MAC 2313, Section 06 with Dr. Hurdal
Fall 2009 Final Exam Answers

1. Possible extreme values occur at $(0,1)$, $(0,-1)$, $\left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$.
The maximum occurs at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$ and is $\frac{2}{3\sqrt{3}}$. The minimum occurs at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$ and is $-\frac{2}{3\sqrt{3}}$.

2. Compute the line integral directly (i.e. compute $\int_C \mathbf{F} \cdot d\mathbf{r}$) AND compute $\int\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.
Total work is -9π.
Computing the line integral directly requires parameterizing a circle of radius 3 in the xy-plane:

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 0\rangle \text{ for } 0 \leq t \leq 2\pi.$$

Using Stokes' Theorem requires parameterizing a hemisphere of radius 3 and using the normal of the hemisphere. One possibility is to parameterize the surface as

$$\mathbf{r}(\theta, \phi) = \langle 3\cos(\theta)\sin(\phi), 3\sin(\theta)\sin(\phi), 3\cos(\phi)\rangle \text{ for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \pi/2$$

and then $\mathbf{n} = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle 9\sin(\phi)$.
An alternate surface parameterization is

$$\mathbf{r}(x, y) = \langle x, y, \sqrt{9 - x^2 - y^2}\rangle \text{ for } -\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2} \text{ and } -3 \leq x \leq 3$$

with $\mathbf{n} = \langle -g_x, -g_y, 1\rangle = \left\langle \frac{x}{\sqrt{9-x^2-y^2}}, \frac{y}{\sqrt{9-x^2-y^2}}, 1 \right\rangle$. For this latter parameterization, the surface integral can be converted to polar coordinates with $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 3$.

3. $\mathbf{v}(t) = \langle 3t^2 + 1, 4t^3 - 1, -3t^2 + 3\rangle$; $\mathbf{r}(t) = \langle t^3 + t + 1, t^4 - t + 2, -t^3 + 3t - 1\rangle$

4. Use the curl test in 3D to show $\mathbf{F}$ is conservative. Here, curl $\mathbf{F} = \langle e^z - e^z, 0, e^y - e^y\rangle = \mathbf{0}$. Since $\mathbf{F}$ is conservative then $\mathbf{F}$ has a potential function and the Fundamental Theorem of Line Integrals can be used to find the work done (i.e. to evaluate the line integral). The potential function is $f(x, y, z) = xe^y + ye^z + 2z + C$ and the work done $= f(4, 0, 3) - f(0, 2, 0) = 8$.

5. Region is the front half of a sphere (x is positive) of radius 2, so $0 \leq \rho \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$ and $0 \leq \phi \leq \pi$. The value of the integral is zero.

6. $\frac{6}{5} - \cos(1) - \sin(1)$

7. a) The surface is a hyperboloid of one sheet, with $z \geq 0$.
b) $z = 2x - 1$
c) 6/5
d) 2

8. $\frac{9\pi}{2}$

9. $\dfrac{e-1}{2}$

Bonus: a) This is one possible solution. In the yz-plane, curl $\mathbf{F} = \left\langle \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}, 0, 0 \right\rangle$ and $\mathbf{n} = \langle 1, 0, 0 \rangle$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_D \text{curl } \mathbf{F} \cdot \mathbf{n}\, dA \text{ by Stokes' Theorem} \\
&= \int\int_D \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, 0, 0 \right\rangle \cdot \langle 1, 0, 0 \rangle\, dA \\
&= \int\int_D \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dA.
\end{aligned}$$

b) $\dfrac{1}{6}$