$\not{1}$ 95

MAC 2313, Section 04 with Dr. Hurdal
Fall 2011 – Test 1

Name: SOLUTIONS

Calculators are allowed. Notebooks and textbooks are NOT allowed. There will be 10 marks allocated for clear and well written mathematics solutions. This test will be graded out of ~~100~~. 95

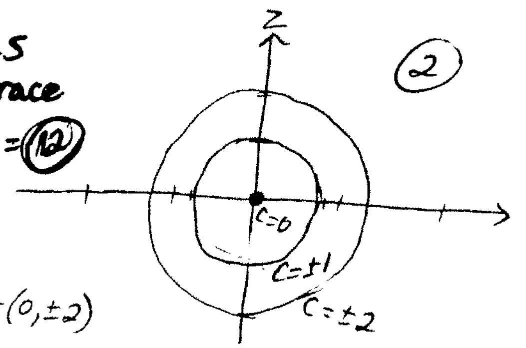
12.6 #25, 14 Like 12.6 #11 1. (20 marks) + 2 mathematical writing (a) Sketch at least four traces in one plane (your choice) of the surface given by $y^2 = 2x^2 + z^2$.
6 (b) Sketch and describe the 3D surface.

Traces in only one of the planes are required:

a) xz-plane

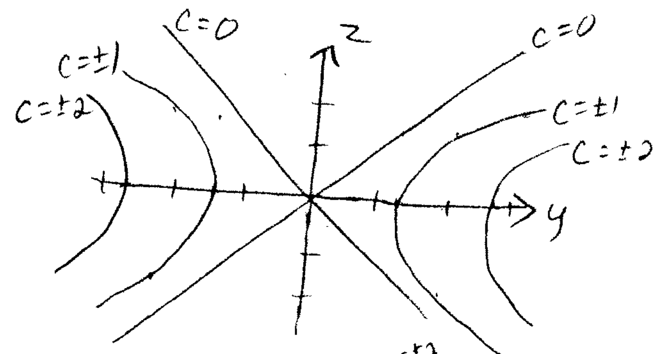
③ marks per trace ×4 = ⑫

$y = c$	$2x^2+z^2=c^2 \Rightarrow x^2+\frac{z^2}{2}=\frac{c^2}{2}$
$c=0$	$2x^2+z^2=0 \Rightarrow$ pt $(0,0)$
$c=\pm1$	$2x^2+z^2=1 \Rightarrow$ ellipse pts $(x,z)=(0,\pm1)+(\pm\frac{1}{\sqrt{2}},0)$
$c=\pm2$	$2x^2+z^2=4 \Rightarrow \frac{x^2}{2}+\frac{z^2}{4}=1 \Rightarrow$ ellipse pts $(x,z)=(\pm\sqrt{2},0)+(0,\pm2)$

② 

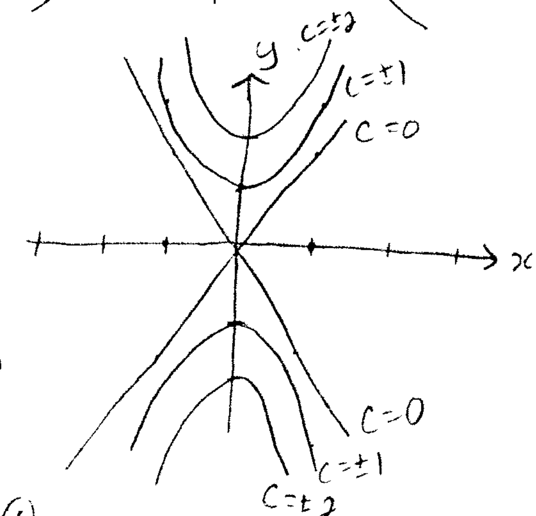
yz-plane

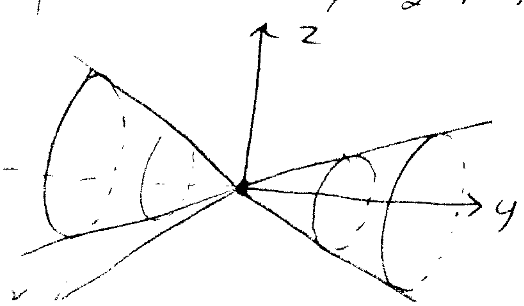
$x=c$	$2c^2+z^2=y^2 \Rightarrow y^2-z^2=2c^2$
$c=0$	$y^2-z^2=0 \Rightarrow y^2=z^2 \Rightarrow y=\pm z$
$c=\pm1$	$y^2-z^2=2 \Rightarrow$ hyperbola pts $(y,z)=(\pm\sqrt{2},0)$
$c=\pm2$	$y^2-z^2=8 \Rightarrow$ hyperbola pts $(y,z)=(\pm\sqrt{8},0)$



xy-plane

$z=c$	$2x^2+c^2=y^2 \Rightarrow y^2-2x^2=c^2$
$c=0$	$y^2-2x^2=0 \Rightarrow y^2=2x^2 \Rightarrow y=\pm\sqrt{2}x$
$c=\pm1$	$y^2-2x^2=1$ hyperbola pts $(x,y)=(0,\pm1)$
$c=\pm2$	$y^2-2x^2=4 \Rightarrow \frac{y^2}{4}-\frac{x^2}{2}=1$ hyperbola pts $(x,y)=(0,\pm2)$



b) 

②

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① ① ① ①
Elliptical Cone along y-axis with major axis of ellipse along z-axis

+1 mathematical writing

Ch 12 Review #5 2. (10 marks) Find the values of x such that the vectors $\langle 3, 2, x\rangle$ and $\langle 2x, 4, x\rangle$ are orthogonal.

Like 12.3 #26

To be perpendicular,

$$\langle 3,2,x\rangle \cdot \langle 2x,4,x\rangle = 0$$

$$\Rightarrow \quad 6x + 8 + x^2 = 0$$

$$(x+4)(x+2) = 0$$

$$x = -4 \quad \text{or} \quad x = -2$$

10

+2 mathematical writing

ike 12.5 #39 5 3. (20 marks) Consider the two planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

a) Sketch the second plane.

ike 12.5 #55 [5 b) At what angle do the planes intersect?

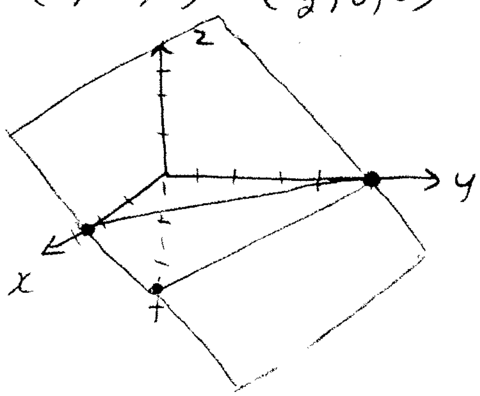
5 c) Find the parametric equations for the line of intersection of the planes.

ike 12.5 #37 5 d) Find the equation of a new plane that contains the line of intersection and the point $(1, 0, -1)$.

a) $2x + y - 2z = 5$

Intercepts: $(0,0,-\frac{5}{2})$

③ $(0,5,0)$ + $(\frac{5}{2},0,0)$

② 

b) angle of intersection = angle between normal vectors

① $\vec{n}_1 = \langle 3,-6,-2\rangle$

① $\vec{n}_2 = \langle 2,1,-2\rangle$

① $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

① $= \dfrac{(3)(2)+(-6)(1)+(-2)(-2)}{\sqrt{(3)^2+(-6)^2+(-2)^2}\sqrt{(2)^2+(1)^2+(-2)^2}}$

$= \dfrac{4}{\sqrt{49}\cdot\sqrt{9}}$

$= \dfrac{4}{21}$

① $\theta \approx 79.0°$

c) line of intersection:

- need direction vector: $\vec{v} = \vec{n}_1 \times \vec{n}_2$
- need point on both planes

② $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$

$= \vec{i}(12-(-2)) - \vec{j}(-6-(-4)) + \vec{k}(3-(-12))$

$= \langle 14, 2, 15\rangle$

To find a point, set the planes equal to each other.

② $2x+y-2z=5 \Rightarrow y = 5-2x+2z$

Sub in $3x-6y-2z=15$

$3x-6(5-2x+2z)-2z=15$

$3x-30+12x-12z-2z=15$

$15x-14z=45$

Let $z=0$. Then $x=3$ + $y=5-2(3)-0 = -1$

Pt is $(3,-1,0)$.

Equation of Line: $\vec{r} = \vec{r}_0 + t\vec{v}$

$\vec{r} = \langle 3,-1,0\rangle + t\langle 14,2,15\rangle$

Parametric Eqns:

① $x = 3+14t,\ y=-1+2t,\ z=15$

d) A second vector on new plane

① given by pts $(1,0,-1)$ + $(3,-1,0)$

① $\vec{a} = \langle 3-1, -1-0, 0-(-1)\rangle = \langle 2,-1,1\rangle$

① $\vec{n} = \vec{v}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 14 & 2 & 15 \\ 2 & -1 & 1 \end{vmatrix} = \langle 17,16,-18\rangle$

① $\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0$

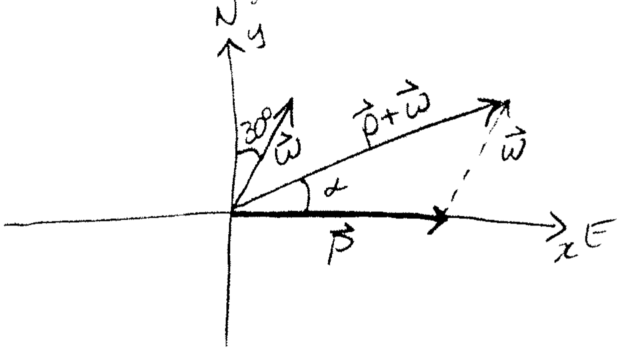
$\langle 17,16,-18\rangle \cdot \langle x,y,z\rangle = \langle 17,16,-18\rangle \cdot \langle 1,0,-1\rangle$

① $17x+16y-18z=35$

5 marks
+5 bonus
+1 mathematical writing

4. ~~(10 marks)~~ A plane is flying due east at 500km/hr and experiences a 70 km/hr tailwind blowing in the direction N30°E. What is the new heading and groundspeed the plane needs to fly in order to maintain a compass heading due east?

Like 12.1 #30



Let $\vec{p}$ be the plane relative to air

Let $\vec{\omega}$ be the tailwind

Given: $|\vec{p}| = 500 \quad |\vec{\omega}| = 70$

(2) $\vec{p} = \langle 500, 0 \rangle$

(2) $\vec{\omega} = \langle 70\cos 60°, 70\sin 60° \rangle$
$= \langle 70(\tfrac{1}{2}), 70(\tfrac{\sqrt{3}}{2}) \rangle$
$= \langle 35, 35\sqrt{3} \rangle$
$= \langle 35, 60.62 \rangle$

(2) $\vec{p}+\vec{\omega} = \langle 500+35, 0+35\sqrt{3} \rangle$
$= \langle 535, 35\sqrt{3} \rangle$
$= \langle 535, 60.62 \rangle$

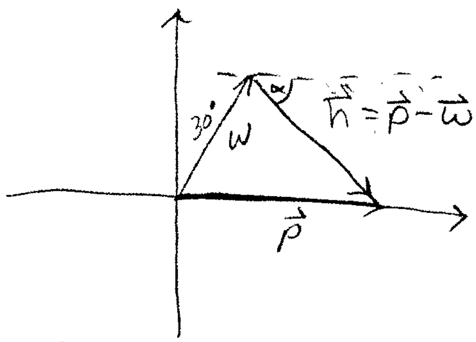
(2) groundspeed $= |\vec{p}+\vec{\omega}|$
$= \sqrt{(535)^2 + (35\sqrt{3})^2}$
$= 538.4$ km/hr

(2) $\tan\alpha = \dfrac{35\sqrt{3}}{535} =$

$\alpha = 6.46°$

The heading is E6.46°N = 6.46° N of E
= N83.54°E = 83.54°E of N

Alternate Interpretation:



$\vec{h} = \vec{p} - \vec{\omega}$
$= \langle 500-35, 0-35\sqrt{3} \rangle$
$= \langle 465, -35\sqrt{3} \rangle$
$= \langle 465, -60.62 \rangle$

groundspeed $= |\vec{\omega} - \vec{p}|$
$= \sqrt{(465)^2 + (-35\sqrt{3})^2}$
$= 468.9$ km/hr

$\tan\alpha = \dfrac{-35\sqrt{3}}{465}$

$\alpha = -7.43°$

The heading is
E7.43°S = 7.43° S of E
S82.57°E = 82.57° E of S

+2 mathematical writing

Like 12.4#35 +#33 5. (10 marks) Consider the points $P(2,4,5)$, $Q(1,5,7)$, $R(-1,6,8)$ and $S(-2,7,9)$.
a) Find the volume of the parallelepiped formed by those points.
Like 12.3#35 b) Find the scalar and vector projections of $\vec{PQ}$ onto $\vec{PR}$.

a) $\vec{PQ} = \langle -1, 1, 2 \rangle$ (1)

$\vec{PR} = \langle -3, 2, 3 \rangle$ (1)

$\vec{PS} = \langle -4, 3, 4 \rangle$ (1)

$$\text{volume} = |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| \quad (1)$$

$$= \left| \begin{vmatrix} -1 & 1 & 2 \\ -3 & 2 & 3 \\ -4 & 3 & 4 \end{vmatrix} \right|$$

$$= |-1(2\cdot 4 - 3\cdot 3) - 1(-3\cdot 4 - (-3)(-4)) + 2(-3\cdot 3 - (-4)(2))|$$

$$= |-(-1) - (0) + 2(-1)|$$

$$= |-1|$$

$$= 1 \text{ units}^3 \quad (1)$$

b) Let $\vec{b} = \vec{PQ} = \langle -1, 1, 2 \rangle$ and $\vec{a} = \vec{PR} = \langle -3, 2, 3 \rangle$ (1)

scalar projection $= \text{comp}_a \vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \quad (1)$$

$$= \frac{3 + 2 + 6}{\sqrt{22}}$$

$$= \frac{11}{\sqrt{22}} \quad (1)$$

vector projection

$$= (\text{comp}_a \vec{b})\, \vec{u}_a \quad (1)$$

$$= \frac{11}{\sqrt{22}} \frac{\langle -3, 2, 3 \rangle}{\sqrt{9+4+9}}$$

$$= \left\langle -\frac{33}{22}, \frac{22}{22}, \frac{33}{22} \right\rangle$$

$$= \left\langle -\frac{3}{2}, 1, \frac{3}{2} \right\rangle \quad (1)$$

$+1$ mathematical writing

6. (10 marks) Show that the line $x = 1 - 2t$, $y = 2 + 5t$, $z = -3t$ is NOT parallel to the plane $-2x + 5y = 8 + 3z$. Then find the point at which the line intersects the plane.

Like 12.5 #43

$$L: \left.\begin{array}{l} x = 1-2t \\ y = 2+5t \\ z = -3t \end{array}\right\} \Rightarrow \vec{r_0} = \langle 1,2,0\rangle + \vec{v} = \langle -2,5,-3\rangle \quad (1)$$

$$P: \left.\begin{array}{l} -2x+5y = 8+3z \\ \Rightarrow -2x+5y-3z = 8 \end{array}\right\} \Rightarrow \vec{n} = \langle -2,5,-3\rangle \quad (1)$$

If the line and plane are parallel, then the direction vector of the line and the normal vector of the plane must be perpendicular. ie $\vec{v}\cdot\vec{n} = 0$

$$\begin{aligned} \vec{v}\cdot\vec{n} &= \langle -2,5,-3\rangle \cdot \langle -2,5,-3\rangle \quad (1) \\ &= 4+25+9 \quad (1) \\ &= 38 \\ &\neq 0 \quad (1) \end{aligned}$$

$\therefore$ not perpendicular

Point of Intersection: sub parametric eqns into eqn of plane

$$\begin{aligned} -2(1-2t) + 5(2+5t) - 3(-3t) &= 8 \quad (1) \\ -2 + 4t + 10 + 25t + 9t &= 8 \\ 38t &= 0 \\ t &= 0 \quad (1) \end{aligned}$$

Then
$$\left.\begin{array}{l} x = 1-2(0) = 1 \\ y = 2+5(0) = 2 \\ z = -3(0) = 0 \end{array}\right\} (3)$$

$\therefore$ pt of intersection is $(1,2,0)$

+1 mathematical writing

Like 12.5 #19, 21

7. (10 marks) Show that the lines
L_1: $x = 1 + t, y = -2 + 3t, z = 4 - t$ and
L_2: $\vec{r} = \langle 2s, 3 + s, -3 + 4s \rangle$
are skew.

$L_1 : \vec{r}_1 = \langle 1, -2, 4 \rangle$ + $\vec{V}_1 = \langle 1, 3, -1 \rangle$ (1)

$L_2 : \vec{r}_2 = \langle 0, 3, -3 \rangle$ + $\vec{V}_2 = \langle 2, 1, 4 \rangle$ (1)

$\vec{V}_1 \neq \lambda \vec{V}_2 \Rightarrow$ lines not parallel (2)

Check if lines intersect:

	L_1		L_2	
x:	$1+t$	$=$	$2s$	(1)
y:	$-2+3t$	$=$	$3+s$	(2)
z:	$4-t$	$=$	$-3+4s$	(3)

(2)

(1)+(3) $\Rightarrow 5 = -3 + 6s$

$s = \frac{8}{6} = \frac{4}{3}$

Sub $s = 4/3$ in (1) $\Rightarrow 1 + t = 2\left(\frac{4}{3}\right)$ (2)

$\Rightarrow t = \frac{5}{3}$

Do $s = \frac{4}{3}$ + $t = \frac{5}{3}$ satisfy (2) ?

$LS = -2 + 3t$
$= -2 + 3\left(\frac{5}{3}\right)$
$= 3$

$RS = 3 + s$
$= 3 + \frac{4}{3}$
$= \frac{13}{3}$
$\neq L.S.$

(2)

Since there are no values for s+t that satisfy all equations, ∴ lines do not intersect

Conclusion: lines are not parallel + do not intersect
$\Rightarrow$ skew lines

2

12.4
pg 788

Bonus (10 marks): If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, use the definitions of cross product, dot product and length of a vector to show

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

for $0 \le \theta \le \pi$.

Showing $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ is the same as showing $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1 \rangle$$

$$\begin{aligned}
L.S. &= |\vec{a}\times\vec{b}|^2 \\
&= (a_2b_3 - a_3b_2)^2 + (-a_1b_3 + a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \\
&\quad + a_1^2b_3^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \\
&= a_1^2(b_2^2+b_3^2) + a_2^2(b_1^2+b_3^2) + a_3^2(b_1^2+b_2^2) - 2(a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_3b_2b_3)
\end{aligned}$$

$$\begin{aligned}
R.S. &= |\vec{a}|^2|\vec{b}|^2\sin^2\theta \\
&= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta) \quad \text{for } 0 \le \theta \le \pi \text{ due to squaring both sides} \\
&= |\vec{a}|^2|\vec{b}|^2\left(1 - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)^2\right) \\
&= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2 \\
&= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + \cancel{a_3^2a_3^2} \\
&\quad - \cancel{a_1^2b_1^2} - \cancel{a_2^2b_2^2} - \cancel{a_3^2b_3^2} - 2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2a_3b_2b_3 \\
&= a_1^2(b_2^2+b_3^2) + a_2^2(b_1^2+b_3^2) + a_3^2(b_1^2+b_2^2) - 2(a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_3b_2b_3) \\
&= R.S. \quad \text{as required.}
\end{aligned}$$