MAC 2313, Section 04 with Dr. Hurdal

Fall 2011 – Test 2

Name: SOLUTIONS

As stated in class, you are allowed to bring to the test one 8.5x11 inch page, written on both sides. Calculators are allowed. Notebooks and textbooks are NOT allowed. Marks will be allocated for clear and well written mathematics solutions. This test will be graded out of 100.

1. (15 marks) Consider the function $f(x, y) = \dfrac{y^2}{x^2 + y^2}$.

like 14.1#9, Ch 14 Ex 1 — 2+2 a) What is the domain and range of $f(x, y)$?

like 14.2#29 — 2 b) Where is $f(x, y)$ continuous?

Ch 14.2 Ex 6, like 14.2#9 — 5 c) Find $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

like 14.3#21 — 4 d) Compute $f_x(x, y)$.

a) Domain: all points except where denominator $= 0$

$x^2 + y^2 = 0 \Rightarrow (x, y) = (0, 0)$

$D = \{(x,y) \mid x \neq y \neq 0\}$ or $D = \{(x,y) \mid (x,y) \neq (0,0)\}$

Range: all values for $f(x,y)$. Observe $f(x,y) \geq 0$ and $f(x,y) \leq 1$

$R = \{f(x,y) \mid 0 \leq f(x,y) \leq 1\}$ OR $R: f(x,y) \in [0, 1]$

Full credit if answer is $z \in [0, \infty)$

b) $f(x,y)$ continuous where function defined.

Continuous everywhere except at $(x,y) = (0,0)$.

c) Limit along x-axis $\Rightarrow y = 0$

$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{0}{x^2 + 0} = 0$$

Limit along y-axis $\Rightarrow x = 0$

$$\lim_{(x,y)\to(0,0)} f(0,y) = \lim_{(x,y)\to(0,0)} \frac{y^2}{0 + y^2} = 1$$

Since the limit has 2 different values along 2 different paths, $\therefore$ limit does not exist

d) $f_x(x,y) = -y^2(x^2+y^2)^{-2}(2x)$

$$= -\frac{2xy^2}{(x^2+y^2)^2}$$

or $f_x(x,y) = \dfrac{0(x^2+y^2) - y^2(2x)}{(x^2+y^2)^2}$

$$= \frac{-2xy^2}{(x^2+y^2)^2}$$

2. (15 marks) Given the surface $z = \sqrt{y - x}$,

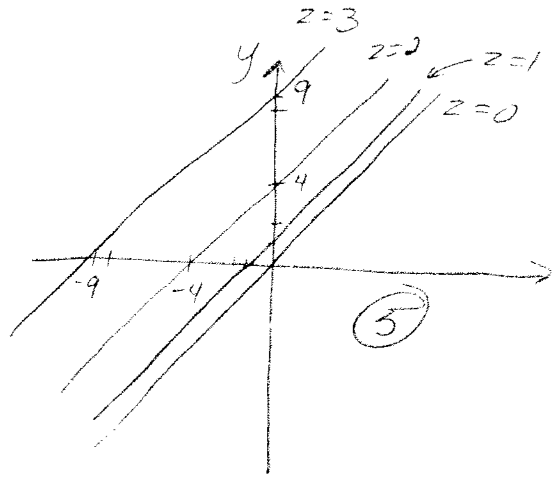
Like 14.1#39 10 a) sketch 4 level curves of the surface,

Like 14.4#3 5 b) find the equation of the plane that is tangent to the surface at $(x, y) = (1, 2)$.

a)

$z = k$	$\sqrt{y-x} = k$
$z=0$	$\sqrt{y-x} = 0 \Rightarrow y = x$
$z=1$	$\sqrt{y-x} = 1 \Rightarrow y - x = 1 \Rightarrow y = x+1$
$z=2$	$\sqrt{y-x} = 2 \Rightarrow y - x = 4 \Rightarrow y = x+4$
$z=-1$	Observe: $z \geq 0$ as $z = \sqrt{y-x}$
$z=3$	$\sqrt{y-x} = 3 \Rightarrow y - x = 9 \Rightarrow y = x+9$

5



5

b) $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$ 1

$f_x = \frac{1}{2}(y-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{y-x}}$ $\quad f_x(1,2) = -\frac{1}{2\sqrt{2-1}} = -\frac{1}{2}$ 1

$f_y = \frac{1}{2}(y-x)^{-1/2} = \frac{1}{2\sqrt{y-x}}$ $\quad f_y(1,2) = \frac{1}{2\sqrt{2-1}} = \frac{1}{2}$ 1

$z_0 = f(1,2) = \sqrt{2-1} = 1$ 1

$z = -\frac{1}{2}(x-1) + \frac{1}{2}(y-2) + 1$

$2z = -x + 1 + y - 2 + 2$

$x - y + 2z = 1$ 1

or $z = -\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}$

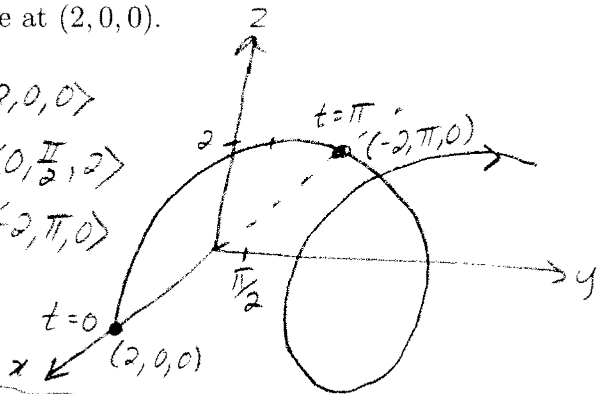
Like 13.1 #19 / Ch 13 Rev #1 5 / Like 13.3 #1 5 / Like 13.4 #33 5 / Like 13.3 #23 5 / Like 13.3 #45 5

3. (25 marks) For the curve $\mathbf{r}(t) = \langle 2\cos t, t, 2\sin t \rangle$,
a) sketch the curve (make sure you indicate the direction and label a few points),
b) find the arc length of the curve for $0 \le t \le \pi/4$,
c) find the tangential and normal components of acceleration for the curve,
d) find the equation of the tangent line to the curve at $(2, 0, 0)$,
e) find the equation of the osculating plane of the curve at $(2, 0, 0)$.

a) $x = 2\cos t$
$y = t$
$z = 2\sin t$
$x^2 + z^2 = 4$

Curve is a helix along y-axis with circular part having radius 2.

$\vec{r}(0) = \langle 2, 0, 0 \rangle$
$\vec{r}(\frac{\pi}{2}) = \langle 0, \frac{\pi}{2}, 2 \rangle$
$\vec{r}(\pi) = \langle -2, \pi, 0 \rangle$



b) $L = \int_0^{\pi/4} |\vec{r}\,'(t)|\,dt$
$= \int_0^{\pi/4} \sqrt{5}\,dt$
$= \sqrt{5}\,t \Big|_0^{\pi/4}$
$= \frac{\pi\sqrt{5}}{4}$

$\vec{r}\,'(t) = \langle -2\sin t, 1, 2\cos t \rangle$
$|\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 1 + 4\cos^2 t}$
$= \sqrt{5}$
$\vec{r}\,''(t) = \langle -2\cos t, 0, -2\sin t \rangle$

$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & 1 & 2\cos t \\ -2\cos t & 0 & -2\sin t \end{vmatrix}$
$= \langle -2\sin t, -4, 2\cos t \rangle$

c) $a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$
$= \frac{4\sin t\cos t + 0 - 4\sin t\cos t}{\sqrt{5}}$
$= 0$

$a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$
$= \frac{\sqrt{4\sin^2 t + 16 + 4\cos^2 t}}{\sqrt{5}}$
$= \frac{\sqrt{20}}{\sqrt{5}} = 2$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
$= \langle -\frac{2}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos t \rangle$
$T'(t) = \langle -\frac{2}{\sqrt{5}}\cos t, 0, -\frac{2}{\sqrt{5}}\sin t \rangle$
$|\vec{T}\,'(t)| = \sqrt{\frac{4}{5}\cos^2 t + \frac{4}{\sqrt{5}}\sin^2 t}$
$= \frac{2}{\sqrt{5}}$
$\vec{N}(t) = \frac{T'(t)}{|T'(t)|}$
$= \langle -\cos t, 0, -\sin t \rangle$

d) At $(2,0,0)$, $t = 0$ since $y = t$.
$\vec{r}_0 = \vec{r}(0) = \langle 2, 0, 0 \rangle$
$\vec{v} = \vec{r}\,'(0) = \langle 0, 1, 2 \rangle$

$\vec{r} = \vec{r}_0 + t\vec{v}$
$= \langle 2, 0, 0 \rangle + t\langle 0, 1, 2 \rangle$
$\vec{r} = \langle 2, t, 2t \rangle$ Tangent Line

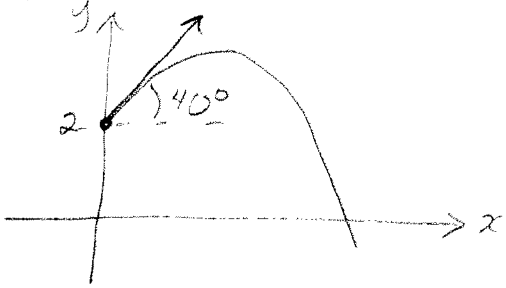
e) For osculating plane: $\vec{n} \parallel \vec{T} \times \vec{N} = \vec{B}$ or $\vec{n} \parallel \vec{r}\,'(t) \times \vec{r}\,''(t)$

Solution 1: $\vec{n} = \vec{r}\,'(t) \times \vec{r}\,''(t)$
$= \langle -2\sin t, -4, 2\cos t \rangle$
At $t = 0$, $\vec{n} = \langle 0, -4, 2 \rangle$
Plane: $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
$\langle 0, -4, 2 \rangle \cdot \langle x, y, z \rangle = \langle 0, -4, 2 \rangle \cdot \langle 2, 0, 0 \rangle$
$-4x + 2y = 0$
$-2x + y = 0$

Solution 2: $\vec{n} = \vec{T} \times \vec{N}$
$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\frac{2}{\sqrt{5}}\cos t, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos t \rangle$ see above
$\vec{N}(t) = \frac{T'(t)}{|T'(t)|} = \langle -\cos t, 0, -\sin t \rangle$
$\vec{B} = \vec{T} \times \vec{N}$
$= \langle -\frac{1}{\sqrt{5}}\sin t, -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos t \rangle$
At $t = 0$, $\vec{B}(0) = \langle 0, -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$

Plane: $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
$-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = 0$
or $-2x + y = 0$

Like 13.4 #23 4. (15 marks) A javelin leaves the thrower's hand 2 m above the ground at a 40° angle and
Like Ch 13 Rev #19 at 30 m/s. How high and how far does the javelin go?



$|\vec{V}_0| = 30$

$(x_0, y_0) = (0, 2)$

$\alpha = 40°$

$g = 9.8$

$$\vec{r}(t) = \left\langle |\vec{V}_0| t \cos\alpha + x_0,\ |\vec{V}_0| t \sin\alpha - \frac{gt^2}{2} + y_0 \right\rangle$$

(3)

$$= \langle 30t\cos 40,\ 30t\sin 40 - 4.9t^2 + 2 \rangle$$

$$= \langle 22.9183t,\ 19.2836t - 4.9t^2 + 2 \rangle$$

Max Height occurs when $V_y = 0$

$V_y = 30\sin 40 - 9.8t = 0 \Rightarrow t = \frac{30\sin 40}{9.8} = 1.9677s$

(6)

Max Height $= r_y(1.97)$

$= 30(1.97)\sin 40 - 4.9(1.97)^2 + 2$

$= 20.97\,m$

Range occurs when $r_y = 0$

$30t\sin 40 - 4.9t^2 + 2 = 0$

$-4.9t^2 + 19.28t + 2 = 0$

(6)

$$t = \frac{-19.28 \pm \sqrt{(19.28)^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$= 4.03655$ or $-0.1011\,s$

Range $= r_x(4.03655)$

$= 30(4.03655)\cos 40$

$= 92.77\,m$

Like 14.4 #19

5. (15 marks) (a) Find the linearization of the function $f(x,y) = e^x \cos y$ at $(0, \pi)$.
(b) Use your result to estimate the value of $f(x,y)$ at $(0.1, 3.2)$.

a) $L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$ 1

$f_x(x,y) = e^x \cos y \qquad f_x(0,\pi) = -1 \qquad f(0,\pi) = -1$ 5

$f_y(x,y) = -e^x \sin y \qquad f_y(0,\pi) = 0$

$L(x,y) = f_x(0,\pi)(x-0) + f_y(0,\pi)(y-\pi) + f(0,\pi)$ 5

$= -1(x-0) + 0(y-\pi) - 1$

$= -x - 1$ 2

b) $L(0.1, 3.2) = -0.1 - 1$

$= -1.1$ 2

Like 14.3 #3, 14.3 #4, Ch 14 Rev #11a) 6. (15 marks) The wave heights (h, in feet) in the open sea depend on the speed of the wind (v, in knots) and the length of time (t, in hours) the wind has been blowing at that speed. Values of the function $h = f(v,t)$ are given in the following table.

		Duration in hours (t)		
		15	20	25
Wind Speed in knots (v)	20	17	18	19
	30	28	31	33
	40	40	45	48

10 (a) Estimate the values of $f_v(30,20)$ and $f_t(30,20)$. NOTE: you only need to estimate the derivative from one side.

5 (b) Interpret the meaning of your answers from part (a).

a) $$f_v(30,20) \approx \frac{f(30+h,20)-f(30,20)}{h} = \frac{f(30+10,20)-f(30,20)}{10} = \frac{45-31}{10} = 1.4 \text{ feet/knot}$$ 5

$$f_t(30,20) \approx \frac{f(30,20+h)-f(30,20)}{h} = \frac{f(30,25)-f(30,20)}{5} = \frac{33-31}{5} = 0.4 \text{ feet/hr}$$ 5

b) When the wind has been blowing at 30 knots for 20 hours, the wave height is increasing by 1.4 feet for every knot the wind speed increases. 2½

When the wind has been blowing at 30 knots for 20 hours, the wave height is increasing by 0.4 feet for every increasing hour. 2½

Alternate solution to a)

$$f_v(30,20) \approx \frac{f(30-h,20)-f(30,20)}{-h} = \frac{f(20,20)-f(30,20)}{-10} = \frac{18-31}{-10} = 1.3 \text{ feet/knot}$$

$$f_t(30,20) \approx \frac{f(30,20-h)-f(30,20)}{-h} = \frac{f(30,15)-f(30,20)}{-5} = \frac{28-31}{-5} = 0.6 \text{ feet/hr}$$

13.3 #49 Bonus (10 marks): At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

Let $\vec{n}_1$ be the normal vector for the normal plane.

$\Rightarrow \vec{n}_1$ is parallel to $\vec{B} \times \vec{N} = \vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ (1) where $\vec{r}'(t) = \langle 3t^2, 3, 4t^3 \rangle$ (1)

Let $\vec{n}_2$ be the normal vector for the plane $6x + 6y - 8z = 1$

$\Rightarrow \vec{n}_2 = \langle 6, 6, -8 \rangle$ (1)

If the two planes are parallel, then $\vec{n}_1$ + $\vec{n}_2$ are parallel

(1) $\Rightarrow \vec{n}_1 = \alpha \vec{n}_2$ for some constant α.

Solution 1:

$\vec{n}_1 = \vec{T}(t)$

$= \dfrac{\langle 3t^2, 3, 4t^3 \rangle}{\sqrt{9t^4 + 9 + 16t^6}}$

$\vec{n}_1 = \alpha \vec{n}_2$

$\Rightarrow \dfrac{\langle 3t^2, 3, 4t^3 \rangle}{\sqrt{9t^4 + 9 + 16t^6}} = \alpha \langle 6, 6, -8 \rangle$ (1)

Equating components gives

$\dfrac{3t^2}{\sqrt{9t^4+9+16t^6}} = 6\alpha$ ①

$\dfrac{3}{\sqrt{9t^4+9+16t^6}} = 6\alpha$ ②

$\dfrac{4t^3}{\sqrt{9t^4+9+16t^6}} = -8\alpha$ ③

Eqn ② gives $\alpha = \dfrac{1}{2\sqrt{9t^4+9+16t^6}}$ (1)

Then ① gives $\dfrac{3t^2}{\sqrt{9t^4+9+16t^6}} = 6\left(\dfrac{1}{2\sqrt{9t^4+9+16t^6}}\right)$ (1)

$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

Also ③ gives $\dfrac{4t^3}{\sqrt{9t^4+9+16t^6}} = -8\left(\dfrac{1}{2\sqrt{9t^4+9+16t^6}}\right)$ (1)

$\Rightarrow 4t^3 = -4 \Rightarrow t = -1$

$\therefore$ planes are parallel at $t = -1$ (1)

or at $(x, y, z) = (-1, -3, 1)$ (1)

Solution 2:

$\vec{n}_1$ is parallel to $\dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\Rightarrow \vec{n}_1$ is parallel to $\vec{r}'(t)$ (since $|\vec{r}'(t)|$ is a scalar).

So, $\vec{n}_1 = \vec{r}'(t) = \langle 3t^2, 3, 4t^3 \rangle$

$\vec{n}_1 = \alpha \vec{n}_2$

$\Rightarrow \langle 3t^2, 3, 4t^3 \rangle = \alpha \langle 6, 6, -8 \rangle$

Equating components gives

$3t^2 = 6\alpha$ ④

$3 = 6\alpha$ ⑤

$4t^3 = -8\alpha$ ⑥

Eqn ⑤ gives $\alpha = \frac{1}{2}$.

Then ④ gives $3t^2 = 6(\frac{1}{2})$

$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

Also ⑥ gives $4t^3 = -8(\frac{1}{2})$

$\Rightarrow 4t^3 = -4 \Rightarrow t = -1$

$\therefore$ the planes are parallel at $t = -1$

or at $x = (-1)^3 = -1$

$y = 3(-1) = -3$

$z = (-1)^4 = 1$

ie at $(x, y, z) = (-1, -3, 1)$