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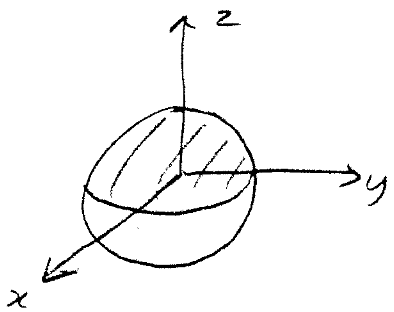
MAC 2313, Section 04 with Dr. Hurdal
Fall 2011 – Test 3

Name: SOLUTIONS

As stated in class, you are allowed to bring to the test one 8.5x11 inch page, written on both sides. Calculators are allowed. Notebooks and textbooks are NOT allowed. Marks will be allocated for clear and well written mathematics solutions. This test will be graded out of ~~100.~~ 92

Like 15.4 Ex#2,
like 15.4 #23

1. (10 marks) Use integration to find the volume bounded by the surface $z = \sqrt{4 - x^2 - y^2}$ and the plane $z = 0$.

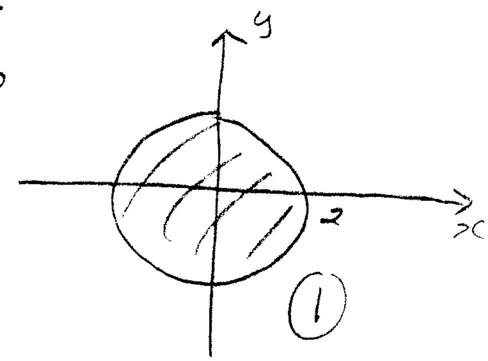


Region of Integration:

$z = \sqrt{4-x^2-y^2}$ & $z=0$

$\sqrt{4-x^2-y^2} = 0$

$\Rightarrow x^2+y^2=4$ (2)



$$V = \text{Volume of Top} - \text{Volume of Bottom}$$

$$= \iint_R \left(\sqrt{4-x^2-y^2} - 0\right) dA \quad (1)$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{4-r^2}\; r\,dr\,d\theta \quad (1)\,(1)\,(1)\,(1)$$

$$= \int_0^{2\pi} (4-r^2)^{\frac{3}{2}}\left(\frac{2}{3}\right)\left(-\frac{1}{2}\right)\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\left(0 - 4^{\frac{3}{2}}\right) d\theta$$

$$= \frac{8}{3}\int_0^{2\pi} d\theta \quad (1)$$

$$= \frac{8}{3}\theta \Big|_0^{2\pi}$$

$$= \frac{16\pi}{3} \quad (1)$$

Like 14.7 #11 15

Like 14.6 #21 [3 2]

2. (20 marks) For the function $f(x,y) = x^3 + y^3 + 3x^2 + 3y^2 - 8$,
a) find all critical points of $f(x,y)$ and use the second derivative test to classify them;
b) what is the maximum rate of change of $f(x,y)$ at the point $(1,-1)$;
c) in which direction does $f(x,y)$ change most rapidly?

a) $\nabla f = 0$ (1)

$f_x = 0$ $\qquad$ $f_y = 0$

$3x^2 + 6x = 0$ (1) $\qquad$ $3y^2 + 6y = 0$ (1)

$3x(x+2) = 0$ $\qquad$ $3y(y+2) = 0$

$x = 0$ (1) OR $x = -2$ (2) (1) $\qquad$ $y = 0$ (3) or $y = -2$ (4) (1)

(1) + (3)
(1) + (4)
(2) + (3)
(2) + (4)

Critical Points at
(0,0)
(0,-2)
(-2,0)
(-2,-2)
(4)

$D = f_{xx} f_{yy} - (f_{xy})^2$ (1)

$= (6x+6)(6y+6) - 0$ (1)

$= 36(x+1)(y+1)$

$f_{xx} = 6x+6$
$f_{yy} = 6y+6$
$f_{xy} = 0$

$D(0,0) = 36$ + $f_{xx}(0,0) = 6$ $\Rightarrow (0,0)$ is a local minimum
$D(0,-2) = -36$ $\Rightarrow (0,-2)$ is a saddle point
$D(-2,0) = -36$ $\Rightarrow (-2,0)$ is a saddle point
$D(-2,-2) = 36$ + $f_{xx}(-2,-2) = -6$ $\Rightarrow (-2,-2)$ is a local maximum
(4)

b) $\nabla f(x,y) = \langle 3x^2+6, 3y^2+6y \rangle$

$\nabla f(1,-1) = \langle 9, -3 \rangle$

(3) max rate of change at $(1,-1)$ $= |\nabla f(1,-1)| = \sqrt{81+9} = \sqrt{90} = 3\sqrt{10}$

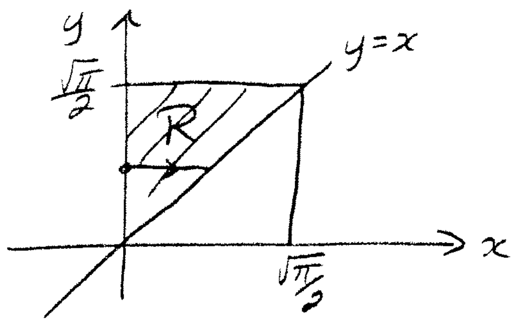
(2) c) direction of max rate of change $= \nabla f(x,y) = \langle 3x^2+6, 3y^2+6y \rangle$

Like 15.3 #45-50

3. (10 marks) Evaluate $\int_0^{\sqrt{\pi}/2}\int_x^{\sqrt{\pi}/2} 2y^2\sin(xy)\,dy\,dx$ by reversing the order of integration.

$x \le y \le \frac{\sqrt{\pi}}{2}$

$0 \le x \le \frac{\sqrt{\pi}}{2}$



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$$\overset{2}{\int_0^{\sqrt{\pi}/2}} \overset{2}{\int_0^{\sqrt{\pi}/2}} 2y^2\sin(xy)\,dy\,dx$$

$$= \int_0^{\sqrt{\pi}/2}\int_0^{y} 2y^2\sin(xy)\,dx\,dy$$

$$= \int_0^{\sqrt{\pi}/2} -\frac{2y^2\cos(xy)}{y}\Big|_0^{y}\,dy \quad 1$$

$$= \int_0^{\sqrt{\pi}/2} -2y(\cos y^2 - \cos 0)\,dy \quad 1$$

$$= \int_0^{\sqrt{\pi}/2} (-2y\cos y^2 + 2y)\,dy$$

$$= \left(-\sin y^2 + y^2\right)\Big|_0^{\sqrt{\pi}/2} \quad 1$$

$$= \left(-\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{\sqrt{2}}$$

$$= \frac{\pi - 2\sqrt{2}}{4} \quad 1$$

14.8 #29 4. (15 marks) Use Lagrange multipliers to find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Let the point be given by (x, y, z)

$d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$

$d^2 = (x-4)^2 + (y-2)^2 + z^2$ ①

Minimizing d is equivalent to minimizing d^2.

$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

$z^2 = x^2 + y^2$

$z^2 - x^2 - y^2 = 0$

① $g(x,y,z) = z^2 - x^2 - y^2 = 0$

$\nabla f = \lambda \nabla g$

$f_x = \lambda g_x$ ① $\quad f_y = \lambda g_y$ ① $\quad f_z = \lambda g_z$ ① $\quad g = c$

② $2(x-4) = \lambda(-2x)$ ① $\quad$ ② $2(y-2) = \lambda(-2y)$ ② $\quad$ ② $2z = \lambda(2z)$ ③ $\quad z^2 - x^2 - y^2 = 0$ ④

③ $\Rightarrow \lambda = 1$

Sub $\lambda = 1$ in ① + ③

$2(x-4) = -2x$

$4x - 8 = 0$

$x = 2$

$2(y-2) = -2y$

$4y - 4 = 0$

$y = 1$ ②

Sub $x = 2$ + $y = 1$ into ④

$z^2 - 4 - 1 = 0$

$z^2 = 5$

$z = \pm\sqrt{5}$

The points are given by $(2, 1, \sqrt{5})$ + $(2, 1, -\sqrt{5})$.
②

$d = \sqrt{(2-4)^2 + (1-2)^2 + (\pm\sqrt{5})^2}$

$= \sqrt{4 + 1 + 5}$

$= \sqrt{10}$

The distance is $\sqrt{10}$.

5. (20 marks) The temperature at a point (x, y) is given by $T(x,y) = \sqrt{x^2+3y^2}$ where T is measured in °C and x, y are in meters.

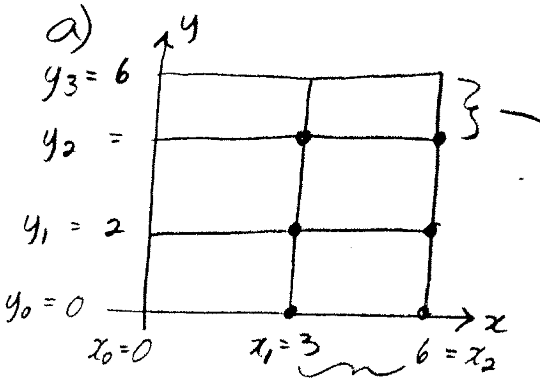
Like 15.1 9 #1, 4 a) Estimate the average temperature in the region $R = [0,6] \times [0,6]$ using sample points to be lower right corners with $m = 2$ and $n = 3$.

Like 14.4 #11 3 b) Where is $T(x, y)$ differentiable?

Like 14.6 #19 5 c) Find the rate of change of temperature at the point $(2, -2)$ in the direction toward the point $(3, 0)$.

Like 14.5 #7 3 d) If $x = e^s\cos(t)$ and $y = e^s\sin(t)$, use the chain rule to find $\dfrac{\partial T}{\partial t}$.

a)



$n = 3 \rightarrow \Delta y = \frac{6-0}{3} = 2$ (1)

$m = 2 \quad \Delta x = \frac{6-0}{2} = 3$ (1)

area $(R) = (6)(6) = 36$ (1)

$\Delta A = \Delta x \Delta y = (3)(2) = 6$ (1)

$$\text{Ave Temp} = \frac{1}{\text{area}(R)} \iint_R T(x,y)\,dA \quad (1)$$

$$\approx \frac{1}{36} \sum_{i=1}^{m=2} \sum_{j=1}^{n=3} T(x_i, y_j)\,\Delta A$$

$$= \frac{1}{36}\Big[T(3,0) + T(3,2) + T(3,4) \quad (3)$$
$$+ T(6,0) + T(6,2) + T(6,4)\Big](6)$$

$$= \frac{1}{6}\Big[3 + \sqrt{9+12} + \sqrt{9+48} + 6 + \sqrt{36+12} + \sqrt{36+48}\Big]$$

$$= \frac{1}{6}\left(9 + \sqrt{21} + \sqrt{57} + \sqrt{48} + \sqrt{84}\right)$$

$$= \frac{1}{6}(37.2258) = 6.2043 \quad (1)$$

b) $T(x,y)$ is differentiable at (a,b) if T_x + T_y exist near (a,b) and are continuous at (a,b)

$$T_x(x,y) = \frac{1}{2}(x^2+3y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+3y^2}} \quad (1)$$

$$T_y(x,y) = \frac{1}{2}(x^2+3y^2)^{-1/2}(6y) = \frac{3y}{\sqrt{x^2+3y^2}} \quad (1)$$

T_x + T_y exist and continuous everywhere except where $\sqrt{x^2+3y^2} = 0$ ie except @ $(0,0)$. $\therefore T(x,y)$ differentiable everywhere except at (1) $(0,0)$.

c) $\vec{d} = \langle 3-2, 0+2\rangle = \langle 1, 2\rangle$ (1)

$\vec{u}_d = \left\langle \frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right\rangle$ (1)

$$D_u T(2,-2) = \nabla T(2,-2) \cdot \vec{u}_d \quad (1)$$

$$= \left\langle \frac{2}{\sqrt{4+12}}, \frac{-6}{\sqrt{4+12}}\right\rangle \cdot \left\langle \frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right\rangle$$

$$= \left\langle \frac{1}{2}, -\frac{3}{2}\right\rangle \cdot \left\langle \frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right\rangle \quad (1)$$

$$= \frac{1}{2\sqrt5} - \frac{6}{2\sqrt5} = -\frac{5}{2\sqrt5} = -\frac{\sqrt5}{2} \quad (1)$$

d) T; x, y; s, t, s, t

$x = e^s\cos(t)$

$\frac{\partial x}{\partial t} = -e^s\sin(t)$

$y = e^s\sin(t)$

$\frac{\partial y}{\partial t} = e^s\cos(t)$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial t} \quad (1)$$

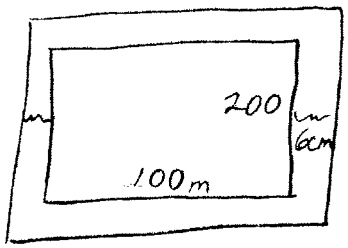
$$= \frac{x}{\sqrt{x^2+3y^2}}(-e^s\sin t) \quad (1)$$
$$+ \frac{3y}{\sqrt{x^2+3y^2}}\, e^s\cos t \quad (1)$$

$$= \frac{-e^{2s}\sin t\cos t + 3e^{2s}\sin t\cos t}{\sqrt{e^{2s}\cos^2 t + 3e^{2s}\sin^2 t}}$$

Bonus (1) $\rightarrow = \dfrac{2e^{2s}\sin t\cos t}{e^s\sqrt{\cos^2 t + 3\sin^2 t}} = \dfrac{2e^s\sin t\cos t}{\sqrt{1+2\sin^2 t}}$

14.4
#37

6. (10 marks) A boundary stripe 6 cm wide is painted around a rectangle whose dimensions are 100 m by 200 m. Use differentials to approximate the number of square meters of paint in the stripe. Warning: be careful with your units.



$l = 100\ m$ 1

$w = 200\ m$ 1

$dl = 2(6)\ cm$

$= 0.12\ m$ 1

$dw = 2(6)\ cm$

$= 0.12\ m$ 1

$A(l, w) = lw$

$dA = A_l\, dl + A_w\, dw$ 1

$= w\, dl + l\, dw$ 2

$= 200\,(0.12) + 100\,(0.12)$ 2

$= 24 + 12$

$= 36\ m^2$ 1

Like 15.5 #1, #10 7. (15 marks) If two streets forming the boundary a suburb D are given by the parabolas $8y = x^2$ and $x = y^2$, and the population density function of D is given by $\rho(x, y) = \sqrt{x}$ in thousands/km^2, find the total population of D.

Intersection of

$8y = x^2$ & $x = y^2$

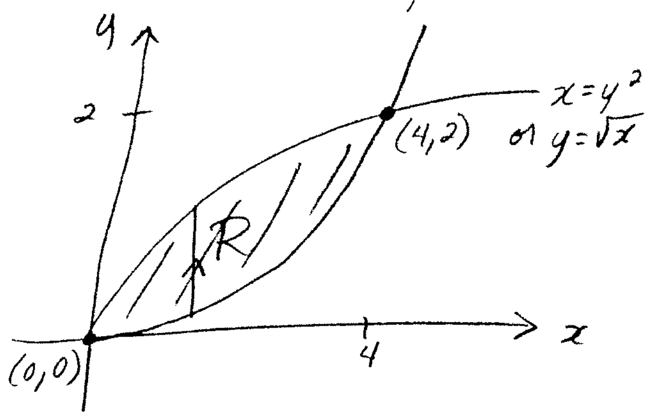
$8y = (y^2)^2$

$y^4 - 8y = 0$

$y(y^3 - 8) = 0$

$y = 0$ or $y = 2$ (2)

$\Rightarrow x = 0 \qquad x = 4$



(2)

$$\text{Total Population} = \iint_R \sqrt{x}\, dA$$

$$= \int_0^4 \int_{\frac{x^2}{8}}^{\sqrt{x}} \sqrt{x}\, dy\, dx \quad (2)\ (2)$$

$$= \int_0^4 \sqrt{x}\, y \Big|_{\frac{x^2}{8}}^{\sqrt{x}} dx \quad (1)$$

$$= \int_0^4 \left(x - \frac{x^{5/2}}{8}\right) dx \quad (2)$$

$$= \left[\frac{x^2}{2} - \frac{x^{7/2}}{8}\left(\frac{2}{7}\right)\right]_0^4 \quad (2)$$

$$= \left(8 - \frac{2^7}{4 \cdot 7}\right) - 0$$

$$= 8 - \frac{32}{7} \quad (2)$$

$$= \frac{24}{7}$$

$= 3.429$ thousands of people

≈ 3400 people

OR

$$= \int_0^2 \int_{y^2}^{\sqrt{8y}} \sqrt{x}\, dx\, dy$$

$$= \int_0^2 x^{3/2}\left(\frac{2}{3}\right)\Big|_{y^2}^{\sqrt{8y}} dy$$

$$= \int_0^2 \frac{2}{3}\left[\left((8y)^{1/2}\right)^{3/2} - y^3\right] dy$$

$$= \frac{2}{3}\int_0^2 \left((8y)^{3/4} - y^3\right) dy$$

$$= \frac{2}{3}\left[(8y)^{7/4}\left(\frac{4}{7}\right)\left(\frac{1}{8}\right) - \frac{y^4}{4}\right]_0^2$$

$$= \frac{2}{3}\left[2^7\left(\frac{4}{7}\right)\left(\frac{1}{8}\right) - \frac{16}{4}\right]$$

$$= \frac{2}{3}\left[\frac{2^6}{7} - 4\right]$$

$$= \frac{2}{3}\left[\frac{64 - 28}{7}\right]$$

$$= \frac{2}{3}\left(\frac{36}{7}\right)$$

$$= \frac{24}{7}$$

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#6

Bonus (10 marks): Assuming all the following integrals exist, the following integral property holds:

If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$mA(D) \leq \int\int_D f(x,y)\,dA \leq MA(D)$$

where $A(D) = \int\int_D dA$.

Use this property to determine if the statement

$$\int_1^4 \int_0^1 (x^2 + \sqrt{y}) \sin(x^2y^2)\,dx\,dy \leq 9$$

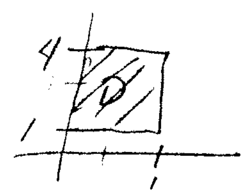
is true or false. If the statement is true, explain why. If it is false, explain why or give an example that disproves this statement.

Here, $f(x,y) = (x^2 + \sqrt{y}) \sin(x^2y^2)$

and D is the rectangle $0 \leq x \leq 1$, $1 \leq y \leq 4$

Note that $\sin(x^2y^2) \leq 1$ for all x, y.

Thus $f(x,y) \leq (x^2 + \sqrt{y})(1)$



Over D, $f(x,y) \leq (1 + \sqrt{4}) = 3 = M$

Also, $A(D) = \iint_D dA = \int_1^4 \int_0^1 dx\,dy = 3$

$\therefore \int_1^4 \int_0^1 (x^2 + \sqrt{y}) \sin(x^2y^2) \leq M\,A(D) = (3)(3) = 9$ as required.

$\therefore$ the statement is true.