MAC 2313, Section 02 with Dr. Hurdal
Spring 2007 – Final Exam

Name: _________________________ SSN: _________________________

As stated in class, you are allowed to bring to the test one 8.5x11 inch page, written on both sides. Calculators are allowed. Notebooks and textbooks are NOT allowed. This test will be graded out of 100.

1. (10 marks) Sketch the region of integration and reverse the order of integration to evaluate

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy.$$

2. (10 marks) Directly evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by the vector function $\mathbf{r}(t) = t\mathbf{i} + \sin(t)\mathbf{j} + \cos(t)\mathbf{k}$, $0 \leq t \leq \pi$ and $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} - x\mathbf{k}$.

3. (10 marks) Use Lagrange multipliers to find the extrema of the function $f(x, y) = x^2+3xy+y^2$ subject to the constraint $x^2 + y^2 \leq 1$.

4. (10 marks) Use Stokes' Theorem to find the work done by the force field $F(x, y, z) = x\mathbf{i} + z^2\mathbf{j} + (x^2 + y^2)\mathbf{k}$ in moving a particle along the curve C where C is oriented counterclockwise as viewed from above and is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

5. (10 marks) a) Sketch at least 3 level curves of the function $f(x, y) = 4x^2 + y^2 + 1$.
b) Find an equation of the tangent plane to this surface at the point $(0, 2)$.
c) Find the directional derivative of $f(x, y)$ at $(0, 2)$ in the direction of $(3, 6)$.
d) Find the maximum rate of change of $f(x, y)$ at $(0, 2)$.

6. (15 marks) Verify that the Divergence Theorem is true for the vector field $F(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + 2xz\mathbf{k}$ on the region E where E is the solid half cylinder given by $x^2 + y^2 = 4, x \geq 0$ and bounded by the planes $z = 0$ and $z = 1$.

7. (10 marks) a) Find a parametric equation for the line through $(1, 0, 1)$ and $(4, -2, 2)$.
b) Where does this line intersect the plane $x + y + z = 6$?
c) If the line in part (a) represents the acceleration vector of a particle, find the velocity and position vectors of this particle if the initial velocity and position of the particle are given by $\mathbf{v}(0) = \mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$ and $\mathbf{r}(1) = 5\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$.

8. (10 marks) Calculate the flux of $\mathbf{F}$ across the open surface S where S is the part of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$ with downward orientation and $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$.

9. (15 marks) Verify that Green's Theorem is true for the line integral

$$\int_C y^2\,dx + x^2\,dy$$

where C consists of the region lying between the graphs of $y = x$ and $y = x^2/4$.

Bonus (10 marks): Flow lines or streamlines for a vector field are useful for studying the flow of fields such as electric and magnetic fields. A flow line or a stream line of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a path $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ whose velocity vector equals $\mathbf{F}$. Thus, $\mathbf{r}'(t) = \mathbf{F}$ and so $x'(t) = P$ and $y'(t) = Q$. These two differential equations can be solved to a give a parameterization of the flow line.

For example, if a constant velocity vector field $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j}$ passes through the point $(1, 2)$ at time $t = 0$, then

$$x'(t) = 3 \qquad \text{and} \qquad y'(t) = 4.$$

Thus,

$$x(t) = 3t + x_0 \qquad \text{and} \qquad y(t) = 4t + y_0.$$

Since the path passes through the point $(1, 2)$ at $t = 0$, we have $x_0 = 1$ and $y_0 = 2$ and so

$$x(t) = 3t + 1 \qquad \text{and} \qquad y(t) = 4t + 2.$$

Thus, the path of the flow line is the line given parametrically by

$$\mathbf{r}(t) = (3t + 1)\mathbf{i} + (4t + 2)\mathbf{j}.$$

Use the above definition of a flow line to answer the following problem.

A velocity vector field is given by $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$. Find the path of the flow of an object that is at the point $(-2, 2)$ at time $t = 0$. On the same picture, sketch the vector field and sketch the path of the flow line beginning at $(-2, 2)$.

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