

As in the last paragraph, we are stuck and at this point we must use guess and check.

10. We first determine the VEP expression for the AV. If we start with the first payment, we get $AV = 8(1+i)^{14} + 8(1.0608)(1+i)^{13} + \cdots$ (15 terms). Then factor out the first term, and using the numeric value of the AV given, we get $239.5728222 = 8(1+i)^{14}\left[1 + \left(\frac{1.0608}{1+i}\right) + \cdots \text{(15 terms)}\right]$. As mentioned in the solution to the last problem, even though we know the expression in the bracket is either an "$\ddot{a}$" or an "s", since the factor in front of the brackets also has the unknown i, then we will not be able to use TVM to complete the calculation. We're stuck again.

Instead of proceeding with guess and check, reverse the VEP expression for the AV. We get $AV = 8(1.0608)^{14} + 8(1.0608)^{13}v + \cdots$ (15 terms). Continuing, we get $239.5728222 = 8(1.0608)^{14}\left[1 + \left(\frac{1+i}{1.0608}\right) + \cdots \text{(15 terms)}\right]$. As in the previous problem, we see that the sum in the brackets must be less than 15. Therefore, the common ratio $r = \frac{1+i}{1.0608} < 1$ and so the sum in the brackets is $\ddot{a}_{\overline{n}|j}$ (we're using j in the annuity symbol since i has already been used) with n = 15 and $j = \frac{1}{r} - 1 = \frac{1.0608}{1+i} - 1$. Using TVM and solving for I/Y, we get j = 0.02, and so i = 0.04.

11. By VEP, we have $PV = 5(1+j) + 5(1+j)^2 v_{.06} + \cdots$ (25 terms) and therefore we get $93.19470323 = 5(1+j)\left[1 + \left(\frac{1+j}{1.06}\right) + \cdots \text{(25 terms)}\right]$. It looks like we will have to guess and check at this point, but we can tweak our approach as follows. Notice that the common ratio of the geometric sum in brackets is $r = \frac{1+j}{1.06}$. Multiply the factor in front of the bracket by $\frac{1.06}{1.06}$ and rearrange the resulting expression to get $5(1.06)\cdot\frac{1+j}{1.06}$. Now distribute the factor $\frac{1+j}{1.06}$ through the brackets, obtaining the equation $93.19470323 = 5(1.06)\cdot\left[\left(\frac{1+j}{1.06}\right) + \left(\frac{1+j}{1.06}\right)^2 + \cdots \text{(25 terms)}\right]$. Using the same logic as before, the sum of the terms in the brackets must be less than 25, which means the common ratio $r = \frac{1+j}{1.06} < 1$. We therefore recognize the sum in the brackets as $a_{\overline{n}|i}$, not $\ddot{a}_{\overline{n}|i}$, where n = 25 and $i = \frac{1}{r} - 1 = \frac{1.06}{1+j} - 1$. Using TVM and solving for I/Y, we get i = 0.029126214, and so j = 0.03.

12. For this perpetuity, we have $PV \overset{VEP}{=} 18 + 18(0.96)v + 18(0.96)^2v^2 + \cdots$. Since this is a geometric series with common ratio $r = 0.96v = \frac{0.96}{1.05}$ then we have

$$PV = \frac{First\ Term}{1-r} = \frac{18}{1 - \dfrac{0.96}{1.05}} = 210$$

13. For this perpetuity, we have $PV = 500 \overset{VEP}{=} 10v + 10(1.08)v^2 + 10(1.08)^2v^3 + \cdots$. Since this is a geometric series with common ratio $r = 1.08v = \frac{1.08}{1+i}$ then we have

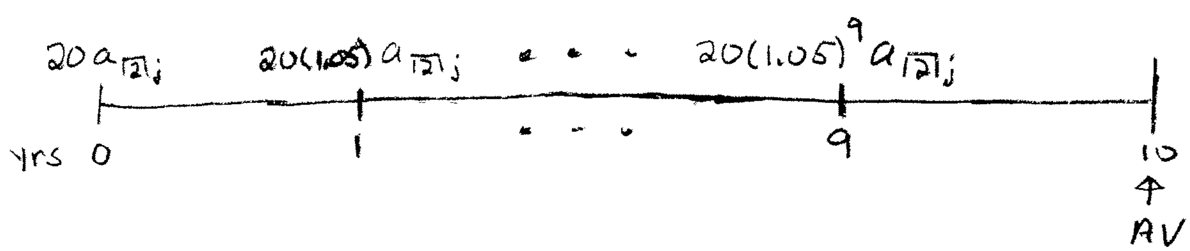
$$500 = \frac{First\ Term}{1-r} = \frac{10v}{1-1.08v} = \frac{\left(\frac{10}{1+i}\right)}{1-\left(\frac{1.08}{1+i}\right)}$$

Simplifying the complex fraction in the last expression gives $500 = \frac{10}{i-0.08}$ which implies $i = 0.10$.

14. This is a tough problem. The idea is to group together each year's level payments and value them first at an appropriate valuation date. For example, replace the monthly payments of 20 during the first year with a single payment of $20a_{\overline{12}|j}$ at the beginning of year 1, where j is the meir that's equivalent to the given aeir of 0.06. That is, $j = 1.06^{\frac{1}{12}} - 1$

A couple of comments are needed here. First, the payments of 20 are made monthly, and so we need to use the meir to value the payments. Next, we are using the "a" symbol because we have chosen to replace the payments of 20 with a single payment at the beginning of the year, which is one payment period before the first payment. If the payments were made at the beginning of each month, then we would use the symbol "$\ddot{a}$". Finally, we could have chosen to use a single payment at the end of the year. Since the payments are made at the end of the month, the end of the year would be at the time of the last payment. In this case we would use the "s" symbol to replace the monthly payments with a single payment.

Now, continue as we did with the first year's payments and replace the monthly payments of 20(1.05) during the second year with a single payment of $20(1.05)a_{\overline{12}|j}$ at the beginning of year 2. Continue. We get the equivalent timeline:



Recognize that this is the timeline for a geometric annuity. We determine the AV using the 3 step process. First, $AV \overset{VEP}{=} 20a_{\overline{12}|j}(1.06)^{10} + 20(1.05)a_{\overline{12}|j}(1.06)^9 + \cdots$ (10 terms). Factoring out the first term and recognizing the annuity VEP formula, we get $AV = 20a_{\overline{12}|j}(1.06)^{10}\left[1 + \frac{1.05}{1.06} + \cdots (10 \text{ terms})\right] = 20a_{\overline{12}|j}(1.06)^{10}\ddot{a}_{\overline{10}|i}$ where $i = \frac{1.06}{1.05} - 1$. We get $AV = 3992.638209$.

15. It is convenient to view the perpetuity as 3 separate perpetuities; one with payments of 2, 2(0.9), ..., one with payments of 4, 4(0.9), ..., and one with payments of 7, 7(0.9), ... Then $PV \overset{VEP}{=} [2 + 2(0.9)v^3 + \cdots] + [4v + 4(0.9)v^4 + \cdots] + [7v^2 + 7(0.9)v^5 + \cdots]$ The sum in each of the three sets of brackets is a geometric series, and in each case the common ratio is $r = 0.9v^3 = \frac{0.9}{1.025^3}$. Using the fact that, in each case,

$$\Sigma = \frac{First\ Term}{1 - r}$$

we get

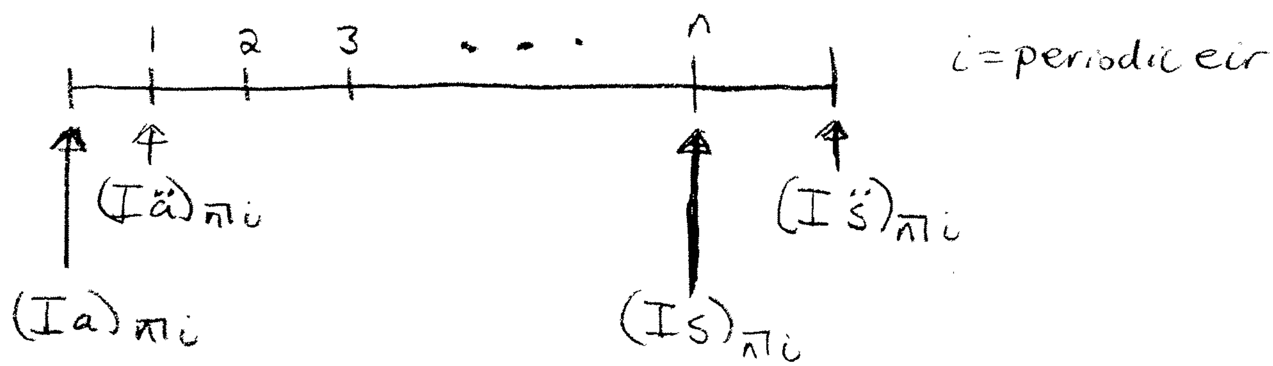
$$PV = \left(\frac{2}{1-r}\right) + \left(\frac{4v}{1-r}\right) + \left(\frac{7v^2}{1-r}\right) = \frac{2 + 4v + 7v^2}{1-r} = \frac{2 + \frac{4}{1.025} + \frac{7}{1.025^2}}{1 - \frac{0.9}{1.025^3}} = 76.4952$$

Section 6: Arithmetic Annuities

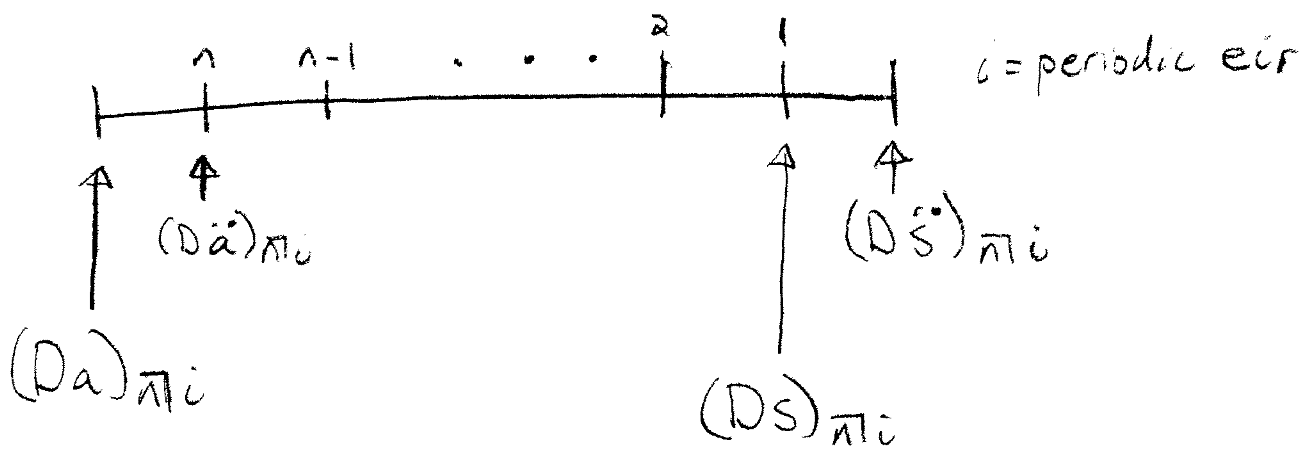
A sequence of terms forms an arithmetic progression if there is a "common difference" between consecutive terms of the sequence. This means that given any term in the sequence, we can get the next term by adding the common difference, denoted by d. Note that d may be negative, in which case we get from one term to the next by subtracting the common value.

An **arithmetic annuity** is an annuity for which the payments form an arithmetic progression. Unlike geometric annuities, there are special actuarial annuity symbols for the present and accumulated values of arithmetic annuities, which we'll get to below. Note that if $d < 0$ then the payments are decreasing, whereas if $d > 0$ the payments are increasing. If the first payment is 1 and $d = 1$, the payments are 1, 2, 3, ... n, and we call this annuity a basic increasing annuity. If the first payment is n and $d = -1$, the payments are n, $n - 1$, ..., 1, and we call this annuity a basic decreasing annuity.

Timelines and Notation:
(Basic Increasing Annuity)



(Basic Decreasing Annuity)



VEP's and CRF's for Basic Increasing and Decreasing Annuities

$$\boxed{(Ia)_{\overline{n|}} \overset{VEP}{=} v + 2v^2 + \cdots + nv^n \overset{CRF}{=} \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}}$$

$$(I\ddot{a})_{\overline{n|}} \overset{VEP}{=} 1 + 2v + 3v^2 + \cdots + nv^{n-1} \overset{CRF}{=} \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = (Ia)_{\overline{n|}} \cdot (1+i)$$

$$(Is)_{\overline{n|}} \overset{VEP}{=} (1+i)^{n-1} + 2 \cdot (1+i)^{n-2} + \cdots + n \overset{CRF}{=} \frac{\ddot{s}_{\overline{n|}} - n}{i} = (Ia)_{\overline{n|}} \cdot (1+i)^n$$

$$(I\ddot{s})_{\overline{n|}} \overset{VEP}{=} (1+i)^n + 2 \cdot (1+i)^{n-1} + \cdots + n \cdot (1+i) \overset{CRF}{=} \frac{\ddot{s}_{\overline{n|}} - n}{d} = (Is)_{\overline{n|}} \cdot (1+i)$$

$$\boxed{(Da)_{\overline{n|}} \overset{VEP}{=} nv + (n-1) \cdot v^2 + \cdots + v^n \overset{CRF}{=} \frac{n - a_{\overline{n|}}}{i}}$$

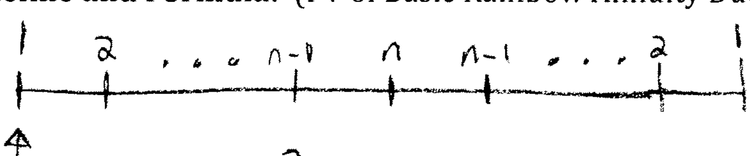
$$(D\ddot{a})_{\overline{n|}} \overset{VEP}{=} n + (n-1) \cdot v + \cdots + v^{n-1} \overset{CRF}{=} \frac{n - a_{\overline{n|}}}{d} = (Da)_{\overline{n|}} \cdot (1+i)$$

$$(Ds)_{\overline{n|}} \overset{VEP}{=} n \cdot (1+i)^{n-1} + (n-1) \cdot (1+i)^{n-2} + \cdots + 1 \overset{CRF}{=} \frac{n(1+i)^n - s_{\overline{n|}}}{i} = (Da)_{\overline{n|}} \cdot (1+i)^n$$

$$(D\ddot{s})_{\overline{n|}} \overset{VEP}{=} n \cdot (1+i)^n + (n-1) \cdot (1+i)^{n-1} + \cdots + (1+i) \overset{CRF}{=} \frac{n(1+i)^n - s_{\overline{n|}}}{d} = (Ds)_{\overline{n|}} \cdot (1+i)$$

Note that by knowing the two boxed formulas, we can easily derive the others by using the relationship in the last equality of each formula. The following two formulas are used often on exams.

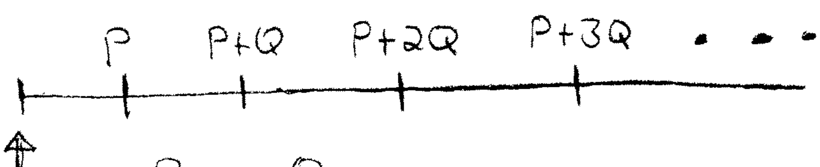
Timeline and Formula: (PV of Basic Rainbow Annuity Due)



peak = n
i = periodic eir

$$PV = (\ddot{a}_{\overline{n|}i})^2$$

Timeline and Formula: (PV of General Increasing Perpetuity Immediate)



i = periodic eir

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

Module 2 Section 6 Problems:

1. Determine $12v + 11v^2 + \cdots + v^{12}$ using $i = 0.03$.

2. Determine $v + 2v^2 + 3v^3 \cdots + 12v^{12}$ using $i = 0.03$.

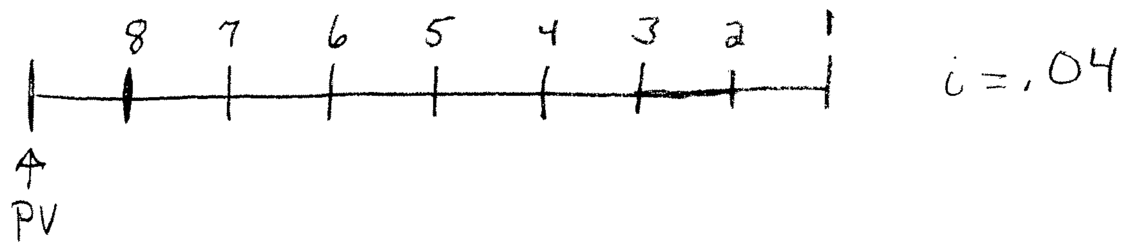
3. Determine $(1.025)^{25} + 2 \cdot (1.025)^{24} + 3 \cdot (1.025)^{23} \cdots + 25 \cdot (1.025)$

4. Determine $6 + 7v + 8v^2 + \cdots + 27v^{21}$ using $i = 0.05$

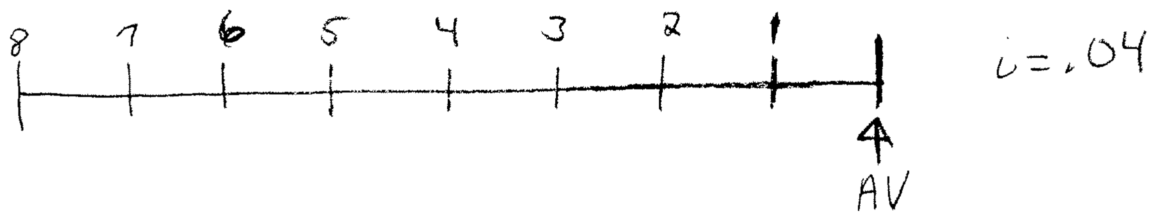
5. Determine $37 + 42(1.07) + 47(1.07)^2 + \cdots + 92(1.07)^{11}$

For Numbers 6 through 14, determine the value of the given annuity at the given valuation date using the given interest rate.

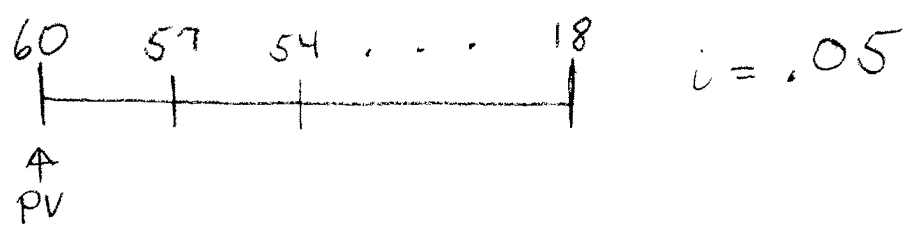
6.



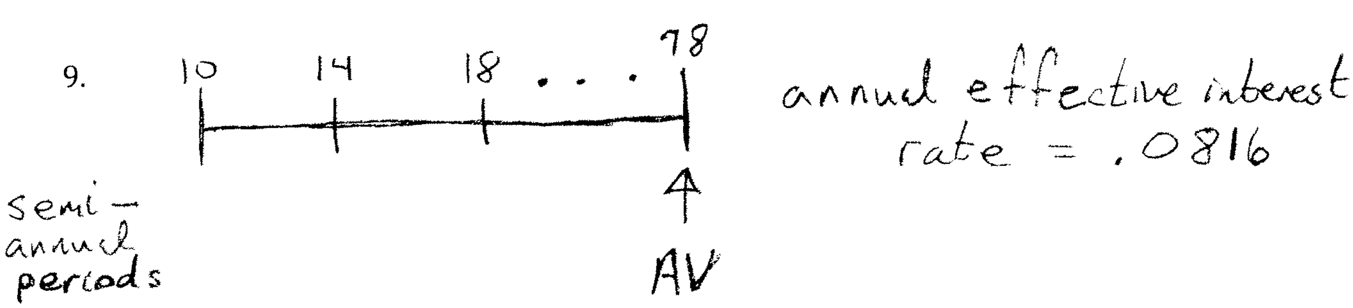
7.



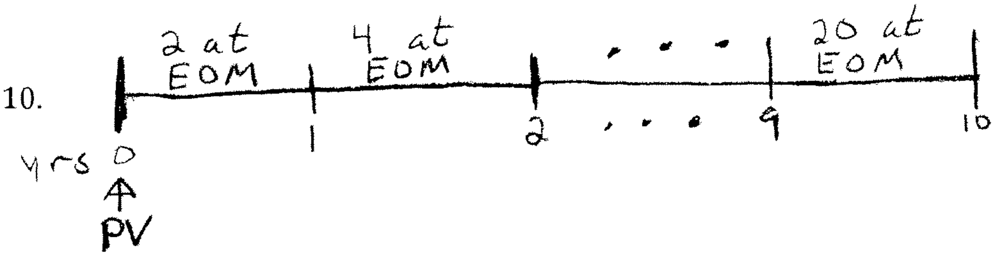
8.



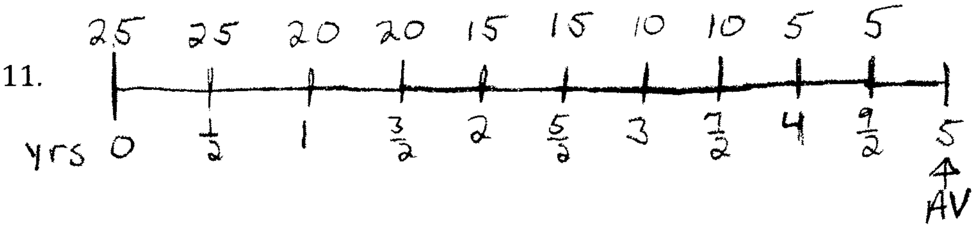
9.



EOM - End of Month

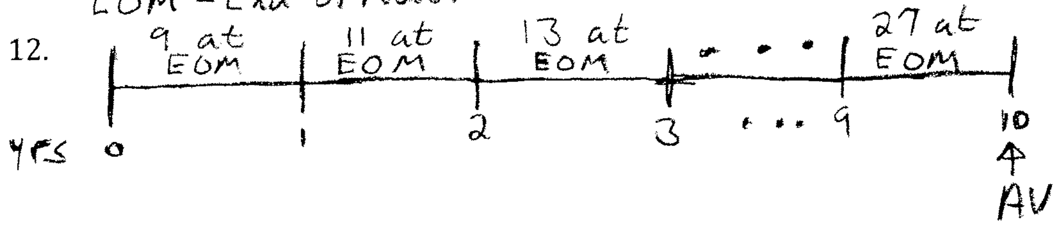
10. 

$i = .05$

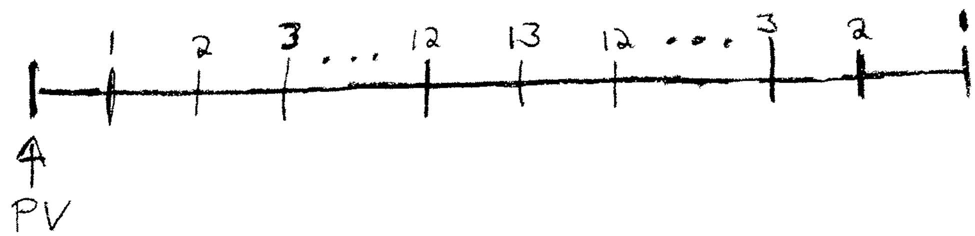
11. 

annual effective interest rate = .06

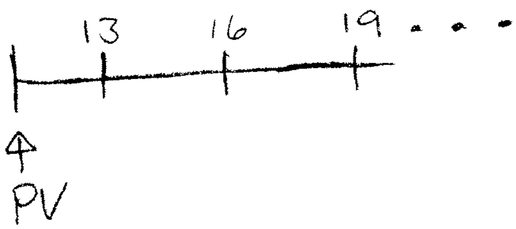
EOM - End of Month

12. 

annual effective interest rate = .05

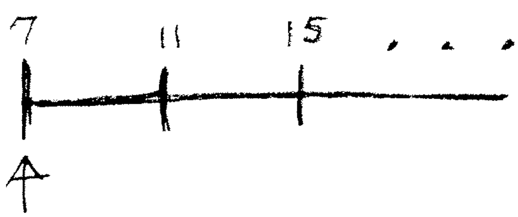
13. 

periodic eir = .03

14. 

periodic eir = .025

15. Determine the periodic effective interest rate, *i*, given:



PV = 10557 using i

Solutions to Module 2 Section 6 Problems:

1. Recognize this sum, $12v + 11v^2 + \cdots + v^{12}$, as the VEP expression for $(Da)_{\overline{12|}}$. Letting Σ denote the value of the sum, we have
$\Sigma = (Da)_{\overline{12|}.03} = \frac{12 - a_{\overline{12|}}}{.03} = 68.1999$

The sum in this problem, $12v + 11v^2 + \cdots + v^{12}$, and the other problems in this section, are what I (half-jokingly) refer to as georithmic sums. I say "half-jokingly" because you won't see that word in books, or anywhere else for that matter. I call it a georithmic sum because if we ignore the coefficients, then the sum would be a geometric sum, but the coefficients themselves form an arithmetic progression. When you see a georithmic sum, you can use the techniques of this section (arithmetically increasing/decreasing annuities) to value the sum.

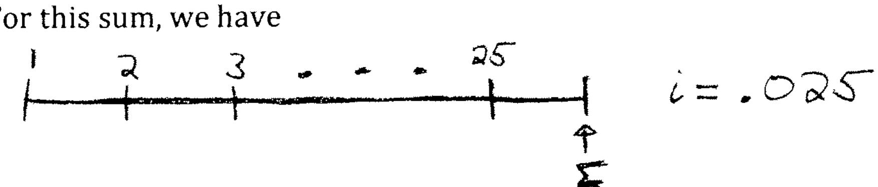
2. We recognize this sum, $v + 2v^2 + 3v^3 \cdots + 12v^{12}$, as the VEP expression for $(Ia)_{\overline{12|}}$. Letting Σ denote the value of the sum, we have

$$\Sigma = (Ia)_{\overline{12|}.03} = \frac{\ddot{a}_{\overline{12|}} - 12v^{12}}{.03} = 61.2022$$

There is a TVM shortcut for calculating the numerator of the *Ia* CRF expression. For example, with the numerator above, we think of $\ddot{a}_{\overline{12|}} - 12v^{12}$ as equal to a present value. We can rewrite as $PV - \ddot{a}_{\overline{12|}} + 12v^{12} = 0$. The coefficient of 12 in the last term on the left side of the equal sign can be thought of as a *FV* since it's being discounted *N* (=12) periods, the same as the number of payments of the annuity, and at the same *I/Y* (3%) interest rate as the annuity is being valued. So we can compute the numerator in one step as follows: $\boxed{\text{BGN}}$ 12 $\boxed{\text{N}}$ 3 $\boxed{\text{I/Y}}$ 1 $\boxed{+/-}$ $\boxed{\text{PMT}}$ 12 $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$.

Finally, consider the sum in Problem 1, $12v + 11v^2 + \cdots + v^{12}$, together with the sum in this problem, $v + 2v^2 + 3v^3 \cdots + 12v^{12}$. If we add the two expressions, then we get the level annuity $13v + 13v^2 + 13v^3 \cdots + 13v^{12}$. Generalizing to an arbitrary *n*, we get the formula, $(Da)_{\overline{n|}} + (Ia)_{\overline{n|}} = (n+1) \cdot a_{\overline{n|}}$ Other similar formulas can be established this way.

3. Sometimes it's easier to recognize arithmetic annuities if we draw the corresponding timeline. For this sum, we have



We now recognize the sum as the VEP expression for $(I\ddot{s})_{\overline{25|}.025}$ Therefore

$$\Sigma = (I\ddot{s})_{\overline{25|}.025} = \frac{\ddot{s}_{\overline{25|}} - 25}{.025} \cdot 1.025 = 410.4800$$

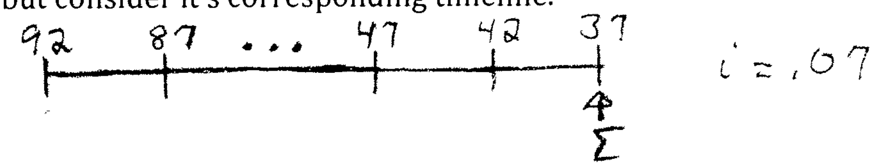
4. Since the coefficients are increasing arithmetically with common difference equal to 1, rewrite the expression as the sum of an increasing arithmetic annuity and level annuity as follows:

$$6 + 7v + 8v^2 + \cdots + 27v^{21} = (1 + 2v + 3v^2 + \cdots + 22v^{21}) + (5 + 5v + \cdots + 5v^{21})$$

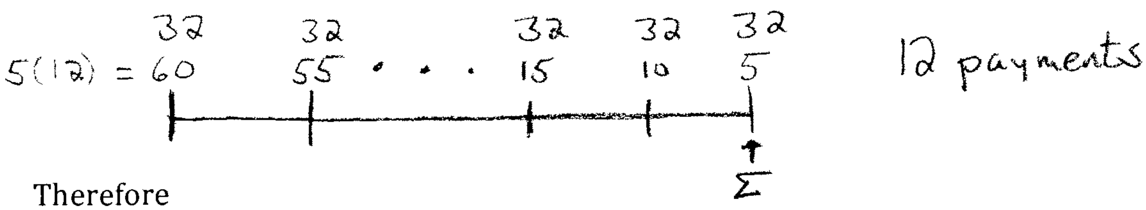
The sum in the first set of parenthesis on the right side of the equal sign is the VEP expression for $(I\ddot{a})_{\overline{22}|}$ and the sum in the second set of parenthesis is the VEP expression for $5\ddot{a}_{\overline{22}|}$. So we have $6 + 7v + 8v^2 + \cdots + 27v^{21} = (I\ddot{a})_{\overline{22}|.05} + 5\ddot{a}_{\overline{22}|,05}$
Therefore

$$\Sigma = (I\ddot{a})_{\overline{22}|.05} + 5\ddot{a}_{\overline{22}|,05} = \frac{\ddot{a}_{\overline{22}|} - 22v^{22}}{.05}(1.05) + 5\ddot{a}_{\overline{22}|} = 201.4153$$

5. Be careful. This sum appears to correspond with an arithmetically increasing annuity, but consider it's corresponding timeline.



We now recognize the sum as the VEP expression for an arithmetically *decreasing* annuity. Since the coefficients are decreasing arithmetically with common difference equal to 5, rewrite the timeline as follows:



Therefore

$$\Sigma = 32s_{\overline{12}|.07} + 5(Ds)_{\overline{12}|.07} = 32s_{\overline{12}|.07} + 5 \cdot \frac{12(1.07)^{12} - s_{\overline{12}|}}{.07} = 1225.1339$$

6. This is the timeline for a standard decreasing annuity. Since the valuation date is one period before the first payment, we have

$$PV = (Da)_{\overline{8}|.04} = \frac{8 - a_{\overline{8}|}}{.04} = 31.6814$$

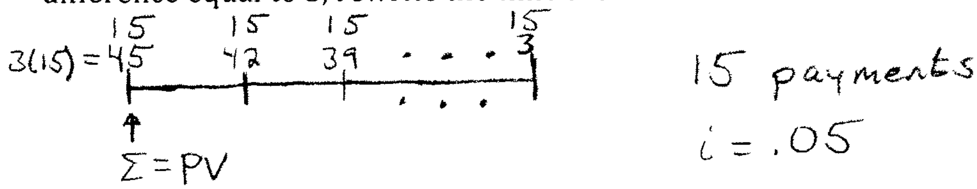
7. The valuation date is one period after the last payment, so

$$AV = (D\ddot{s})_{\overline{8}|.04} = \frac{8(1.04)^8 - s_{\overline{8}|}}{.04}(1.04) = 45.0925$$

Alternatively, note that this AV is the PV from #6, accumulated for 9 periods. So,

$$AV = PV \cdot (1 + i)^9 = 31.6814 \cdot (1.04)^9 = 45.0925$$

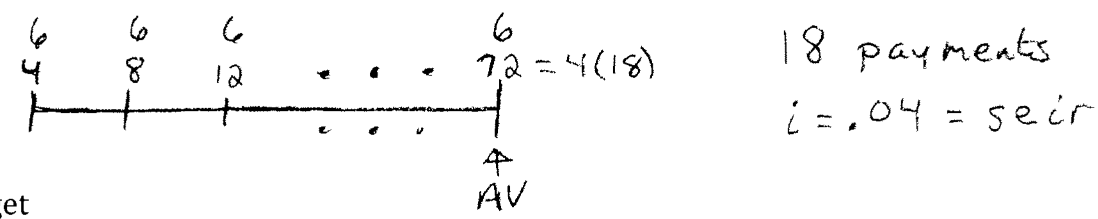
8. This is similar to #5. Since the payments are decreasing arithmetically, with a difference equal to 3, rewrite the timeline as follows:



We get

$$\Sigma = 15\ddot{a}_{\overline{15}|.05} + 3(D\ddot{a})_{\overline{15}|.05} = 15\ddot{a}_{\overline{15}|.05} + 3 \cdot \frac{15 - a_{\overline{15}|}}{.05}(1.05) = 454.5612$$

9. Since the payments are made semiannually, we first convert the given aeir of 8.16% to its equivalent seir, getting $i = 1.0816^{\frac{1}{2}} - 1 = 0.04$. The payments are increasing arithmetically with a common difference of 4. Therefore, rewrite the timeline as:

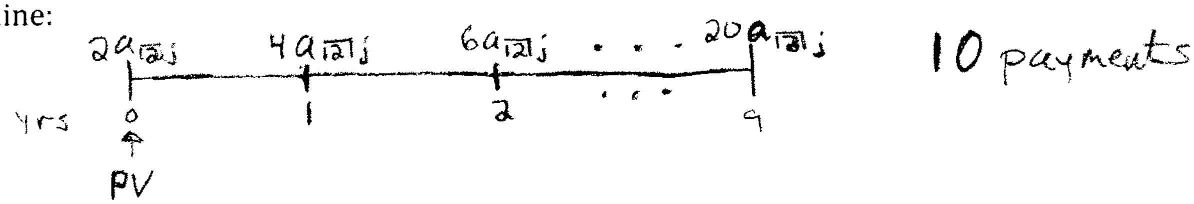


We get

$$AV = 6s_{\overline{18}|.04} + 4(Is)_{\overline{18}|.04} = 6s_{\overline{18}|.04} + 4 \cdot \frac{\ddot{s}_{\overline{18}|} - 18}{.04} = 1020.9954$$

10. This is similar to #14 from the previous section, except that the increase in each year's payments in this problem is arithmetic, instead of geometric like in the last section. We proceed in a similar way, though. The idea is to group together each year's level payments and value them first at an appropriate valuation date. For example, replace the monthly payments of 2 during the first year with a single payment of $2a_{\overline{12}|j}$ at the beginning of year 1, where j is the meir that's equivalent to the given aeir of 0.05. That is, $j = 1.05^{\frac{1}{12}} - 1$.

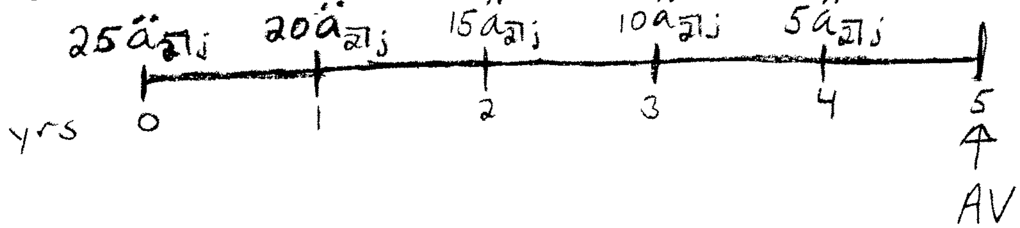
Continue, replacing the monthly payments of 4 during the second year with a single payment of $4a_{\overline{12}|j}$ at the beginning of year 2, and so on. We get the equivalent timeline:



Recognize that the payments are increasing arithmetically with a common difference equal to $2a_{\overline{12}|j}$. Therefore, we get

$$PV = 2a_{\overline{12}|j} \cdot (I\ddot{a})_{\overline{10}|.05} = 2a_{\overline{12}|j} \cdot \frac{\ddot{a}_{\overline{10}|.05} - 10v^{10}}{.05}(1.05) = 966.4356$$

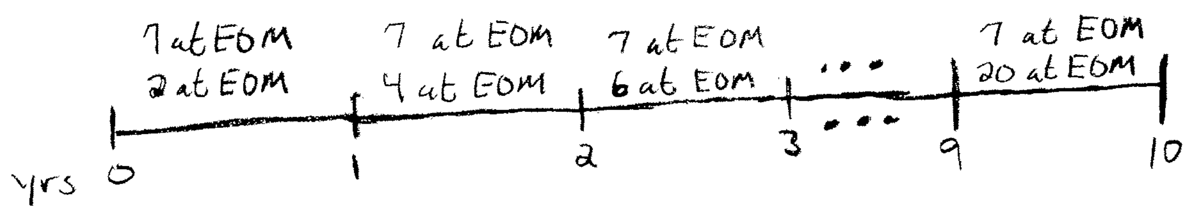
11. Since the payments are made semiannually, we first convert the given aeir of 6% to its equivalent seir, getting $j = 1.06^{\frac{1}{2}} - 1$. Again, we group together each year's level payments and value them first at an appropriate valuation date. For example, replace the semiannual payments of 25 during the first year with a single payment of $25\ddot{a}_{\overline{2}|j}$ at the beginning of year 1, the semiannual payments of 20 during the second year with a single payment of $20\ddot{a}_{\overline{2}|j}$ at the beginning of year 2, and so on. We get the equivalent timeline:



Recognize that the payments are decreasing arithmetically with a difference equal to $5\ddot{a}_{\overline{2}|j}$. After factoring out $5\ddot{a}_{\overline{2}|j}$ from each payment, the resulting payment stream is 5, 4, 3, 2, 1. This is the payment stream for the standard 5-payment arithmetically decreasing annuity. We get

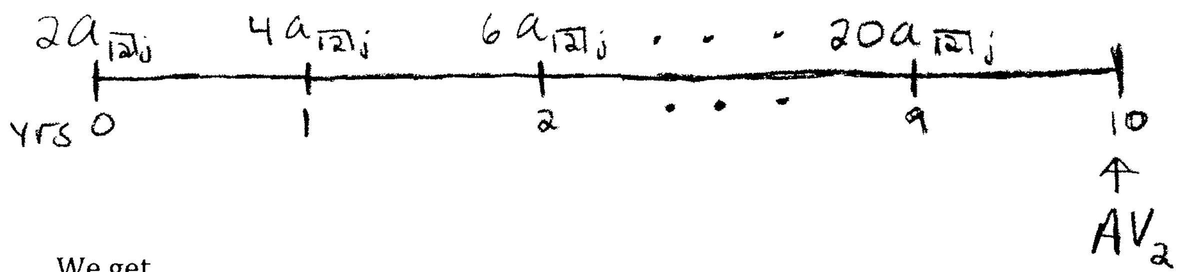
$$AV = 5\ddot{a}_{\overline{2}|j} \cdot (D\ddot{s})_{\overline{5}|.06} = 5\ddot{a}_{\overline{2}|j} \cdot \frac{5(1.06)^5 - s_{\overline{5}|}}{.06}(1.06) = 183.5394$$

12. The monthly payments are level each year, and increase arithmetically with a common difference of 2. Therefore, rewrite the timeline as follows:



Note that this annuity can be thought of as two separate annuities; one with monthly payments of 2 during the first year, monthly payments of 4 during the second year, and so on, and the other annuity being level with monthly payments of 7. The easiest way to value the latter level annuity is to convert the given 5% aeir to its equivalent meir and note that there are 120 payments of 7; that is, $j = 1.05^{\frac{1}{12}} - 1$ and so the accumulated value of the latter annuity is $AV_1 = 7s_{\overline{120}|j} = 1080.5421$.

The other annuity is exactly the annuity in #10. Using the technique in the solution to #10, we get the equivalent timeline:



We get

$$AV_2 = 2a_{\overline{12}|j} \cdot (I\ddot{s})_{\overline{10}|.05} = 2a_{\overline{12}|j} \cdot \frac{\ddot{s}_{\overline{10}|.05} - 10}{.05}(1.05) = 1574.2218$$

Note that to get this accumulated value, we could have just accumulated the PV obtained in #10, getting $AV_2 = PV \cdot (1+i)^{10} = 966.4356 \cdot (1.05)^{10} = 1574.2218$.

Finally, the total accumulated value is $AV = 1080.5421 + 1574.2218 = 2654.7639$.

13. This is a rainbow annuity with a peak of 13. Note that the formula for the present value of a rainbow annuity due with a peak of n is $PV = (\ddot{a}_{\overline{n}|i})^2$. Since the valuation date for this calculation is one period before the first payment, we must discount the value obtained from the above formula for one period. We get

$$PV = (\ddot{a}_{\overline{n}|i})^2 \cdot v = \frac{\left(\ddot{a}_{\overline{13}|.03}\right)^2}{1.03} = 116.4953$$

14. The first payment is $P = 13$ and each payment increases by $Q = 3$. The present value, one period before the first payment, which is the valuation date, is

$$PV = \frac{P}{i} + \frac{Q}{i^2} = \frac{13}{0.025} + \frac{3}{0.025^2} = 5320$$

15. The first payment is $P = 7$ and each payment increases by $Q = 4$. The present value, one period before the first payment is

$$PV = \frac{P}{i} + \frac{Q}{i^2} = \frac{7}{i} + \frac{4}{i^2}$$

Since the valuation date for this problem is at the time of the first payment, we accumulate the above expression for one period. We get

$PV = 10557 = \left(\frac{7}{i} + \frac{4}{i^2}\right) \cdot (1+i)$

Multiplying this equation by i^2 to clear fractions give $10557i^2 = 7i(1+i) + 4(1+i)$ which simplifies to the quadratic $10550i^2 - 11i - 4 = 0$. Solving, (and ignoring the extraneous solution) we get $i = 0.02$.

Section 7: Advanced Upper m and Continuous Annuities

Although technically on the syllabus, the material in this section is very rarely tested, and when it is there are methods to solve problems that bypass the material in this section. The examples at the end of the section will illustrate this point. We will present the material here for completeness, but if pressed for time, the material in this section can be safely omitted from your studies.

Advanced Upper m Annuities

By advanced upper m notation, we mean combining upper m annuity notation with arithmetically increasing/decreasing annuity notation to get symbols like

$(Ia)_{\overline{n}|i}^{(m)}$ $\quad(I\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad(Is)_{\overline{n}|i}^{(m)}$ $\quad(I\ddot{s})_{\overline{n}|i}^{(m)}$ $\quad(Da)_{\overline{n}|i}^{(m)}$ $\quad(D\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad(Ds)_{\overline{n}|i}^{(m)}$ $\quad(D\ddot{s})_{\overline{n}|i}^{(m)}$

$(I^{(m)}a)_{\overline{n}|i}^{(m)}$ $\quad(I^{(m)}\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad(I^{(m)}s)_{\overline{n}|i}^{(m)}$ $\quad(I^{(m)}\ddot{s})_{\overline{n}|i}^{(m)}$ $\quad(D^{(m)}a)_{\overline{n}|i}^{(m)}$ $\quad(D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)}$

$(D^{(m)}s)_{\overline{n}|i}^{(m)}$ $\quad(D^{(m)}\ddot{s})_{\overline{n}|i}^{(m)}$

Instead of examining these symbols separately and deriving VEP and CRF's for each of the symbols, let's discuss the notation. Just as with basic upper m notation, the (m) above the symbol $\overline{n}|i$ represents the fact that there are m payments per (interest conversion) period. In order to simplify matters, but without loss of generality, let's assume that i is an aeir, and so the period is in years. Now, the symbol I represents the fact that the payments are increasing each year, starting with a payment of 1 for the first year, then 2 for the second year, and so on. Putting both these symbols together then implies there are m payment during the first year totaling 1, m payments during the second year totaling 2, and so on. That is, the payments during the first year are level at $1/m$ each, the payments during the second year are level at $2/m$ each, and so on. Then a symbol like $(Ia)_{\overline{n}|i}^{(m)}$ represents the present value of this payment stream with a valuation date being one payment period before the first payment. The CRF's for these advanced upper m symbols are obtained from the CRF's for the corresponding basic upper m symbols by replacing the periodic rate in the denominator by it's equivalent "upper m" nominal rate. For example, since $(D\ddot{a})_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{d}$ then $(D\ddot{a})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}}{d^{(m)}}$ Likewise for the others CRF's.

Now consider placing the (m) next to the I or D, getting $I^{(m)}$ or $D^{(m)}$. This implies that the increase, or decrease, takes place m times per year also. The upper m next to the I or D is always accompanied by the upper m over the $\overline{n}|i$. That is, you will not see symbols such as $(I^{(2)}s)_{\overline{n}|i}^{(4)}$. For these "double upper m"annuities, the payments are made m times per year and the increase is made m times per year. That is, the increase occurs with each payment. For the basic increasing double upper m annuity, the first payment is $\frac{1}{m^2}$, the second payment is $\frac{2}{m^2}$, and so on.

The CRF's for these double upper m annuities are obtained from the CRF's for the corresponding advanced upper m annuities above by replacing the annuity symbol in the numerator by its corresponding upper m annuity symbol. For example, since $(Is)_{\overline{n}|i}^{(m)} = \frac{\ddot{s}_{\overline{n}|}-n}{i^{(m)}}$ then $(I^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{\ddot{s}_{\overline{n}|}^{(m)}-n}{i^{(m)}}$. Similarly, since $(D\ddot{a})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}}{d^{(m)}}$ then $(D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}^{(m)}}{d^{(m)}}$. Likewise for the other CRF's. Although these are the CRF's for double upper m annuities, there is an easier way to determine the value for these symbols. If we factor out the first term in the VEP formulas, then the second factor will just be a basic increasing or decreasing annuity with mn payments, although we must adjust the periodic effective interest rate to match the period of the payments. For example, for the two formulas above, we have $(I^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{1}{m^2}(Is)_{\overline{mn}|j}$ and $(D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)} = \frac{1}{m^2}(D\ddot{a})_{\overline{mn}|j}$ where j is the periodic effective interest rate and where there are m periods per year.

Advanced Continuous Annuities

The annuities discussed thus far are called discrete annuities since the payments are made at discrete times during the year. We get advanced continuous annuities by taking a limiting process, as $m \to \infty$, of advanced upper m annuities. Note that as $m \to \infty$, then $i^{(m)}$ and $d^{(m)} \to \delta$, and with annuity symbols, upper ∞ is replaced by using a "bar". This allows us to derive advanced continuous annuity CRF's from advanced upper m CRF's. For example,

$$(Ds)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}}{i^{(m)}} \quad \Longrightarrow \quad (D\bar{s})_{\overline{n}|i} = \lim_{m\to\infty}(Ds)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}}{\delta}$$

$$(D^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - s_{\overline{n}|}^{(m)}}{i^{(m)}} \quad \Longrightarrow \quad (\bar{D}\bar{s})_{\overline{n}|i} = \lim_{m\to\infty}(D^{(m)}s)_{\overline{n}|i}^{(m)} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}$$

Let's discuss the difference between the continuous annuities that are being valued with the two equations on the right hand side of the implications above. The continuous annuity corresponding to the symbols with a "bar" over the a or s, *and* a "bar" over the I or D, is an annuity in which the payments are made continuously, and the increase or decrease is made continuously. For example, the symbol $(\bar{I}\bar{a})_{\overline{n}|i}$ represents the present value of an annuity in which payments are made continuously and the payment rate at time t is equal to t. Technically, we say that the payment rate function is $f(t) = t$, which means we can approximate the amount the annuity pays from time t to time $t + \Delta t$ by the expression $t \cdot \Delta t$. Thus we have $(\bar{I}\bar{a})_{\overline{n}|i} \overset{VEP}{=} \int_0^n t \cdot v^t \, dt$ and $(\bar{I}\bar{s})_{\overline{n}|i} \overset{VEP}{=} \int_0^n t \cdot e^{\delta(n-t)} \, dt$. Notice we can factor the exponential in the integrand of the last integral as $e^{\delta n} \cdot e^{-\delta t} = e^{\delta n} \cdot v^t$. Since $e^{\delta n} = (1+i)^n$ is constant with respect to t, we can factor it out. The result is that $(\bar{I}\bar{s})_{\overline{n|}i} = (1+i)^n \cdot \int_0^n t \cdot v^t \, dt = (1+i)^n \cdot (\bar{I}\bar{a})_{\overline{n|}i}$ which is what we already know; namely, that $AV = PV \cdot (1+i)^n$.

The continuous annuity corresponding to the symbols with a "bar" over the a or s, but no "bar" over the I or D, is an annuity in which the payments are made continuously, but the increase or decrease is made discretely. For example, the symbol $(I\bar{a})_{\overline{n|}i}$ represents the present value of an annuity in which payments are made continuously and total 1 during the first year, 2 during the second year, etc. So the payment rate function is the piecewise-defined function

$$f(t) = \begin{cases} 1 & \text{if } 0 < t \leq 1 \\ 2 & \text{if } 1 < t \leq 2 \\ & \vdots \\ n & \text{if } n-1 < t < n \end{cases}$$

Since the payment rate function is defined as such, the VEP (integral) expression for such annuities will actually be a sum of n integrals. For example, we have

$$(I\bar{a})_{\overline{n|}i} \overset{VEP}{=} \int_0^1 1 \cdot v^t \, dt + \int_1^2 2 \cdot v^t \, dt + \cdots + \int_{n-1}^n n \cdot v^t \, dt$$

General Payment Rate Functions

We generalize the payment rate function to a general $f(t)$ to get present values and accumulated values of general continuous annuities. There is no actuarial notation in this general form, and so we just write

$$PV \overset{VEP}{=} \int_0^n f(t) \cdot v^t \, dt$$
$$AV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\delta(n-t)} \, dt$$

General Payment Rate Functions and General Force of Interest

Finally, the most general situation is if we have a general payment rate function $f(t)$ and a general force of interest δ_t (as opposed to a constant force of interest δ). Then

$$PV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\int_t^0 \delta_s ds} dt$$

$$AV \overset{VEP}{=} \int_0^n f(t) \cdot e^{\int_t^n \delta_s ds} dt$$

Module 2 Section 7 Problems:

1. Determine the present value of a 10-year annuity immediate with level monthly payments of 2 for the first year, level monthly payments of 4 for the second year, level monthly payments of 6 for the third year, and so on. Use an annual effective interest rate of 5%.

2. Determine the accumulated value of a 5-year annuity due with level semiannual payments of 25 for the first year, level semiannual payments of 20 for the second year, level semiannual payments of 15 for the third year, and so on. Use an annual effective interest rate of 6%.

3. Determine the accumulated value of a 10-year annuity immediate with level monthly payments of 9 for the first year, level monthly payments of 11 for the second year, level monthly payments of 13 for the third year, and so on. Use an annual effective interest rate of 5%.

4. Determine the accumulated value of a 9-year annuity immediate with semiannual payments that start at 10 and increase each semiannual period by 4. Use an annual effective interest rate of 8.16%.

5. Determine the present value of a 15-year continuous annuity in which the payment rate during the first year is 1, the payment rate during the second year is 2, and so on. Use a force of interest of 3%.

6. Determine the accumulated value of a 20-year decreasing continuous annuity in which the payment rate at time t is $20 - t$. Use a force of interest of 4%.

7. Determine the accumulated value of a 3-year continuous annuity with payment rate function $f(t) = 9t^2$ where the force of interest at time t is given by $\delta_t = \frac{t^2}{9}$.

Solutions to Module 2 Section 7 Problems:

1. This is exactly #10 from Section 6, and if presented with this problem on an actuarial exam, you may wish to use the solution presented in Section 6. As an alternative, we can use advanced upper *m* notation. This annuity is a 10-year annuity immediate with payments made monthly, increasing arithmetically each year, to be valued using an annual effective interest rate of 5%. So the expression used to denote the present value of this annuity will include the symbol $(Ia)^{(12)}_{\overline{10|}.05}$ Recall that when using the symbol, $(Ia)^{(m)}_{\overline{n|}}$, the total annual payment for the first year is 1, and since with this annuity the total annual payment for the first year is 24, then we have

$$PV = 24(Ia)^{(12)}_{\overline{10|}.05} = 24\frac{\ddot{a}_{\overline{10|}.05} - 10v^{10}_{.05}}{i^{(12)}} = 966.4356$$

Note that $i^{(12)} = 12\left[(1.05)^{\frac{1}{12}} - 1\right]$.

2. This is exactly #11 from Section 6, and if presented with this problem on an actuarial exam, you may wish to use the solution presented in Section 6. As an alternative, we can use advanced upper *m* notation. This annuity is a 5-year annuity due with payments made semiannually, decreasing arithmetically each year, to be valued using an annual effective interest rate of 6%. So the expression used to denote the accumulated value of this annuity will include the symbol $(D\ddot{s})^{(2)}_{\overline{5|}.06}$ Recall that when using the symbol, $(D\ddot{s})^{(m)}_{\overline{n|}}$, the total annual payment for the first year is *n* (*n* = 5 in this problem) and since with this annuity the total annual payment for the first year is 50, then we have

$$AV = 10(D\ddot{s})^{(2)}_{\overline{5|}.06} = 10\frac{5(1.06)^5 - s_{\overline{5|}.06}}{d^{(2)}} = 183.5394$$

Note that $d^{(2)} = 2\left[1 - (1.06)^{\frac{-1}{2}}\right]$.

3. This is exactly #12 from Section 6, and if presented with this problem on an actuarial exam, you may wish to use the solution presented in Section 6. As an alternative, we can use advanced upper *m* notation. This annuity is a 10-year annuity immediate with payments made monthly, increasing arithmetically each year, to be valued using an annual effective interest rate of 5%. So the expression used to denote the accumulated value of this annuity will include the symbol $(Is)^{(12)}_{\overline{10|}.05}$ In this problem, the amount of the monthly increase each year is 2. Therefore, it is convenient to pull off a 7 from each of the payments, and think of each monthly payment during the first year as 2 + 7, each monthly payment during the second year as 4 + 7, and so on until.

Now think of the annuity as two different 10-year annuities immediate, one with monthly payments of 2 during the first year, monthly payments of 4 during the second year, and so on, and the other annuity being level with monthly payments of 7. The easiest way to value the latter level annuity is to convert the given 5% aeir to its equivalent meir and note that there are 120 payments of 7, just as we did in Section 6. Then $j = 1.05^{\frac{1}{12}} - 1$ and the accumulated value is $7s_{\overline{120}|j} = 1080.5421$.

The former arithmetically increasing annuity is exactly the annuity in #1 of this section. There we computed the present value of this annuity as 966.4356, and so the accumulated value of the annuity is $966.4356(1.05)^{10} = 1574.222$. Using upper m notation and the arguments in the solution to #1, we see that, symbolically, the accumulated value of this annuity is $24(Is)^{(12)}_{\overline{10}|.05}$. Then using the CRF, we again get

$$24(Is)^{(12)}_{\overline{10}|.05} = 24\frac{\ddot{s}_{\overline{10}|.05} - 10}{i^{(12)}} = 1574.222$$

Finally, the total accumulated value is $AV = 1080.5421 + 1574.222 = 2654.764$.

4. This is exactly #9 from Section 6. The payments are made semiannually and the increase is done semiannually. For these problems, the method in Section 6 is much easier than using advanced upper m notation. If presented with this problem on an actuarial exam, I suggest you use the method in the solution presented in Section 6.

As an alternative, we present advanced upper m notation here. Since the payments are made semiannually and the increase is done semiannually, this is a "double upper m" problem with $m = 2$. The expression for the accumulated value will include the symbol $(I^{(2)}s)^{(2)}_{\overline{9}|.0816}$. Since the amount of the increase is 4, similar to the previous problem it is convenient to pull off a 6 from each of the payments, just as we did in Section 6.

As with the previous problem, we think of the annuity as two different 9-year annuities immediate, one with semiannual payments starting at 4 and increasing by 4 each semiannual period, and the other annuity being level with semiannual payments of 6. The easiest way to value the latter level annuity is to convert the given 8.16% aeir to its equivalent seir, resulting in $j = .04$ (seir). Then note that there are 18 payments of 6 and so the accumulated value is $6s_{\overline{18}|.04} = 153.8725$.

The accumulated value of the former annuity is expressed using the double upper 2 symbol. Recall that the annuity that is valued by the standard symbol $(I^{(m)}s)^{(m)}_{\overline{n}|}$ has a first payment of $\frac{1}{m^2}$. With $m = 2$ this value is ¼. Since the first payment of the annuity here is 4, we need to multiply the standard symbol by 16, so that the first payment is 16(¼) = 4 as needed. Then the accumulated value of this annuity is

$$16(I^{(2)}s)^{(2)}_{\overline{9}|.0816} = 16\frac{\ddot{s}^{(2)}_{\overline{9}|.0816} - 9}{i^{(2)}} = 867.1229$$

Finally, the total accumulated value is $AV = 153.8725 + 867.1229 = 1020.995$.

5. Since we're determining the present value of a continuous annuity, we will be using $\bar{a}$ notation. The payment rates are increasing discretely (that is, 1 per year for the first year, 2 per year for the second year, etc.) and so we add the I notation. Therefore, the present value of this annuity will be denoted by the symbol $(I\bar{a})_{\overline{15}|}$. The CRF for $(I\bar{a})_{\overline{15}|}$ is obtained as follows: the CRF for $(Ia)_{\overline{15}|}$ is

$$(Ia)_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15v^{15}}{i}$$

Then $(Ia)^{(m)}_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15v^{15}}{i^{(m)}}$. Since $(I\bar{a})_{\overline{15}|} = \lim_{m\to\infty}(Ia)^{(m)}_{\overline{15}|}$ then

$$(I\bar{a})_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15v^{15}}{\delta}$$

We calculate the numerator above by first determining the equivalent aeir to $\delta = .03$ and then using the TVM shortcut previously discussed. The equivalent aeir is $i = e^{\delta} - 1 = e^{.03} - 1$. We conclude $(I\bar{a})_{\overline{15}|} = 89.89103928$.

6. Since we're determining the accumulated value of a continuous annuity, we will be using $\bar{s}$ notation. The payment rate function is the continuous decreasing function $f(t) = 20 - t$, which is the payment rate function associated to the $\bar{D}$ symbol. Therefore, the accumulated value of this annuity will be denoted by the symbol $(\bar{D}\bar{s})_{\overline{20}|}$. We can get the value of $(\bar{D}\bar{s})_{\overline{20}|}$ as follows: the CRF for $(Da)_{\overline{20}|}$ is

$$(Da)_{\overline{20}|} = \frac{20 - a_{\overline{20}|}}{i}$$

Then we have $(Da)^{(m)}_{\overline{20}|} = \frac{20 - a_{\overline{20}|}}{i^{(m)}}$ and then $\left(D^{(m)}a\right)^{(m)}_{\overline{20}|} = \frac{20 - a^{(m)}_{\overline{20}|}}{i^{(m)}}$
Since $(\bar{D}\bar{a})_{\overline{20}|} = \lim_{m\to\infty}\left(D^{(m)}a\right)^{(m)}_{\overline{20}|}$ then

$$(\bar{D}\bar{a})_{\overline{20}|} = \frac{20 - \bar{a}_{\overline{20}|}}{\delta}$$

With $\delta = .03$, we have $\bar{a}_{\overline{20}|} = \frac{1 - v^{20}}{\delta} = \frac{1 - e^{-.03(20)}}{.03}$ and $(\bar{D}\bar{a})_{\overline{20}|} = 165.3462623$. Then

$$(\bar{D}\bar{s})_{\overline{20}|} = (\bar{D}\bar{a})_{\overline{20}|} \cdot (1+i)^{20} = (\bar{D}\bar{a})_{\overline{20}|} \cdot e^{.03(20)} = 301.2805332$$

7. We have

$$AV = \int_0^3 f(t)e^{\int_t^3 \delta_s ds}dt = \int_0^3 9t^2 e^{\int_t^3 (\frac{1}{9}s^2)ds}dt$$

The exponential in the integrand reduces to $e^{\frac{s^3}{27}|_t^3} = e^{1-\frac{t^3}{27}}$ We can use the substitution $u = 1 - \frac{t^3}{27}$ to complete the integration. The result is

$$AV = \int_0^3 9t^2 e^{1-\frac{t^3}{27}}dt = 81(e-1)$$

Section 8: Summary of Annuity Formulas

You have now been baptized into the world of actuarial notation. Consider all the annuity symbols we've seen and/or discussed in this module:

$a_{\overline{n}|i}$ $\quad \ddot{a}_{\overline{n}|i}$ $\quad s_{\overline{n}|i}$ $\quad \ddot{s}_{\overline{n}|i}$ $\quad a_{\overline{n}|i}^{(m)}$ $\quad \ddot{a}_{\overline{n}|i}^{(m)}$ $\quad s_{\overline{n}|i}^{(m)}$ $\quad \ddot{s}_{\overline{n}|i}^{(m)}$

$(Ia)_{\overline{n}|i}$ $\quad (I\ddot{a})_{\overline{n}|i}$ $\quad (Is)_{\overline{n}|i}$ $\quad (I\ddot{s})_{\overline{n}|i}$ $\quad (Da)_{\overline{n}|i}$ $\quad (D\ddot{a})_{\overline{n}|i}$ $\quad (Ds)_{\overline{n}|i}$ $\quad (D\ddot{s})_{\overline{n}|i}$

$(Ia)_{\overline{n}|i}^{(m)}$ $\quad (I\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad (Is)_{\overline{n}|i}^{(m)}$ $\quad (I\ddot{s})_{\overline{n}|i}^{(m)}$ $\quad (Da)_{\overline{n}|i}^{(m)}$ $\quad (D\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad (Ds)_{\overline{n}|i}^{(m)}$ $\quad (D\ddot{s})_{\overline{n}|i}^{(m)}$

$(I^{(m)}a)_{\overline{n}|i}^{(m)}$ $\quad (I^{(m)}\ddot{a})_{\overline{n}|i}^{(m)}$ $\quad (I^{(m)}s)_{\overline{n}|i}^{(m)}$ $\quad (I^{(m)}\ddot{s})_{\overline{n}|i}^{(m)}$ $\quad (D^{(m)}a)_{\overline{n}|i}^{(m)}$ $\quad (D^{(m)}\ddot{a})_{\overline{n}|i}^{(m)}$

$(D^{(m)}s)_{\overline{n}|i}^{(m)}$ $\quad (D^{(m)}\ddot{s})_{\overline{n}|i}^{(m)}$ $\quad \bar{a}_{\overline{n}|i}$ $\quad \bar{s}_{\overline{n}|i}$ $\quad (I\bar{a})_{\overline{n}|i}$ $\quad (I\bar{s})_{\overline{n}|i}$ $\quad (\bar{I}\bar{a})_{\overline{n}|i}$ $\quad (\bar{I}\bar{a})_{\overline{n}|i}$

$(D\bar{a})_{\overline{n}|i}$ $\quad (D\bar{s})_{\overline{n}|i}$ $\quad (\bar{D}\bar{a})_{\overline{n}|i}$ $\quad (\bar{D}\bar{a})_{\overline{n}|i}$

For the OVERWHELMING majority of problems you'll see on the exam, you will only need the first four basic annuity symbols in the first row, and the eight basic increasing/decreasing annuity symbols in the second row (12 total symbols). On rare occasions you may see the four basic upper *m* annuity symbols in the first row, and possibly the two basic level continuous annuity symbols in the last row (another 6 symbols, for a total of 18). It is very unlikely that you'll see any of the other symbols and, although the material is on the syllabus for the exam, you may safely omit these from your studies.

For each of the 18 annuity symbols to be focused on from the last paragraph, you should know what payments are being valued and the location of the valuation date. With this knowledge, the VEP formulas should be intuitive. For CRF's, note that all the other CRF's can be easily derived from the following three basic CRF's as discussed below.

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|}-nv^n}{i} \qquad (Da)_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{i}$$

We derive the other CRF's by knowing how to "accumulate" and "decorate" the above three formulas. For example, if we want *s* instead of *a* then we "accumulate" the above formula by multiplying by $(1+i)^n$. If we want double-dot, then we replace the periodic effective interest rate *i* in the denominator by its equivalent periodic effective discount rate *d*. If we want upper *m* then we replace the periodic effective rate in the denominator with its equivalent upper *m* nominal rate. For example, from the basic *Da* formula, we have $(D\ddot{s})_{\overline{n}|i}^{(m)} = \frac{n-a_{\overline{n}|}}{d^{(m)}} \cdot (1+i)^n = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{d^{(m)}}$ (although this is one I said you can omit!)

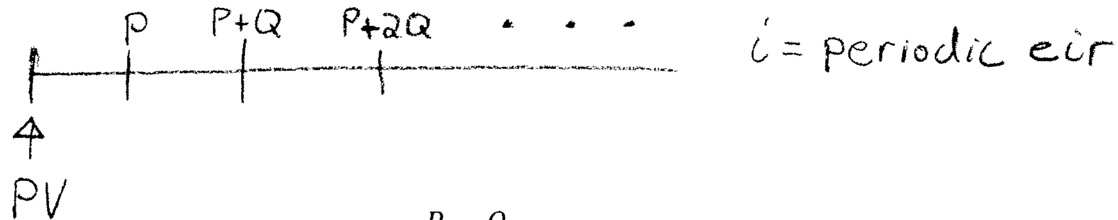
Perpetuities

Note that there is no such thing as the accumulated value of a perpetuity (think about it). Except for arithmetic perpetuities, the present values of all the perpetuities in this module are convergent geometric series. Therefore we can use the following general fact, where r represents the common ratio of the geometric series:

$$PV = \frac{\text{First Term}}{1 - r}$$

This is the basic fact used to derive CRF's such as $a_{\overline{\infty}|} = \frac{1}{i}$ and $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$ and others.

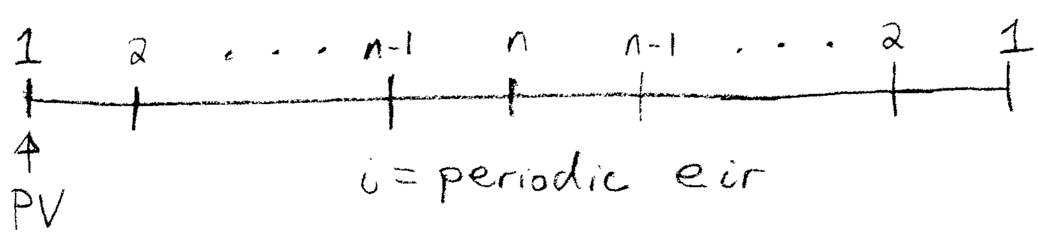
For arithmetic perpetuities, it is worthwhile to commit to memory the following formula for the present value of a perpetuity with initial payment of P and common difference of Q, where the valuation date is one payment period before the first payment:



$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

If the valuation date is not one period before the first payment, then multiply this expression by the appropriate accumulation or discount factor to get to the valuation date.

Finally, there is one more formula that is worth committing to memory; namely, the present value of a rainbow annuity due. We have



$$PV = (\ddot{a}_{\overline{n}|})^2$$