MAC 2313, Section 04 with Dr. Hurdal
Fall 2011 – Test 4

Name: SOLUTIONS

/100

As stated in class, you are allowed to bring to the test one 8.5x11 inch page, written on both sides. Calculators are allowed. Notebooks and textbooks are NOT allowed. Marks will be allocated for clear and well written mathematics solutions. This test will be graded out of 100.

Like 15.9 #15

1. (20 marks) Use the transformation $T: x = u,\ y = \dfrac{v}{u}$ to evaluate $\displaystyle\iint_R 2x^3y\,dA$ where R is the region in the first quadrant bounded by the lines $y = x$, $y = 4x$ and the hyperbolas $xy = 1$ and $xy = 5$.

$y = x \xrightarrow{T} \frac{v}{u} = u \Rightarrow v = u^2 \Rightarrow u = \sqrt{v}$ (2)

$y = 4x \xrightarrow{T} \frac{v}{u} = 4u \Rightarrow v = 4u^2 \Rightarrow u = \frac{\sqrt{v}}{2}$ (2)

$xy = 1 \xrightarrow{T} u\left(\frac{v}{u}\right) = 1 \Rightarrow v = 1$ (2)

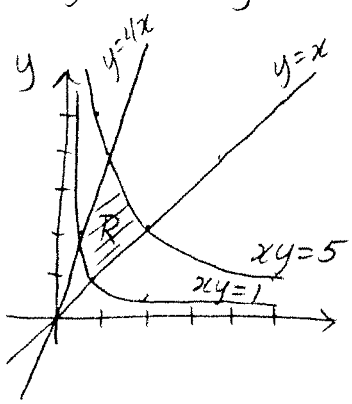
$xy = 5 \xrightarrow{T} u\left(\frac{v}{u}\right) = 5 \Rightarrow v = 5$ (2)

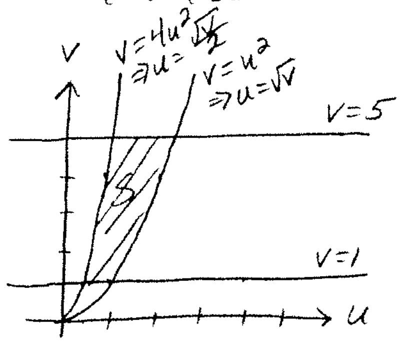
$f(x,y) = 2x^3y \xrightarrow{T} f(u,v) = 2(u)^3\left(\frac{v}{u}\right) = 2u^2v$ (2)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ (1) = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} \ (1) = \frac{1}{u} \ (1)$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{u}\right| = \frac{1}{u}$$

Since u is in 1st quadrant





Note: sketches of R + S not needed

$$\iint_R 2x^3y\,dA = \iint_S f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \ (1)$$

$$= \int_{v=1}^{5}\int_{u=\frac{\sqrt{v}}{2}}^{\sqrt{v}} \underbrace{(2u^2v)\left(\frac{1}{u}\right)}_{(1)} du\,dv \ (2)$$

$$= \int_1^5\int_{\frac{\sqrt{v}}{2}}^{\sqrt{v}} 2uv\,du\,dv$$

$$= \int_1^5 u^2v\Big|_{\frac{\sqrt{v}}{2}}^{\sqrt{v}} dv$$

$$= \int_1^5 v\left(v - \frac{v}{4}\right) dv$$

$$= \int_1^5 \frac{3v^2}{4}\,dv \quad (2)$$

$$= \frac{v^3}{4}\Big|_1^5$$

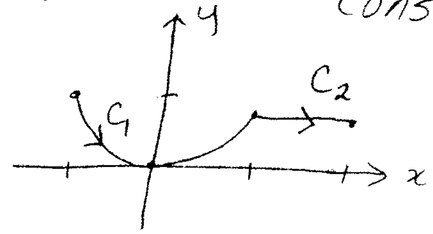
$$= \frac{125}{4} - \frac{1}{4}$$

$$= 31 \quad (1)$$

Like 16.2 #41 16.2 #15

2. (20 marks) Find the work done by the force field $\mathbf{F}(x,y) = \langle xy^2, -x^2 \rangle$ over the curve C where C consists of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and the line segment from $(1,1)$ to $(2,1)$.

$$\text{curl } \vec{F} = \frac{\partial(xy^2)}{\partial y} - \frac{\partial(-x^2)}{\partial x} = 2xy + 2x \neq 0 \quad \therefore \vec{F} \text{ not conservative} \quad (+1B)$$



$$W = \int_C \vec{F}\cdot d\vec{r} \quad (1)$$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} \quad (1)$$

$$= \int_{-1}^{1} \langle t^5, -t^2 \rangle \cdot \langle 1, 2t \rangle \, dt \quad (1)$$

$$+ \int_0^1 \langle 1+t, -(1+t)^2 \rangle \cdot \langle 1, 0 \rangle \, dt \quad (1)$$

$$= \int_{-1}^{1} (t^5 - 2t^3)\,dt \; (1) + \int_0^1 (1+t)\,dt \; (1)$$

$$= \left(\frac{t^6}{6} - \frac{t^4}{4}\right)\Bigg|_{-1}^{1} + \left(t + \frac{t^2}{2}\right)\Bigg|_0^1$$

$$= \left[\left(\frac{1}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right)\right] + \left[\left(1 + \frac{1}{2}\right) - 0\right]$$

$$= 0 \; (1) + \frac{3}{2} \; (1)$$

$$= \frac{3}{2}$$

$C_1: y = x^2$ from $(-1,1)$ to $(1,1)$

$x = t \quad y = t^2 \quad (2)$

$\vec{r}(t) = \langle t, t^2 \rangle \quad -1 \le t \le 1 \quad (1)$

$d\vec{r} = \langle 1, 2t \rangle \, dt \quad (1)$

$\vec{F}(\vec{r}(t)) = \langle t(t^2)^2, -t^2 \rangle$

$= \langle t^5, -t^2 \rangle \quad (2)$

C_2: line segment from $(1,1)$ to $(2,1)$

$\vec{r}(t) = \langle 1, 1 \rangle + t\langle 2-1, 1-1 \rangle$

$= \langle 1+t, 1 \rangle \; (2) \quad 0 \le t \le 1 \quad (1)$

$d\vec{r} = \langle 1, 0 \rangle \, dt \quad (1)$

$\vec{F}(\vec{r}(t)) = \langle 1+t, -(1+t)^2 \rangle \quad (2)$

Alternate parameterization for C_2:

$x = t \quad y = 1$

$\vec{r}(t) = \langle t, 1 \rangle \quad 1 \le t \le 2$

$d\vec{r} = \langle 1, 0 \rangle \, dt$

$\vec{F}(\vec{r}(t)) = \langle t(1)^2, -t^2 \rangle = \langle t, -t^2 \rangle$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^2 \langle t, -t^2 \rangle \cdot \langle 1, 0 \rangle \, dt$$

$$= \int_1^2 t \, dt = \frac{t^2}{2}\Bigg|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

9 marks for C_1 + $\int_{C_1} \vec{F}\cdot d\vec{r}$

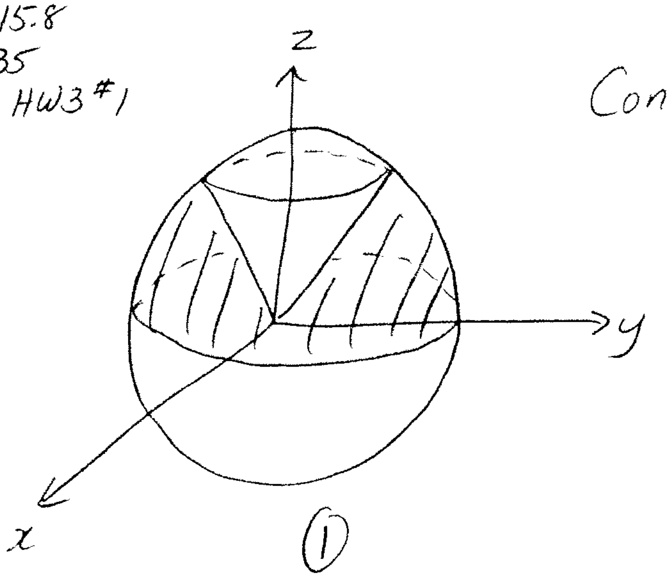
9 marks for C_2 + $\int_{C_2} \vec{F}\cdot d\vec{r}$

15.8 #30,

3. (20 marks) Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Like 15.8 #35

Like HW3 #1



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Cone: $z = \sqrt{x^2+y^2}$

$\rho\cos\phi = \sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi}$

$= \sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$ 2

$\rho\cos\phi = \rho\sin\phi$ 1

$\Rightarrow \tan\phi = 1$

$\phi = \frac{\pi}{4}$ 1

Above xy-plane $\Rightarrow \phi \le \frac{\pi}{2}$ 1

$\therefore \frac{\pi}{4} \le \phi \le \frac{\pi}{2}$

$V = \iiint dV$ 1

2 R 2 2 1

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{\rho^3}{3}\sin\phi \Big|_0^2 \, d\phi \, d\theta \quad 1$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{8}{3}\sin\phi \, d\phi \, d\theta \quad 1$$

$$= \int_0^{2\pi} -\frac{8}{3}\cos\phi \Big|_{\pi/4}^{\pi/2} d\theta \quad 1$$

$$= \int_0^{2\pi} -\frac{8}{3}\left(0 - \frac{1}{\sqrt{2}}\right) d\theta \quad 1$$

$$= \frac{8}{3\sqrt{2}}\theta \Big|_0^{2\pi} \quad 1$$

$$= \frac{16\pi}{3\sqrt{2}} = \frac{8\sqrt{2}\pi}{3} \quad 1$$

16.1 pg1028 Like 16.2 #17,18

11 4. (20 marks) Consider the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$.
a) Sketch $\mathbf{F}$.

2 b) On your sketch draw a curve C_1 such that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} < 0$.

3 c) If C_2 is a circle of radius 1 centered at the origin and oriented counter clockwise, is $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ positive, negative or zero? Explain.

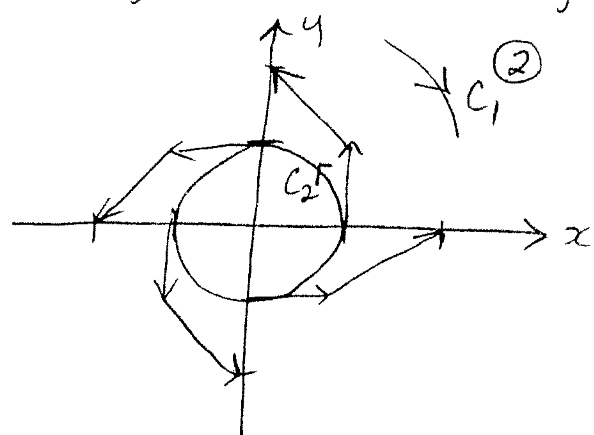
4 d) Using your result from part c) and without using the curl test, do you think $\mathbf{F}$ is a conservative vector field? Explain.

a)

(x,y)	$\vec{F} = \langle -y, x\rangle$
$(1,0)$	$\langle 0,1\rangle$
$(-1,0)$	$\langle 0,-1\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(0,-1)$	$\langle 1,0\rangle$
$(1,1)$	$\langle -1,1\rangle$
$(-1,1)$	$\langle -1,-1\rangle$
$(-1,-1)$	$\langle 1,-1\rangle$
$(1,-1)$	$\langle 1,1\rangle$
$(0,0)$	$\langle 0,0\rangle$

$|\vec{F}| = \sqrt{y^2+x^2}$
$\Rightarrow |\vec{F}|^2 = x^2+y^2$ $\Rightarrow$ All vectors on the same radius circle have the same magnitude.

11

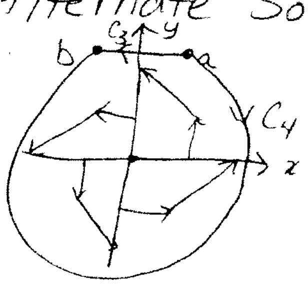


b) Since C_1 is going against $\vec{F}$, $\int_{C_1} \vec{F}\cdot d\vec{r} < 0$.

c) Since C_2 is going in the same direction as $\vec{F}$ everywhere,
(3) $\int_{C_2} \vec{F}\cdot d\vec{r} > 0$.

d) If $\vec{F}$ were a conservative/path independent vector field,
(4) then for any closed curve C, $\int_C \vec{F}\cdot d\vec{r} = 0$. Since C_2 is a closed curve and $\int_{C_2} \vec{F}\cdot d\vec{r} > 0$, $\therefore$ $\vec{F}$ is not conservative.

Alternate Solution:



Consider the two paths C_3 + C_4 from a to b.
$\int_{C_3} \vec{F}\cdot d\vec{r} > 0$ as C_3 in same direction as $\vec{F}$.
$\int_{C_4} \vec{F}\cdot d\vec{r} < 0$ as C_4 against $\vec{F}$. Two paths from a to b give different results $\Rightarrow$ $\vec{F}$ is path dependent.

16.3 #3 $\begin{cases} 4 \\ 5 \end{cases}$ 5. (20 marks) a) Show that $\mathbf{F}(x,y) = \langle 2x - 3y, -3x + 4y - 8 \rangle$ is a conservative vector field.

b) Find a potential function f for $\mathbf{F}$.

Like 4+7, 16.3 #13, 16.2 #41 c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ two different ways where C is the portion of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$.

a) $\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x} \overset{(1)}{=} \dfrac{\partial(2x-3y)}{\partial x} - \dfrac{\partial(-3x+4y-8)}{\partial x} = \overset{(1)}{-3} - \overset{(1)}{(-3)} = 0$ (1)

Since curl $\vec{F} = 0$, $\therefore \vec{F}$ is conservative

b) $f_x = P = 2x - 3y$ (1) $\qquad f_y = Q = -3x + 4y - 8$

$f(x,y) = x^2 - 3xy + g(y)$ (1)

$f_y = -3x + g'(y)$ (1)

$-3x + g'(y) = -3x + 4y - 8$

$g'(y) = 4y - 8$ (1)

$g(y) = 2y^2 - 8y + C$

$\therefore f(x,y) = x^2 - 3xy + 2y^2 - 8y + C$ (1)

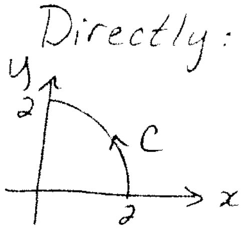
c) $\int_C \vec{F} \cdot d\vec{r} = \overset{(1)}{f(0,2)} - \overset{(1)}{f(2,0)}$ by F.T. of L.I.

$= (0 - 0 + 2 \cdot 4 - 8 \cdot 2 + C) - (4 - 0 + 0 - 0 + C)$

$= 8 - 16 - 4$ (1)

$= -12$ (1)

Directly:



(1) $x = 2\cos t$ $\quad 0 \le t \le \frac{\pi}{2}$ (1)

(1) $y = 2\sin t$

$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

(1) $d\vec{r} = \langle -2\sin t, 2\cos t \rangle dt$

(1) $\vec{F}(\vec{r}(t)) = \langle 4\cos t - 6\sin t, -6\cos t + 8\sin t - 8 \rangle$

$\int_C \vec{F} \cdot d\vec{r}$

$= \int_0^{\pi/2} \langle 4\cos t - 6\sin t, -6\cos t + 8\sin t - 8 \rangle \cdot \langle -2\sin t, 2\cos t \rangle dt$

$= \int_0^{\pi/2} (-8\sin t\cos t + 12\sin^2 t - 12\cos^2 t + 16\sin t\cos t - 16\cos t) dt$

$= \int_0^{\pi/2} \left(8\sin t\cos t - 12\underbrace{(\cos^2 t - \sin^2 t)}_{\cos 2t} - 16\cos t\right) dt$ (1)

$= \left(4\sin^2 t - \dfrac{12\sin 2t}{2} - 16\sin t\right)\Big|_0^{\pi/2}$

$= (4 - 6 \cdot 0 - 16) - 0$

$= -12$ (1)

15.9 pg 1019
16.5 pg 1064

Bonus:

A) (5 marks) Derive the "conversion" factor for spherical coordinates.

B) (5 marks) If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q and R all exist, then the curl of $\mathbf{F}$ is defined as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}.$$

Recall that $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Show that the curl test holds in $\mathbb{R}^3$. That is, show that if $\mathbf{F} = \nabla f$ then curl $\mathbf{F} = \mathbf{0}$.

A) $x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \quad 2$$

$$= \cos\theta\sin\phi(-\rho^2\cos\theta\sin^2\phi)$$
$$+ \rho\sin\theta\sin\phi(-\rho\sin\theta\sin^2\phi - \rho\sin\theta\cos^2\phi)$$
$$+ \rho\cos\theta\cos\phi(0 - \rho\cos\theta\sin\phi\cos\phi)$$

$$= -\rho^2\cos^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin\phi - \rho^2\cos^2\theta\sin\phi\cos^2\phi$$
$$= -\rho^2\cos^2\theta\sin\phi(\sin^2\phi + \cos^2\phi) - \rho^2\sin^2\theta\sin\phi \quad 2$$
$$= -\rho^2\sin\phi(\cos^2\theta + \sin^2\theta)$$
$$= -\rho^2\sin\phi$$

"Conversion Factor" $= \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = |-\rho^2\sin\phi| = \rho^2\sin\phi$ for $0 \le \phi \le \pi$ 1

B) $\text{curl } \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla f$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \quad 2$$

$$= \left\langle \frac{\partial^2 f}{\partial z\partial y} - \frac{\partial^2 f}{\partial y\partial z}, -\left(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\right), \frac{\partial^2 f}{\partial y\partial x} - \frac{\partial^2 f}{\partial x\partial y} \right\rangle \quad 1$$

$$= \left\langle \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}, \frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}, \frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y} \right\rangle$$ 1 by Clairaut's Theorem

$$= \langle 0, 0, 0 \rangle$$

$$= \vec{0}. \quad 1$$