MAC 2313, Section 03 with Dr. Hurdal
Spring 2005 Final Exam

Name: ________________________ SSN: ________________________

As stated in class, you are allowed to bring to the exam one 8.5x11 inch page, written on both sides. Calculators are allowed. Notebooks and textbooks are NOT allowed. This exam will be graded out of 100.

1. (15 marks) Use the divergence theorem to compute the flux of $\vec{F} =< 2xy+z, y^2, -(x+3y) >$ taken over the region bounded by $2x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.

2. (10 marks) A surface is given by the parametric equations $x = u^2$, $y = v^2$ and $z = uv$.
(a) Find an equation of the tangent plane to this surface at the point $u = 1$, $v = 1$.
(b) Express this surface as $z = f(x, y)$.
(c) What is the directional derivative of this surface as one leaves the point $(x, y) = (1, 1)$ and heads toward the point $(x, y) = (8, 2)$?
(d) What is the maximum rate of change at the point $(x, y) = (8, 2)$?

3. (12 marks) Verify Stokes' theorem for $\vec{F} =< -2y, 2x, -z^2 >$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

4. (10 marks) A moving particle starts at an initial position $\vec{r}(0) =< 1, 0, 0 >$ with initial velocity $\vec{v}(0) =< 1, -1, 1 >$. If its acceleration is $\vec{a}(t) =< 4t, 6t, 1 >$, find its position at time t.

5. (15 marks) Verify Green's theorem $\int_C (2x - y + 4)dx + (y + 3x - 6)dy$ where C is the curve $y = 3x^2 + 1$ from $(0, 1)$ to $(2, 13)$ and the line from $(2, 13)$ to $(0, 1)$.

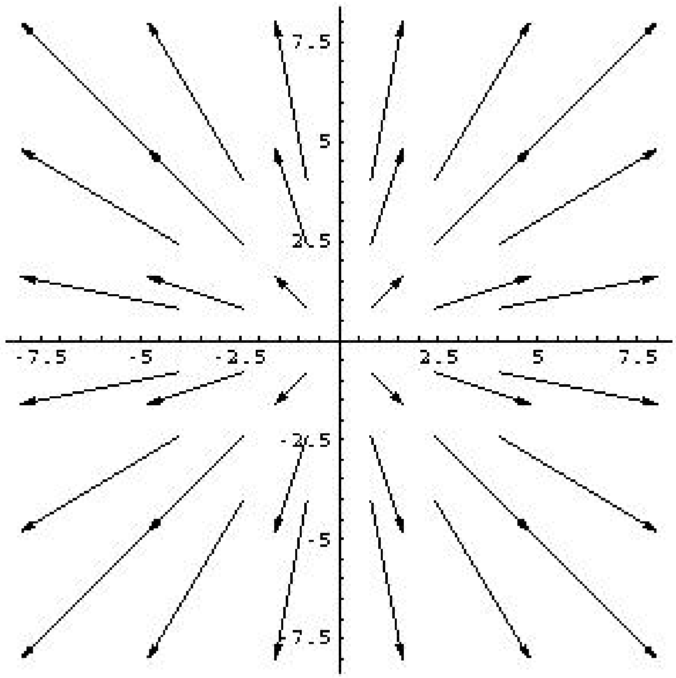
6. (12 marks) (a) Sketch and describe the region of integration of $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$.

(b) Evaluate this integral.

(c) What is an interpretation of the meaning of this integral?

7. (13 marks) Show, without computing the potential function, that the vector field $\vec{F} = 2xe^{-y}\vec{\imath} + (\cos(z) - x^2e^{-y})\vec{\jmath} - y\sin(z)\vec{k}$ is conservative. Then, compute the potential function and find the work done in moving a particle along the line from $(1, 2, 0)$ to $(2, 3, \pi)$.

8. (8 marks) Consider the following vector field.



(a) Is the divergence at $(2.5, 2.5)$ positive, negative or zero? Explain.

(b) If C_1 is the line from $(-5, 5)$ to $(-5, 0)$, is $\int_{C_1} \vec{F} \cdot d\vec{r}$ positive, negative or zero? Explain.

(c) Sketch a curve C_2 that is an example of a line integral whose sign is zero and explain your reasoning.

(d) Do you think this vector field is a conservative vector field? Explain.

9. (5 marks) Evaluate $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} =< xy, 4x^2, yz >$ where S is the surface $z = xe^y$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$, with upward orientation.

Bonus #1 (5 marks): If $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ is a smooth vector field, then Green's Theorem says that

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where C is a simple closed curve surrounding a region R in the xy-plane and C is oriented counter clockwise.
Alternatively, Stokes' Theorem indicates that

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S \text{curl}\vec{F} \cdot d\vec{S}$$

where S is a smooth surface with oriented closed boundary C and $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$. Show that Green's Theorem is a special case of Stokes' Theorem.

Bonus #2 (5 marks): Use Green's Theorem to show that the line integral of $\vec{F} = -\frac{y}{2}\vec{\imath} + \frac{x}{2}\vec{\jmath}$ around a simple closed curve in the xy-plane measures the area enclosed by the curve.