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Conflict and rent-seeking success functions: Ratio vs. difference models of relative success*

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Abstract. The rent-seeking competitions studied by economists fall within a much broader category of conflict interactions that also includes military combats, election campaigns, industrial disputes, lawsuits, and sibling rivalries. In the rent-seeking literature, each party's success p_i (which can be interpreted either as the probability of victory or as the proportion of the prize won) has usually been taken to be a function of the *ratio* of the respective resource commitments. Alternatively, however, p_i may instead be a function of the *difference* between the parties' commitments to the contest. The Contest Success Function (CSF) for the difference form is a logistic curve in which, as is consistent with military experience, increasing returns apply up to an inflection point at equal resource commitments. A crucial flaw of the traditional ratio model is that neither one-sided submission nor two-sided peace between the parties can ever occur as a Cournot equilibrium. In contrast, both of these outcomes are entirely consistent with a model in which success is a function of the difference between the parties' resource commitments.

1. Introduction

Following the seminal contributions of Gordon Tullock (1967, 1980), a number of papers[1] have explored various aspects of rent-seeking competitions. In such contests, each of N players invests effort C_i ($i = 1, \ldots, N$) in the hope of gaining a prize of value V. Existing analyses have mainly explored the nature of equilibrium with varying numbers of contestants, the central issue addressed being whether or not under- or over-dissipation of rents will occur.

The fundamental notion of competitions in which relative success is a function of the parties' respective resource commitments applies far beyond the rent-seeking context. Military combats, election campaigns, industrial struggles (strikes and lockouts), legal conflicts (lawsuits), and even rivalries among siblings or between spouses within the family all fall under this heading. Owing perhaps to failure to perceive these wider implications, the papers in the rent-seeking literature generally do not adopt a *general-equilibrium* approach which would make explicit provision for the alternative productive or consumptive

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uses of resources employed in rent-seeking competitions. Also, what is very important, a general-equilibrium model would typically make the value of the prize an endogenous variable rather than an exogenously given parameter. I have attempted to provide such a general-equilibrium analysis in Hirshleifer (1988).

This note has a more limited aim, however. My main purpose is to point out that Tullock's basic equation for success in rent-seeking competition represents only one of two canonical families of possibilities, the second and at least equally interesting family having been totally ignored in the existing literature. Specifically, in Tullock's formula each party's success is a function of the *ratios* of the respective efforts or inputs C_i. As will be shown, a number of significantly different results are obtained when, alternatively, relative success is determined by the *differences* among the inputs. I will also be allowing for possibly different prize valuations $V_i \neq V_j$.[2]

2. Contest success functions

For $N=2$ players, in Tullock's basic model the proportionate outcomes p_i depend in a simple way upon the contest inputs or efforts C_i:

$$p_1/p_2 = (C_1/C_2)^m \tag{1}$$

Here each p_i may be interpreted either as the party's respective *probability of success* in a discrete either-or competition or else as the *proportionate share* of the prize won in a continuous-outcome contest. Since $p_1 + p_2 = 1$, equation (1) is equivalent to:

$$p_i = \frac{C_1^m}{C_1^m + C_2^m} \tag{2}$$

For given C_2, this may be called the Contest Success Function (CSF) for player #1; the CSF for the other player is defined correspondingly. (I have implicitly been assuming that the two sides' resources have equal effectiveness in the contest. More generally, it would be possible to adjust each side's C_i by an effectiveness coefficient; this straightforward generalization will be omitted here.)

The effect of the "mass effect parameter" m upon the shape of player #1's Contest Success Function is displayed in Figure 1, in which player #2's resource input has been arbitrarily fixed at $C_2 = 100$. Regardless of the level of m, we see that $p_1 = p_2 = .5$ when $C_1 = C_2$. If $m \leq 1$, diminishing returns to competitive effort hold throughout. But for $m > 1$, an initial range of increasing returns exists instead. More specifically, taking the second derivative in the usual way, the inflection point along the CSF of player #1 is determined by the condition:

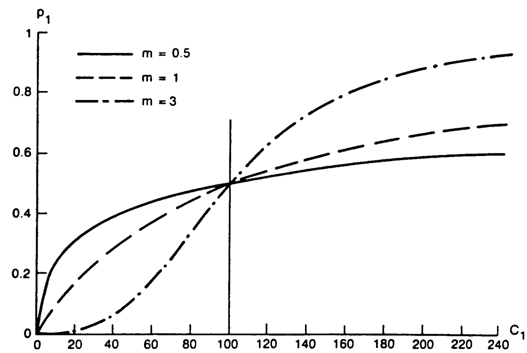

Figure 1. Contest success function: Ratio form

$$\frac{C_1}{C_2} = \left(\frac{m-1}{m+1}\right)^{1/m} \tag{3a}$$

or, equivalently,

$$p_1 = (m-1)/2m \tag{3b}$$

Since m cannot meaningfully be zero or negative we see that, for given C_2, there is a point of inflection in the positive range of C_1 only if $m > 1$.[3]

While it is often plausible to assume that contest power is a function of the *ratio* of the forces or efforts committed, this is by no means the only possibly valid functional relation. Nor are all the implications of the ratio form always reasonable. One implication, for example, is that a side investing zero effort must lose everything so long as the opponent commits any finite amount of resources at all, however small, to the struggle. When, alternatively, the outcome is assumed to be a function of the *difference* between the two sides' efforts, a player can have some chance or share of success even without committing resources to the contest. In struggles between nations, for example, one side may surrender rather than resist. While the hope may sometimes be to appease the aggressor, it might make sense to surrender to a totally unappeasable opponent if the submitting nation does not expect to lose absolutely everything by giving up the struggle. And this is reasonable, since in general it will be costly for the victor, even in the absence of resistance, to locate and extract all the possible spoils.

There is one other factor to consider, namely, the location of the inflection point of the CSF. When it comes to military interactions, "God is on the side of the larger battalions." There is an enormous gain when your side's forces increase from just a little smaller than the enemy's to just a little larger.[4] This implies that the range of increasing returns to player #1's commitment C_1 extends up to $C_1 = C_2$, or equivalently up to $p_1 = p_2$.[5] But, we have seen, when the ratio form of the CSF is used, increasing returns, if present at all (that is, if $m > 1$), can only hold up to some $C_1 < C_2$.

Postulating that contest success depends upon the *difference* between the resource commitments, the required conditions – that $C_1 = 0$ need not imply $p_1 = 0$, and that the inflection point occurs at $C_1 = C_2$ – are met by the *logistic* family of curves:

$$p_1 = \frac{1}{1 + \exp\{k(C_2 - C_1)\}} \tag{4}$$

where p_2 is defined correspondingly. (As is logically required $p_1 + p_2 = 1$.) In particular, when $C_1 = 0$ player #1 still retains a share of success $p_1 = 1/(1 + \exp\{kC_2\})$. Figure 2 shows several CSF curves for varying k, where k is the "mass effect parameter" applicable to the logistic function.

In a military context we might expect the *ratio form* of the Contest Success Function to be applicable when clashes take place under close to "idealized" conditions such as: an undifferentiated battlefield, full information, and unflagging weapons effectiveness. In contrast, the *difference form* tends to apply where there are sanctuaries and refuges, where information is imperfect, and where the victorious player is subject to fatigue and distraction. Given such "imperfections of the combat market," the defeated side need not lose absolutely everything. (For the sake of concreteness I have been using military metaphors and examples, but analogous statements can evidently be made about non-military struggles – e.g., lawsuits or political campaigns or rent-seeking competitions.)

The generalization of equation (2) for any number of players N was provided in Tullock's initial paper. For the i^{th} contestant, the probability of success becomes:

$$p_i = \frac{C_i^m}{C_1^m + C_2^m + \ldots + C_N^m} = \frac{C_i^m}{\Sigma_j C_j^m} \tag{5}$$

Of course, the p_i's sum to unity.

Employing the difference (logistic) form instead, the corresponding generalization of equation (4) is:

$$p_i = \frac{\exp\{kC_i\}}{\Sigma_j \exp\{kC_j\}} \tag{6}$$

It is evident from the form of the last fraction on the right that, as required, the sum of these p_i's will also be unity.[6]

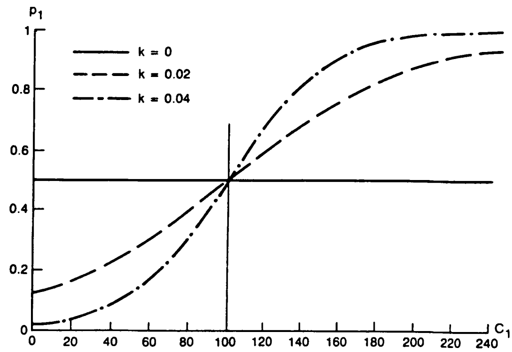

Figure 2. Contest success function: Difference (logistic) form

To illustrate, if N = 3 and i = 1, equation (6) becomes:

$$p_1 = \frac{1}{1+\exp\{k(C_2-C_1)\} + \exp\{k(C_3-C_1)\}} \tag{6a}$$

Both (5) and (6) fall within the more general category of *logit* functions.[7]

3. Symmetrical Nash-Cournot equilibrium

As has been mentioned, when the ratio form of the CSF applies each side will surely always commit some resources to the contest. If peace is defined by the condition $C_1 = C_2 = 0$, then peace can never occur as a Cournot equilibrium under the traditional ratio model!

The demonstration is simple. Side #1 will be seeking to maximize its "profit":

$$Y_1 = Vp_1 - C_1 \tag{7}$$

where V is the given value of the prize and p_1 is determined as in equation (2). A similar equation holds of course for player #2. Suppose momentarily it were the case that $C_1 = C_2 = 0$, the parties sharing the prize equally without fighting. Then, assuming only that $V > 0$, under the Cournot assumption either player would be motivated to defect, since even the smallest finite commitment of resources makes the defector's relative success jump from 50% to 100%. In effect, the marginal profitability of i's contest contribution is infinite when $C_i = 0$.

In contrast, when the logistic Contest Success Function applies, two-sided peace may easily hold as a stable Cournot solution. Since the player who defects from $C_1 = C_2 = 0$ does not get the benefit of a discrete jump from 50% to 100% success, there is a finite marginal gain to be balanced against the marginal cost of contest effort.[8]

Numerical Example 1

Player #1 seeks to maximize his profit as in equation (7), with p_1 defined by the logistic CSF equation (4) above. If $C_2 = 0$, then finding the derivative in the usual way leads to:

$$\frac{k \exp\{-kC_1\}}{(1+\exp\{-kC_1\})^2} = \frac{1}{V}$$

For $C_1 = 0$ to be a solution, we must have $V = 4/k$. By symmetry, an analogous equation will hold for player #2. So if, for example, $k = .04$ and $V = 100$, then (as claimed) $C_1 = C_2 = 0$ will indeed be a Cournot equilibrium. In this equilibrium $p_1 = p_2 = .5$ so that the parties each have profit of 50.

4. Asymmetrical equilibrium

What about the possibility of *one-sided submission* rather than two-sided peace? This means that player #1 (say) chooses $C_1 > 0$ while player #2 sets $C_2 = 0$. For such an outcome, some kind of asymmetry must be introduced – in the parties' valuations of the prize, in the effectiveness of their respective contest efforts, or possibly in the costs of such efforts. But regardless of any such asymmetries, under the ratio model one-sided submission as a Cournot equilibrium can no more occur than could two-sided peace!

We need look only at asymmetries due to inequalities in valuations of the prize. Specifically, suppose $V_1 > V_2$, suggesting that there might be a Cournot equilibrium with $C_1 > 0$ while $C_2 = 0$. Using the profit equation (7) for player #1, and equation (2) for the Contest Success Function in ratio form, the first-order condition is:

$$\frac{\partial Y_1}{\partial C_1} = \frac{V_1^m C_1^{m-1} (C_2^m)}{(C_1^m + C_2^m)^2} - 1 \tag{8}$$

Evidently, whenever $C_2 = 0$ the marginal profit of contest effort to player #1 is always negative. So under the ratio form of the CSF, it will never be possible to have an asymmetrical contest outcome with one party having zero and the other having positive commitment of resources.

For the difference form (logistic CSF), however, the asymmetrical outcomes are quite different. First of all, taking the partial derivatives of the respective Contest Success Functions leads to:

$$\frac{\partial Y_1}{\partial C_1} = \frac{k\exp\{k(C_1+C_2)\}}{(\exp\{kC_1\} + \exp\{kC_2\})^2} = \frac{\partial Y_2}{\partial C_2} \tag{9}$$

This possibly surprising proposition states that at any pair of C_1, C_2 choices, the partial derivatives (the "marginal products" of the respective contest efforts) are always the same for both sides. It follows immediately that, if the valuations V_1 and V_2 are unequal, it is impossible to simultaneously satisfy (as the respective first-order conditions for a profit maximum would require):

$$\frac{\partial Y_1}{\partial C_1} = 1/V_1 \quad \text{and} \quad \frac{\partial Y_2}{\partial C_2} = 1/V_2 \tag{10}$$

Thus, when the difference form of the CSF applies, there cannot be an *interior* asymmetrical Nash-Cournot solution. (Whereas, we have just seen, using the ratio form there cannot be a *corner* asymmetrical solution.)

This impossibility theorem for the difference form is somewhat too strong, since it is an artifact of the assumption implicit in equation (7) that the Marginal Cost of contest effort is constant. If the Marginal Cost of contest effort is rising, equations (10) might be satisfied so as to permit an interior Nash-Cournot equilibrium. More generally, with rising Marginal Cost there could be either a corner or an interior asymmetrical solution, depending upon the numerical parameters and the exact functional form.[9]

Assuming for simplicity that Marginal Cost is constant as in equation (7), the Reaction Curves RC_1 and RC_2 associated with the logistic CSF are parallel straight lines of 45° slope, up to a point of discontinuity. More specifically, in the continuous range the Reaction Curve equations are:[10]

$$\begin{aligned} C_1 &= C_2 + A_1, \text{ where } A_1 = (2/k)\cosh^{-1}\{.5\mathrm{sqrt}(kV_1)\} \\ C_2 &= C_1 + A_2, \text{ where } A_2 = (2/k)\cosh^{-1}\{.5\mathrm{sqrt}(kV_2)\} \end{aligned} \tag{11}$$

The discontinuities fall into three distinct patterns – depending upon the relative positions of the points C_i^0, C_i', and C_i'' as sketched in Figures 3 through 5 – each leading to a particular class of Cournot solution.

The pattern of Reaction Curves RC_1 and RC_2 pictured in Figure 3 represents the "strong asymmetry" case, which stems from a relatively large difference $V_1 - V_2$ between the parties' valuations of the prize. Here RC_2, the Reaction Curve for the lower-valuing player, rises as C_1 increases – but only

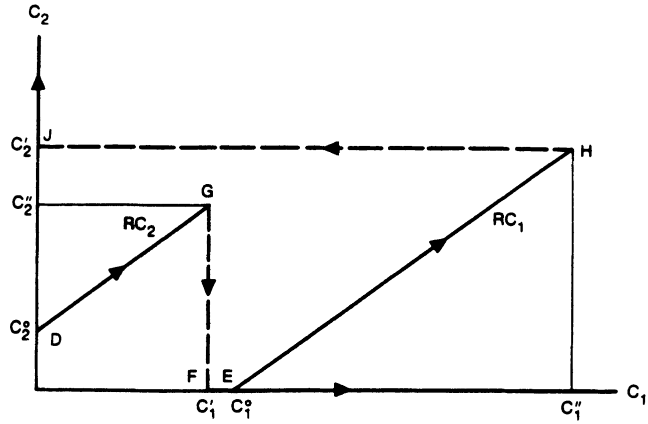

Figure 3. Logistic reaction curves: Strong asymmetry

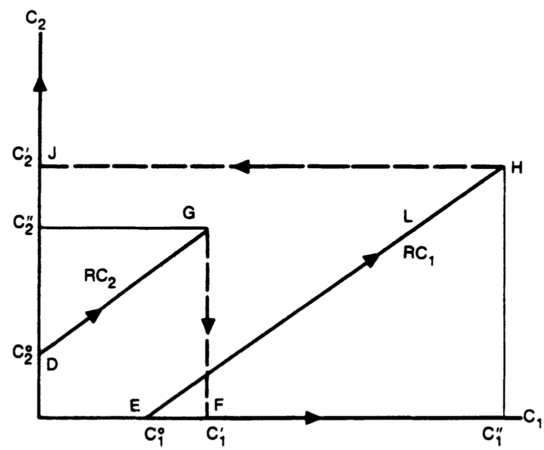

Figure 4. Logistic reaction curves: Moderate asymmetry

up to point G where the opponent's effort has reached a certain critical value C_1'. At G, player #2's optimum drops off discretely to $C_2 = 0$ (point F), and of course remains at zero for all higher values of C_1. (The explanation is that, given a logistic CSF, the lower-valuing player can always take home *some* profit by investing zero effort. Hence doing so always remains a viable alternative, and eventually becomes more advantageous than trying to keep up with very large contest efforts on the part of his higher-valuing opponent.) If, as in Figure 3, $C_1' < C_1^0$ – that is, point F is to the left of point E, the latter being the point where the higher valuing player's Reaction Curve RC_1 intercepts the horizontal axis – then the Nash-Cournot equilibrium is at E, where (C_1, C_2)

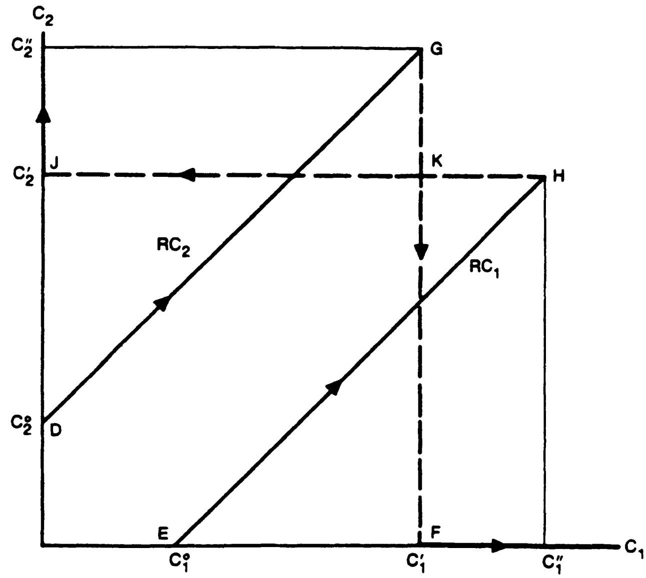

Figure 5. Logistic reaction curves: Symmetry or near-symmetry

$= (C_1^0, 0)$. It is easy to verify that, at point E, each player's effort is a *best response* to the opponent's choice. This solution represents one-sided submission: the lower-valuing player has abandoned the struggle.

Numerical Example 2

Once again each player seeks to maximize his profit $Y_1 = Vp_1 - C_1$, where p_1 is defined by equation (4) above. Let the required asymmetry be in the valuations of the prize, where specifically $V_1 = 400$ and $V_2 = 100$. Assuming $k = .04$, the Reaction Curves are as pictured in Figure 3, with $C_1^0 = A_1 = 65.848$ and $C_2^0 = A_2 = 0$. If the higher-valuing player #1 takes $C_2 = 0$ as given, his profit-maximizing solution C_1 equals $A_1 = 65.848$ (point E). Turning to player #2, with $C_1 = 65.848$ taken as given the profit-maximizing[11] C_2 is indeed $C_2 = 0$.

The expectations on each side as to the other party's behavior being mutually consistent, this is a Cournot equilibrium. The associated shares are $p_1 = .933$, and $p_2 = .067$, and the profits are $Y_1 = 307.4$ and $Y_2 = 6.699$. Note that the higher-valuing player does *disproportionately* better: not only is his prize worth more, but he fights harder for it.

Figure 4 illustrates a "moderate asymmetry" pattern. Here, the difference between V_1 and V_2 being smaller, point E (the horizontal intercept of RC_1) lies to the left of point F (at the discontinuity along RC_2). In consequence, point

E, where player #2 unilaterally submits, is no longer a Cournot equilibrium. (That is, player #2 will no longer choose $C_2 = 0$ as his best response to player #1's choice of $C_1 = C_1^0 = A_1$.) As the prize valuations V_1 and V_2 approach equality, finally, the "symmetrical or near-symmetrical" pattern of Figure 5 is obtained. Here also, it will be evident, unilateral submission will not occur. The actual solutions for both the Figure 4 and the Figure 5 patterns involve mixed strategies on one or both sides,[12] but the specifics of these solutions are not of immediate concern to us.

As the next step, it would be natural to ask whether the ratio versus the difference forms of the Contest Succes Function lead to correspondingly different outcomes in terms of the Stackelberg or other asymmetrical solution concepts. I will not, however, be pursuing these implications here.

5. Conclusion

In analyzing rent-seeking or other conflict competitions, models allowing relative success to respond continuously to changes in contest commitments have heretofore assumed that success must be a function of the *ratio* of the parties' resource commitments. However, this assumption is inconsistent with the observation that two-sided peace or one-sided submission do sometimes occur in the world. When relative success is postulated to stem instead from the *numerical difference* between the respective contest inputs, a Contest Success Function taking the form of a logistic equation is derived. Two-sided peaceful outcomes emerge in Cournot equilibrium when the "mass effect parameter" of the logistic CSF curve is sufficiently low. One-sided submission can also occur when there is a large disparity between the parties' valuation of the prize. As these valuations approach equality, the logistic CSF leads to mixed-strategy Cournot equilibria.

Notes

1. See, e.g., Hillman and Katz (1984), Corcoran and Karels (1985), Higgins, Shughart and Tollison (1985), Appelbaum and Katz (1986), Allard (1988), Hillman and Samet (1987).
2. A recent paper of Hillman and Riley (1988) makes use of still another family of contest payoff functions, in which – in contrast with the sharing rules analyzed here – the entire prize, as in an auction, goes to the high bidder. Their paper also allows for differing prize valuations.
3. In the standard Lanchester equations of military combat (Lanchester, 1916 (1956); Brackney, 1959), the outcome is also assumed to depend upon the *ratio* of the forces committed. But for Lanchester the battle result is always fully deterministic, in the sense that the side with larger forces (adjusted for fighting effectiveness) is 100% certain to win. This makes the CSF a step function, which jumps from $p_1 = 0$ to $p_1 = 1$ when $C_1 = C_2$. So Lanchester's formula can be regarded as the limiting case of equation (2) as the mass effect parameter m goes to infinity.

The same holds also for the auction-style payoffs in Hillman and Riley (1988).

4. As seen in the previous footnote, the Lanchester equations of combat take this to the extreme. The larger force is 100% certain of victory; the smaller force has no chance at all.
5. Compare T.N. Dupuy's study of diminishing returns in combat interactions between Allied and German forces in World War II (Dupuy, 1987: Ch. 11). Dupuy's curves generally show the inflection point displaced slightly from the "equal forces, equal success" point, owing (on his interpretation) to the superior unit effectiveness of the German army.
6. I thank David Levine and Michele Boldrin who independently discovered this generalization of the logistic Contest Success Function.
7. The definition of the logit in this context is:

$$p_i = \frac{f(C_i)}{\Sigma_j f(C_j)}$$

Using only the general properties of logit functions, Dixit (1987) obtained some important qualitative results for "strategic" (non-Nash) behavior in asymmetrical contests.

8. For the analogous result in a general-equilibrium context, see Hirshleifer (1988, Part B).
9. Dixit (1987) appears to assume, incorrectly, that all logit functions do lead to an interior Nash-Cournot asymmetrical equilibrium.
10. Player #1 maximizes $Y_1 = p_1 V_1 - C_1$ where p_1 is given by:

$$p_1 = 1/(1 + \exp\{k(C_2 - C_1)\}) \equiv 1/D \text{ (writing D for the denominator)}$$

For given C_2, the first-order condition $dY_1/dC_1 = 0$ is:

$$V_1 (dp_1/dC_1) \equiv kV_1 \exp\{k(C_2 - C_1)\}/D^2 = 1$$

Rearranging and taking square roots leads to:

$$\text{sqrt}(kV_1) = \exp\{(-k/2)(C_2 - C_1)\} + \exp\{(k/2)(C_2 - C_1)\}$$

Since $\cosh x \equiv .5\{\exp(x) + \exp(-x)\} \equiv \cosh(-x)$, we can write:

$$.5 \text{ sqrt } (kV_1) = \cosh\{-(k/2)(C_2 - C_1)\} \equiv \cosh\{(k/2)(C_2 - C_1)\}$$

Thus: $(k/2)(C_2 - C_1) = \cosh^{-1}\{.5 \text{ sqrt } (kV_1)\}$.

11. As suggested by the preceding discussion, this optimum is not at a smooth maximum (zero first derivative). Instead, player #2's profit function has a negative first derivative throughout, leading him to cut back effort until the limit of zero is reached.
12. The key feature guaranteeing existence of a Nash-Cournot equilibrium is that the payoff functions are continuous, even though the Reaction Curves have discontinuities. See Debreu (1952) and Glicksberg (1952). I thank Eric S. Maskin for this point.

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