"The Influence of the Numerical Strength of Engaged Forces in Their Casualties," by M. Osipov

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TRANSLATOR'S NOTES

This is an English translation of the five-part series of articles that M. Osipov published in 1915 in the Russian journal *Voenniy Sbornik* (*Military Collection*) under the title "Vliyaniye Chislennosti Srazhayushchikhsya Storon na Ikh Poteri" ("The Influence of the Numerical Strength of Opposed Forces on Their Casualties"). These articles appeared in the following issues of *Voenniy Sbornik*:

Part One, Issue No. 6, June 1915, pp. 59–74
Part Two, Issue No. 7, July 1915, pp. 25–36
Part Three, Issue No. 8, August 1915, pp. 31–40
Part Four, Issue No. 9, September 1915, pp. 25–37
Part Five (Addendum), Issue No. 10, October 1915, pp. 93–96

This major work spans a total of 55 pages and contains nine numbered sections, in addition to an unnumbered preface and an addendum. It includes 19 numbered equations, six numbered tables in addition to a list of battles, four numbered examples, and 10 numbered problems.

We have undertaken this translation because we believe that Osipov's work is so important historically and methodologically that it should be made accessible in English. The translators recognize that their work is not perfect, hope that any mistakes will not be seriously misleading, and solicit constructive suggestions for improvement. Some recent and highly laudatory Soviet comments on Osipov's work are included in Appendix B. These appeared in the September 1988 issue of the *Soviet Military-Historical Journal* [12].

The Question of Priority

The Soviets argue that Osipov discovered both the differential equations commonly known as Lanchester's equations and the relation known as Lanchester's square law. We hope the following sketchy remarks will shed some light on the question of priority.

Naval Research Logistics, Vol. 42, pp. 435–490 (1995)

CCC 0894-069X/95/030435-56

To begin with, we note that interest in a scientific theory of warfare was very much in the air during the 19th and early 20th centuries. Writers such as Clausewitz, Jomini, Bloch, Fuller, Mahan, and many others sought to comprehend the basic principles or scientific laws governing warfare and the interaction of combat forces. This concern was sufficiently widespread to be reflected in literary works. For example, Tolstoy [10] wrote "The spirit of an army is the factor which multiplied by the mass gives the resulting foce. To define and express the significance of this unknown factor—the spirit of an army—is a problem for science. Ten men, battalions, or divisions fighting fifteen men, battalions, or divisions, conquer—that is, kill or take captive-all the others, while themselves losing four . . . Consequently, the four were equal to the fifteen, and therefore $4x = 15y$. Consequently $x/y = 15/4$. This equation does not give us the value of the unknown factor but gives us a ratio between two unknowns. And by bringing variously selected historic units (battles, campaigns, periods of war) into such equations, a series of numbers could be obtained in which certain laws should exist and might be discovered." In addition, other investigators worked at compiling statistics to support such explorations (Livermore, Berndt, Bodart, and others).

Dr. Kipp [4] also pointed to earlier Russian interest in military statistics and applied mathematics in military affairs, and in this connection mentioned especially the work of D.A. Miliutin and N.N. Obruchev, who were respectively the Minister of War under Alexander II and Chief of the General Staff under Alexander II, Alexander III, and Nicholas II (see *Voenniya Statistika*—St. Petersburg, *Izdatel'stvo General'nogo Shtaba, 1871*), and that of Nikolai Volotsky on probability theory and the setting of ammunition norms for infantry and artillery (*Voenniy Sbornik, 1903–1904*).

But to focus on more directly related works, we first observe that Bradley A. Fiske's prize-winning work, as reported by Weiss [11], appeared in 1905 [2] and contained the essential ideas of a discrete version of the Lanchester-Osipov equations. Wayne Hughes has recently ferreted out the strange story of J.V. Chase's results, which appeared in 1902 and may have been the first in the field, but were not made available until 1972.[1]

F.W. Lanchester's well-known book *Aircraft in Warfare: The Dawn of the Fourth Arm*, printed by Constable & Co., appeared in London in January of 1916. However, earlier portions (specifically Chapters V and VI) had appeared in the British journal *Engineering* during the months of September through December 1914. The famous N-square law appeared in the *Engineering* issue dated 2 October 1914. Lanchester's original drafts of his book were prepared in the winter of 1913–14.

For comparison, Osipov's articles appeared in June through October of 1915. In our opinion the structure, scope, and astonishing originality of the work itself testify to a lengthy period of intense contemplation. In addition, on the basis of internal evidence, we believe that Osipov had no direct or specific knowledge of either Lanchester's or Fiske's work, and that his achievements are independent of theirs. In support of this view, let us point out that Osipov explicitly claims priority in the last sentence of his Preface (and, less explicitly, elsewhere in his series of articles), and that the absence of

[1]Professor Wayne Hughes of the U.S. Naval Postgraduate School has called attention to a footnote in a book Fiske published in 1916, which mentions the results obtained by Rear Admiral Jehu Valentine Chase using "an application of the calculus." It turns out that Chase's staff paper was written in 1902, but only declassified in 1972. A copy of it appears in Appendix C of the U.S. Naval Institute Press 1988 republication of Bradley A. Fiske's 1916 book, *The Navy as a Fighting Machine*, which includes an Introduction by Professor Wayne P. Hughes, Jr.

references to prior work as well as the tenor of his entire presentation lend credence to this view.

Magnitude of Osipov's Work

Even if it should turn out that Osipov knew of Fiske's or Lanchester's work (e.g., through Lanchester's articles in *Engineering*), Osipov's unique contributions are nevertheless significant and deserve to place him at the forefront of those interested in the theory of combat. For example, we find here for the first time in a published work explicit formulas for the solution to Lanchester's coupled differential equations. Osipov obtains this solution by a unique method that to our knowledge has not appeared elsewhere either before or since. Also, we find a unique method of deriving Lanchester's square law from this solution, as well as the usual derivation by first eliminating the time parameter from Lanchester's differential equations. Osipov also explicitly relates the differential equations to the difference equations and demonstrates that the solutions of the latter approach those of the former as the time step tends to zero. (This result was later rediscovered independently by Engel [11]).

Another unique feature for its time is Osipov's treatment of nonhomogeneous forces. Starting by solving for the survivors in the case of forces consisting of a single type of unit (namely, infantry armed only with rifles), Osipov successively introduces other types of weapons, specifically artillery cannon and machine guns. As they are introduced, Osipov defines conversion factors for relating artillery and machine guns to infantry equivalents, and on the basis of historical information estimates that one cannon is equivalent to about 100 infantrymen. His approach here is conceptually the same as that used in many of today's aggregated-force models [9] except that Osipov strives to obtain numerical estimates for his conversion factors from historical data.

Osipov also knows that real battles seldom last until one side is annihilated, and explicitly hypothesizes that a side will be forced to abandon the battle when it reaches a certain percent casualties—which Osipov estimates on the basis of historical evidence at roughly 20%. This concept too is often used today, even though it is nowadays well known to be inadmissible [3]. In addition, Osipov examines certain optimal allocation of force issues, such as whether it is better to split one's forces to oppose each component of an opponent's divided forces, whether to engage forces piecemeal or all at once, etc.

But Osipov's most unique and important contribution is the explicit and systematic application to quantitative historical data of what, for his time, were fairly advanced formal statistical methods. Osipov tests hypotheses and fits theoretical parameters to empirical observations in a thoroughly modern spirit. The outstanding achievement of this approach is the formulation of Osipov's law. This states that if we let A, A' and B, B' be the initial and final strengths of the sides A and B (respectively), write

$$A^n - A'^n = B^n - B'^n,$$

and consider values of the exponent n equal to either $\frac{3}{2}$ or 2, then we find that the value of n that best fits the empirical data is $n = \frac{3}{2}$. Osipov analyzes 10 different possible reasons why the exponent fails to be equal to 2. As far as we know, nothing comparable to this appeared in the literature for another 40 years, until Joseph Engel published his article analyzing the degree of agreement between Lanchester's equations and the battle of Iwo Jima [1].

Who Was Osipov?

Unhappily, we know nothing of M. Osipov, the author of this remarkable work.[2] We do not even know his full first name; how old he was when he wrote these articles; whether he survived the foreign and domestic wars, social upheavals, and postrevolutionary attacks on intellectuals and bourgeoisie that racked Russia in the first half of this century; or what other materials he may have published. We do not know what his profession was. In these articles, Osipov himself states that he has no practical military experience—but then displays a familiarity with various Russian Field Service Regulations and planning factors such as the percent of a unit's troops that would be committed in the assault echelon, the ratio of cannon to infantry, and the doctrinal spacing of troops in assault ranks. Similarly, while disclaiming any expertise in military history, Osipov is often able to cite pertinent historical examples to illustrate his points and displays a general familiarity with military history. Osipov refers to an engineer's handbook for tables of hyperbolic functions and displays a very solid mathematical and statistical analysis capability bespeaking what for his time would have been a very advanced technical education. He also writes very elegantly and with a large vocabulary, possibly indicating a scholarly background. Osipov complains of a lack of time to develop the subject and a hope to return to it "after peace is restored." Was he, perhaps, a young scholar-turned-officer hastily recording his work for posterity while training his unit and preparing to accompany it to the front? What else would explain his persistent complaints about the "press of events"? We would welcome further information regarding M. Osipov.

General Background

It may be helpful to the general reader to summarize some of the major events that, in Osipov's and Lanchester's day, would have been recent history. To begin with, there had been major developments in military technology. Machine guns, long-range rifled field cannons, better infantry rifles, armored battleships, airplanes, barbed wire, and various noxious or poisonous gases had been introduced into the war by the time their publications appeared in the open literature (poison gas was first used in the battle of Bolimov, on the Eastern Front, 31 January 1915).

The major powers had competed with each other in an arms race and had forged many entangling alliances and interlocking mutual defense agreements. Russia had been defeated in the Russo-Japanese War (1904–1905); President Theodore Roosevelt mediated the peace treaty negotiations. As a result of the Italo-Turkish War (1911–1912), Italy had seized the formerly Turkish possessions of Libya, Rhodes, and the Dodecanese Islands. Serbia was the chief beneficiary of the First (1912–1913) and Second (1913) Balkan Wars, at the expense of Turkey and Bulgaria.

Serbia also made little secret of her designs on the neighboring lands occupied by Serbian ethnic groups but controlled by the Austria-Hungarian Empire. This area specifically included Bosnia, which presently is part of Yugoslavia. Bosnia, which had been

[2]Dr. Jakob Kipp tells us (personal communication dated 2 June 1987) that the only World War I era Osipov with an interest in applied science and mathematics he has been able to identify was Ivan Pavlovich Osipov (born 1854), who was appointed Director of the Kharkov Technological Institute in 1915. He also points out that, while General A. A. Golovin's study of military statistics (*Nauka o Voine*, Paris, Signal, 1938) has a long section on Otto Berndt, it does not even mention Osipov. Dr. Kipp tells us that he has "looked long and hard" for more information on the M. Osipov, who authored these remarkable articles, but without success.

part of the Ottoman Empire since 1463, had been formally assigned to the Austria-Hungarian Empire by the Congress of Berlin at the close of the Russo-Turkish War of 1877–1878. However, this claim was not pressed firmly until 1908, when Austria-Hungary annexed Bosnia and established a new constitution to govern the area. The Slavic and Muslim ethnic groups in Bosnia resented this intrusion of non-Slavic Christians, and were sympathetic to Serbia's Pan-Serbian overtures.

On 28 June 1914 the Austrian heir presumptive (Archduke Franz Ferdinand) and his wife were both assassinated while visiting Sarajevo, the capital of Bosnia, during a military inspection tour of that province. The assassin, although a Bosnian citizen, was an ethnic Serb controlled by the officer heading Serbia's military intelligence organization. Serbia argued that this officer had exceeded his authority and was acting alone, but Austria-Hungary decided to exploit the situation to solve once and for all the tension between it and Serbia. On 23 July 1914, after making sure of Germany's backing, Austria-Hungary delivered an ultimatum to Serbia. The Serbs agreed to most of its provisions but balked at accepting those which called for the dismissal of unnamed Serbian officials who were to be identified later by Austria-Hungary, and for proceedings in Serbia by Austria-Hungarian officials against organizations working against Austria-Hungarian interests. Serbia did offer to submit the whole matter to international arbitration, but on 28 July 1914 Austria-Hungary rejected that and declared war.

One by one other nations were drawn into the conflict. Without trying to trace all of the actions and counteractions, we note that Russia, which was committed to support Serbia, ordered full mobilization on 30 July 1914. Germany, backing Austria-Hungary, declared war on Russia on 1 August 1914. By the end of August, Austria-Hungary and Germany were at war against England, France, Russia, Belgium, and Serbia. Still other countries were drawn into the war as it dragged on.

Russian forces initially advanced against the German forces in East Prussia and defeated them at Gumbinnen (17–20 August 1914). However, throughout the rest of 1914 and most of 1915 they suffered through a series of major defeats or inconclusive but costly actions. Here we list the battles of Tannenburg (26–30 August 1914), Galician Campaign (23 August–2 September 1914), Masurian Lakes I (28 September–17 October 1914), Lodz (11 November–6 December 1914), Masurian Lakes II (7–21 February 1915), and Gorlice-Tarnow (2 May–27 June 1915). The Russians also consumed or lost vast quantities of arms, ammunition, food, and other supplies. This led to their shell famine of 1915. As Dr. Kipp pointed out to us, Osipov's articles appeared during the German breakthrough at Gorlice-Tarnow, and the German advance continued throughout the summer and fall of 1915 until halted by the famous Russian winter weather. A high-command shake-up bears witness to the strain imposed on the Russian army by this crisis. Although the Russian commander in chief (Grand Duke Nicholas) had kept their army from being surrounded by the Germans, in September 1915 he was relieved of his responsibilities on the German front. He was sent to the Caucasas to fight the Turks while his cousin Czar Nicholas II personally assumed the position of commander in chief of Russia's armies.

In March 1917 Czar Nicholas II was deposed, and in November of 1917 the Bolsheviks seized power. They promptly declared a unilateral armistice, which was ratified on 15 December 1917 by a formal agreement signed at Brest-Litovsk. However, this was not the end of the matter, because Germany breached the terms of the armistice by invading the Ukraine and resuming their advance in the Baltic nations and in Belorussia. Finally, on 3 March 1918, the signing of the Treaty of Brest-Litovsk ended all hostilities between Russia and the Central Powers led by Germany and Austria-Hungary.

Nevertheless, civil or international war involving the Russians continued through the Russian Civil War of 1918–1921, the Finnish War of Independence (1918–1920), the Russo-Polish War of 1919–1920, the Latvian War of Independence (1919–1920), the Lithuanian War of Independence (1918–1920), and the Estonian War of Independence (1917–1920) until 1921, when the Bolsheviks were firmly in control of the entire country and external quarrels subsided.

Technical Notes

We corrected several obvious minor errors in the text, formulas, tables, and calculations without individually identifying them. However, the calculational slips in some of the tables were let stand because to correct them would make Osipov's discussion too hard to follow. They make little or no difference in his conclusions. Appendix C furnishes corrected versions of these tables and compares them to the incorrect versions.

It is important to interpret correctly Osipov's use of the term *sistyematichyeskix oschibok*, which we have translated as *systematic errors*. This phrase was deliberately chosen to give some of the flavor of Osipov's slightly old-fashioned terminology. In today's parlance, this would be called the systematic bias, or the average discrepancy between theory and observation.

In all cases, passages emphasized by italics in the translation are emphasized in the original. Translator's interpolations for clarity or asides are placed in square brackets. [like this—Tr.].

The original articles are divided into parts that do not coincide with the start and end of Osipov's sections. As a guide to the original, we have noted where one part starts and ends.

The Russian *poteri*, which in today's Soviet-English dictionaries is usually translated as *losses*, has in this translation been interpreted as *casualties*. This seems to be consistent with Osipov's intention.

We have also translated the Russian *spisochnyya chislennostey* as *roster numbers*, as it appears to correspond roughly to what today's Western military organizations refer to as the total number carried on the unit or muster rolls. Similarly, we translated the Russian *aktivnkh chislennostey* as *active numbers*, which appears to refer to the number actually engaged in combat.

We also translated the Russian *polevoy ystavi* and *ystavi*, which literally mean Field Service Regulations and regulations or manuals, respectively, as *doctrine*, since in these articles that is close to its intent.

"THE INFLUENCE OF THE NUMERICAL STRENGTH OF ENGAGED FORCES ON THEIR CASUALTIES," BY M. OSIPOV

Part I

Part I appeared in Voenniy Sbornik, Issue No. 6, 59–74 (June 1915). The present World War quite naturally raises some very basic questions of military art: What are all of the principal causes or circumstances on which success in battle depends? History shows that usually, though far from always, victory is won by the side with numerical superiority. A closer examination of this question leads to a modification of this statement; that is: More often than not, success is won by the side that managed to engage superior force. Thus, the number of troops is very significant.

Hence, it is clear that we want to understand the role of the numbers that are engaged, and especially the influence of the numerical superiority of a side on its casualties. The latter question is of interest because many basic principles of military art are deducible as corollaries of it (see Section 8, infra).

A fundamental purpose of tactics is to provide a set of rules for achieving the greatest success in an engagement with the smallest forces and the least casualties. Since the effective troops represent the principle value of a military organization, it is clear that the notion of least casualties refers mainly to casualties in personnel killed, wounded, taken as prisoners, and so forth. At the same time, the objective of an engagement consists precisely in inflicting maximum casualties on the enemy. Hence, it is clear that the issue of casualties to contending sides is of great significance for military matters, and it is already possible to foresee that proper resolution of the issue of casualties will turn out to be closely connected to fundamental aspects of tactics and strategy. A full resolution of such issues is impossible due to the extreme diversity of the conditions of particular battles; therefore, from all of these conditions, we select only numerical strength (of riflemen, artillerists, machine gunners, and so forth) and settle for trying to perceive the laws relating casualties to numbers. The simplest hypothesis in this respect is that casualties and strengths are inversely proportional in the course of small time intervals. The deductions from this assumption are carried out by means of a series of formulas connecting strengths and casualties in some ideal conditions. Verification of this hypothesis against the outcomes of historical battles shows that for small armies (not over 75,000) the assumed hypothesis is, generally speaking, indeed close to the truth, but that considerably closer to the truth is the hypothesis that casualties to a side are inversely proportional to the square root of its numerical strength. In any case, however, simple inverse proportionality of casualties and numerical strength predicts an exponent larger than $\frac{3}{4}$ for the casualties on each side, given the casualties to the other. For all cases of inversely related strengths and casualties in accordance with any permissible rule, we can derive a great many basic principles of tactics and strategy, as consequences of such inverse dependence. Since these basic principles are validated by the experience of all military history, then conversely we conclude that an inverse relation of casualties to strengths may be viewed as a consequence of the observations of military history. Thus, the relation between numerical strength and casualties is of great interest from a military point of view. Since this issue apparently has not been dealt with in the military literature, we resolve to set forth some reflections on the matter, along with their implications and consequences; but first we stipulate that, due to the press of current events, there is a possibility of a great many errors and omissions.

Section 1. Preliminary Considerations

The first issue to be addressed is: Does there exist a relationship between the strengths of the opponents and their corresponding casualties in battle? A resolution of this broadly stated question is best approached through data from military history. Below we give a list [List No. 1—Tr.] of 38 battles of the 19th and 20th centuries, predominantly the most significant and notable ones, with the numerical strengths of the engaged forces and their casualties. This list includes neither battles between regular troops and irregular [literally, unorganized—Tr.] detachments of primitive [literally, uncivilized—Tr.] peoples, nor those battles where one side was protected by a fortress or strongly entrenched (for example, Port Arthur, Plevna, Sevastopol, and so forth). In addition, we tried to include examples from all of the major campaigns of the last century.

List No. 1

No.	Battle name	Side	A	a	P	Side	B	b	P
1	***Austerlitz****, 1805	Allies	83	27	–	***French****	75	12	–
2	Jena, 1806	***French****	74	4	–	Prussians	43	12	15
3	***Auerstadt****, 1806	Prussians	48	8	4	***French****	30	7	–
4	Investment of Eylau, 1807	***French****	80	25	–	Russians	64	26	–
5	Friedland, 1807	***French****	85	12	–	Russians	60	15	–
6	Aspern, 1809	***Austrians****	75	25	–	French	70	35	–
7	Wagram, 1809	***French****	160	25	–	Austrians	124	25	–
8	Borodino, 1812	***French****	130	35	–	Russians	103	40	–
9	Berezhina, 1812	***Russians****	75	6	–	French	45	15	20
10	***Lutzen****, 1813	***French****	157	15	–	Allies	92	12	–
11	***Bautzen****, 1813	***French****	163	18	–	Allies	96	12	–
12	***Dresden****, 1813	Allies	160	20	10	***French****	125	15	–
13	Katzbach, 1813	***Allies****	75	3	–	French	65	12	18
14	Kulm, 1813	***Allies****	46	9	–	French	35	10	12
15	***Dennewitz****, 1813	French	70	9	9	***Allies****	57	9	–
16	Leipzig, 1813	***Allies****	300	50	–	French	200	60	30
17	***Hanau****, 1813	***French****	75	15	–	Allies	50	9	–
18	***Craonne****, 1814	***French****	30	8	–	Russians	18	5	–
19	Laon, 1814	***Allies****	100	2	–	French	45	6	3
20	Ligny, 1815	***French****	120	11	–	Prussians	85	18	–
21	Waterloo, 1815	***French****	100	22	–	French	72	32	–
22	Grochow, 1831	***Russian****	72	9	–	Poles	56	12	–
23	Alma, 1854	***Allies****	62	3	–	Russians	34	6	–
24	Chernaya River, 1855	***Allies****	60	2	–	Russians	56	8	–
25	***Inkerman****, 1854	Russians	90	12	–	***Allies****	63	6	–
26	***Magenta****, 1859	Austrians	58	10	–	***French****	54	5	–
27	***Solferino****, 1859	Austrians	170	20	–	***French****	150	18	–
28	Custozza, 1866	***Austrians****	70	8	–	Italians	51	8	–
29	Koeniggratz, 1866	***Prussians****	222	10	–	Austrians	215	43	–
30	***Worth****, 1870	***Germans****	100	10	–	French	45	5	9
31	Mars-la-Tour, 1870	***French****	125	16	–	Germans	65	16	–
32	***Gravelotte****, 1870	***Germans****	220	20	–	French	130	12	–
33	Sedan, 1870	***Germans****	245	9	–	French	124	17	107
34	Metz, 1870	***Germans****	200	6	–	French	173	20	153
35	Aladja, 1877	***Russians****	60	2	–	Turks	36	15	7
36	Liaoyang, 1904	Russians	150	18	–	***Japanese****	120	24	–
37	***Sha-Ho****, 1904	Russians	212	40	–	***Japanese****	157	20	–
38	Mukden, 1905	Russians	330	59	31	***Japanese****	280	70	–
		Total	4652	603	54	Total	3363	692	374
Correction for victorious side crossover:			−260	−37	−54		+260	+37	+54
Total for the victors:			4392	566	–		3623	729	428

Notes: (1) A and B are the number of combat troops on each side, a and b are the numbers of casualties wounded or killed, and P is the number of prisoners (all in thousands). (2) Where casualties on the numerically stronger side were higher than the casualties on the weaker side, the battle name is boldfaced [literally, stressed; these are also italicized and asterisked in the translated copy—Tr.]. (3) The victors are boldfaced [literally, stressed; these are also italicized and asterisked in the translated copy—Tr.]. (4) The numerical values are taken from G.A. Leyer's *Encyclopedia of Military and Naval Science* [Entsiklopediya Voennyx" e Morskyx" Nauk], except for the last three which are taken from the article *History of Russia's Army and Fleet* [Estorey Russkoiy Armey e Flota], published by the Society "Education" [Obrazovaniya]. Thus, there is no question of our picking numbers haphazardly [literally, at random—Tr.].

From the layout of this list of battles, it is obvious that of the 38 cases, in 28 victory fell to the side shown on the left-hand side [of List No. 1—Tr.], that is, to the side with superiority in numbers, and the weaker side won in only 10 cases; this indicates that the numerically weaker side wins only in 1 case out of 4. Examining next the number of battles with names in boldface [asterisks and italics as well as boldface type used in this translation—Tr.], observe that in 14 cases of 38 the larger force suffers the most casualties; while of the remaining 24 cases, 3 have equal casualties on each side, and in 21 cases, the larger force suffers the least casualties; that is, we observe that there is an inverse relationship between casualties and numbers; this is also apparent in the totality of numbers and casualties. Thus, roughly speaking, the casualties are distributed in such a way that generally the larger force suffers the least casualties, rather than the weaker side. However, in each individual battle casualties depend not only on the number of troops but also on a great many other conditions: morale, effective use of one's own forces [tactics—Tr.], better quality armament, heavier artillery, terrain, fortifications, and so forth. The enumerated factors then influence the number of casualties on each side in particular battles; but if we group many battles by some category, for example, by the strength of one of the sides, their aggregate effect on the summed numbers will be influenced by many factors that will increase or decrease casualties, but which favor alternately one side and then the other, and thus by cancellation tend to nullify each other so that the result depends principally on the numerical strengths; that is why this list of battles is ordered by numerical strength.

Section 2. The Simplest Procedure for Calculating Casualties

Having decided that casualties are in some rough way related to numerical strength, we will try to obtain this relation empirically. In order to address this question, it is necessary at first to limit our considerations to definite situations, beginning with the simplest cases, and turning gradually to more complex ones. Therefore, in our investigations we will assume initially that forces having unequal numbers are engaged with each other, but are completely equal in all other respects; that is, each are equally brave, skillful, and armed, located parallel to each other in the form of skirmish lines, with identical local conditions, having the same density, and so forth. If riflemen find a rank of the enemy facing them, then the number of enemies hit by them will depend only on the number of rounds they fire, and not on the number of enemies brought under fire. If we let A be the number of riflemen in our skirmish line, then in a unit of time the enemy casualties will be equal to some percentage of our rifle strength A, and will not depend on the strength B of the enemy's riflemen. Conversely, at the same time our own casualties will equal that same percent of the strength B of the enemy's riflemen, and will not depend on the strength of our riflemen A. However, if these considerations are repeated successively for a whole series of moments following one after the other, then the casualties of each side will depend not only on the strength of its enemy, but also on the strength of its own riflemen, although to a lesser degree. All this is much better explained by numerical examples.

EXAMPLE 1: Suppose that under the above conditions [of equality of circumstances—Tr.] the numbers $A = 1000$ of riflemen and $B = 800$ riflemen engage in a battle, where each rifleman in each unit of time diminishes the enemy by 4% of one man (that is, by one man in 25 units of time). It is required to trace the progress of casualties as time passes. Under the given conditions, in the initial unit of time, side A will lose $0.04B = 0.04 \times 800 = 32$ men, and side B will lose $0.04A \times 1000 = 40$ men. Therefore, at the start of the second unit of time, side A will have left $1000 - 32 = 968$ men, and side B will have left $800 - 40 = 760$ men. In the course of the second unit of time, A will lose $0.04 \times 760 = 30$ men and B will lose $0.04 \times 968 = 39$ men, and so at the start of the third time unit A and B will have left (respectively) $968 - 30$

Table 1.

Time unit	Number on sides at start of time unit		Casualties to sides in the course of this time unit		Total casualties to sides since the start of the battle		Numbers on sides at end of the given time interval	
	A	B	a	b	Σa	Σb	A'	B'
1	1000	800	32	40	32	40	968	760
2	968	760	30	39	62	79	938	721
3	938	721	29	38	91	117	909	683
4	909	683	27	36	118	153	882	647
5	882	647	26	35	144	188	856	612
6	856	612	24	34	168	222	832	578
7	832	578	23	33	191	255	809	545
8	809	545	22	32	213	287	787	513

= 938 and 760 − 39 = 721 men. In the course of the third time unit, casualties to the sides will be 29 and 38 men, and they will have left $A_3' = 909$ and $B_3' = 683$ men, and so forth. The results of such calculations for the first eight time units are given in Table 1.

If we do similar calculations with time units that are four times smaller, so that the number of hits per time interval is four times smaller and is equal to only 0.01, then the results will be as shown in Table 2.

Comparing columns A and B of these two tables, we see that the remaining strengths for A and B at 5, 9, 13, . . . , 29 time units in the second table differ only slightly from the remaining strengths at 2, 3, 4, . . . , 8 time units in the first table. Reducing the chosen time unit still further, we would obtain a third table of casualties and remaining strengths which would hardly differ from those of the second table, and so forth. In general, the remaining strengths for smaller time units will clearly approach some limit. This circumstance suggests that numbers and remaining strengths on a side in the course of a battle are in a fixed relationship to each other, independent of the selection of time

Table 2.

Units of time		Numbers	Casualties	Numbers	Casualties
New	Old	A	$a = 0.01B$	B	$b = 0.01A$
1	1	1000	8.0	800	10.0
2	...	992	7.9	790	9.9
3	...	984	7.8	780	9.8
4	...	976	7.7	770	9.8
5	2	969	...	760	...
⋮	⋮	⋮	⋮	⋮	⋮
9	3	939	...	722	...
13	4	911	...	685	...
17	5	884	...	649	...
21	6	858	...	614	...
25	7	834	...	580	...
29	8	811	...	547	...

unit or the hit rate for each rifleman per unit of time. Obviously, we want to have exact formulas for obtaining accurate results, independent of the choice of time unit and without resorting to tables like the foregoing, which require a lot of time.

In order to proceed, we must admit in advance that a lot of mathematics will be used. But it is necessary to recall that the matter concerning determination of the connections between the numerical strengths of the combatants and their casualties is also expressed in numbers. How shall this information be properly expressed, unless mathematically? Military history can give the basic numbers, but explaining their relation is the domain of mathematics. Nevertheless, for the reader's convenience, all that is of more interest for military affairs will be singled out in standard [literally, large—Tr.] typeface letters; and all that is not directly related to military affairs, although important in the sense of establishing the logical necessity of the conclusions (that is, the mathematics), will be printed in small typeface letters.

Section 3. Derivation of the Simpler Formulas for Determining Casualties

Assuming that, in a very small span of time, casualties a and b to the sides are inversely proportional to their numerical strengths, the remaining strengths of the sides $A' = A - a$ and $B' = B - b$ at the expiration of an arbitrary time will be found in very simple relation to the initial strengths A and B, namely,

$$A'^2 - B'^2 = A^2 - B^2; \tag{1}$$

that is, the difference in the squares of the numbers of combatants stays constant throughout all phases of the battle. For example, if $A = 1000$ and $B = 800$, then the remaining strengths of the sides at the termination of an arbitrary lapse of time and independent of the hits caused by rifles, will always be given by

$$A'^2 - B'^2 = 1000^2 - 800^2 = 600^2.$$

This was approximated by the results for the remaining strengths in Tables 1 and 2 of Section 2 above. Furthermore, as we shall see below (see Sections 4 and 6), when applying formula (1) to examples of military history, one easily convinces oneself that in its place we can take simply:

$$Aa = Bb \tag{1-bis}$$

Although formula (1) can be obtained altogether more easily with the help of the integral calculus,[3] we nevertheless give an alternative demonstration during which other formulas will be obtained incidentally.

[3]If dA and dB are the casualties of the sides in time dt, and α is the hits caused in a unit of time, then $dA = \alpha\, dt \times B$, and $dB = \alpha\, dt \times A$. Eliminating $\alpha\, dt$ from these, we get: $A\, dA = B\, dB$. If A (respectively A') and B (respectively B') are the initial and final numerical strengths of the sides, then integrating the latter equations between their limits we get

$$A^2 - A'^2 = B^2 - B'^2.$$

Let A and B denote the initial numerical strengths of riflemen in the ranks of the combatants, α the hits caused by one rifleman in a unit of time, t the time at the expiration of which must be found the remaining strengths A' and B' of the riflemen of the engaged sides, finally let Δt be an infinitesimally small interval of time. For identical conditions [for both sides—Tr.], the remaining strengths of the engaged sides at the expiration of Δt, $2\,\Delta t$, $3\,\Delta t$, . . . , $n\,\Delta t$ will be as follows:

Time interval	Remaining on side A at its conclusion	Remaining on side B at its conclusion
Δt	$A_1' = A - (\alpha\,\Delta t)B$	$B_1' = B - (\alpha\,\Delta t)A$
$2\,\Delta t$	$A_2' = A - 2(\alpha\,\Delta t)B + (\alpha\,\Delta t)^2A$	$B_2' = B - 2(\alpha\,\Delta t)A + (\alpha\,\Delta t)^2B$
$3\,\Delta t$	$A_3' = A - 3(\alpha\,\Delta t)B + 3(\alpha\,\Delta t)^2A - (\alpha\,\Delta t)^3B$	$B_3' = B - 3(\alpha\,\Delta t)A + 3(\alpha\,\Delta t)^2B - (\alpha\,\Delta t)^3A$

And in this fashion I compose terms similar to those of Newton's binomial formula:

$$n\,\Delta t\;\begin{cases} A_n' = A - n(\alpha\,\Delta t)B + \dfrac{n(n-1)}{1\cdot 2}(\alpha\,\Delta t)^2A - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\alpha\,\Delta t)^3B + \cdots \\ B_n' = B - n(\alpha\,\Delta t)A + \dfrac{n(n-1)}{1\cdot 2}(\alpha\,\Delta t)^2B - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\alpha\,\Delta t)^3A + \cdots \end{cases}$$

Multiplying the terms of the last expression by $(n/n)^0$, $(n/n)^1$, $(n/n)^2$, . . . and observing that $n\,\Delta t = t$, and that the fractions $1/n$, $2/n$, $3/n$, . . . reduce to zero as n tends to infinity, it is not difficult to derive the following formulas:

$$A' = A - B(\alpha t) + \frac{A}{1\cdot 2}(\alpha t)^2 - \frac{B}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{A}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 - \cdots,$$

$$B' = B - A(\alpha t) + \frac{B}{1\cdot 2}(\alpha t)^2 - \frac{A}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{B}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 - \cdots. \tag{2}$$

These formulas are easily simplified if we observe that the series

$$1 + (\alpha t) + \frac{1}{1\cdot 2}(\alpha t)^2 + \frac{1}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 + \cdots,$$

$$1 - (\alpha t) + \frac{1}{1\cdot 2}(\alpha t)^2 - \frac{1}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 + \cdots \tag{2-bis}$$

represents the base of the Naperian logarithm $e = 2.718281828$ . . . raised to the powers $+\alpha t$ and $-\alpha t$. If in (2) we combine separately the terms containing A and the terms containing B, then we get the formulas:

$$A' = A\frac{e^{\alpha t} + e^{-\alpha t}}{2} - B\frac{e^{\alpha t} - e^{-\alpha t}}{2},$$

$$B' = B\frac{e^{\alpha t} + e^{-\alpha t}}{2} - A\frac{e^{\alpha t} - e^{-\alpha t}}{2}. \tag{3}$$

In the last formula (3) the quantities $\frac{1}{2}(\exp(\alpha t) + \exp(-\alpha t))$ and $\frac{1}{2}(\exp(\alpha t) - \exp(-\alpha t))$ are well known in mathematics as the hyperbolic cosine and sine of the quantity (αt) and formula (3) may be written as

$$A' = A\cosh(\alpha t) - B\sinh(\alpha t),$$

$$B' = B\cosh(\alpha t) - A\sinh(\alpha t). \tag{4}$$

It is necessary for calculations to use values of the hyperbolic sines and cosines, which we give in the tables immediately below (taken from the book *Handbook for Engineers*, Hutte).

Values of $\cosh(\alpha t) = \frac{1}{2}(e^{\alpha t} + e^{-\alpha t})$

(αt)	0	1	2	3	4	5	6	7	8	9
0.0	1.0000	0001	0002	0005	0008	0013	0018	0025	0032	0041
0.1	1.0050	0061	0072	0085	0098	0113	0128	0145	0162	0181
0.2	1.0201	0221	0243	0266	0289	0314	0340	0367	0395	0423
0.3	1.0453	0484	0516	0549	0584	0619	0655	0692	0731	0770
0.4	1.0811	0852	0895	0939	0984	1030	1077	1125	1174	1225
0.5	1.1276	1329	1383	1438	1494	1551	1609	1669	1730	1792

Values of $\sinh(\alpha t) = \frac{1}{2}(e^{\alpha t} - e^{-\alpha t})$

(αt)	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0100	0200	0300	0400	0500	0600	0701	0801	0901
0.1	0.1002	1102	1203	1304	1405	1506	1607	1708	1810	1911
0.2	0.2013	2115	2218	2320	2423	2526	2629	2733	2837	2941
0.3	0.3045	3150	3255	3360	3466	3572	3678	3785	3892	4000
0.4	0.4108	4216	4325	4434	4543	4653	4764	4875	4986	5098
0.5	0.5211	5324	5438	5552	5666	5782	5897	6014	6131	6248

For example, suppose $A = 1000$, $B = 800$, $\alpha = 0.04$, and $t = 5$ units of time. Then $\alpha t = 0.20$, hence from the tables above $\cosh(0.20) = 1.0201$, $\sinh(0.20) = 0.2013$, $A' = 1020.1 - 161.0 = 859$, $B' = 816.1 - 201.3 = 615$. The corresponding remaining strengths obtained by us in the first table of §2 were 856 and 612, that is, very close to the exact results. In conclusion, formulas (4) have no need for approximate calculations and are the exact theoretical expressions for determining the remaining strengths on the sides A and B. Formulas (4) are inconvenient for making comparisons to data from military history since they contain α and t, which generally are not known. In order to avoid this difficulty, we add and subtract formulas (3) to obtain:

$$\left.\begin{aligned} A' + B' &= (A + B)e^{-\alpha t} \\ A' - B' &= (A - B)e^{+\alpha t} \end{aligned}\right\} \tag{4-bis}$$

Multiplying one of these expressions by the other, we obtain formula (1), namely

$$A'^2 - B'^2 = A^2 - B^2.$$

For example, if we have $A = 1000$ riflemen and the casualties to side B are equal to 185 men, then since $B' = B - b = 800 - 185 = 615$ men, it follows from formula (1) that $A' = \sqrt{A^2 - B^2 + B'^2} = \sqrt{1000^2 - 800^2 + 615^2} = 859$ men, which is completely in accord with the computations mentioned above using formula (4).

EXAMPLE 2: What is the remaining strength of the stronger side, if the weaker is destroyed without any survivors?

To solve this problem set $B' = 0$ and solve for A' from (1), finding: $A' = \sqrt{A^2 - B^2}$. For example, if $A = 1000$, $B = 800$, then $A' = \sqrt{1000^2 - 800^2} = 600$ men.

Thus a constant difference between the squared numerical strengths A and B is equal to the square of the untouched surviving strength of the side.

In deriving formula (1) we did not take into account that with significantly different frontages the flanks of the stronger side will not have any targets facing them. Although this case would certainly be complicated to solve, its possibility should be taken into account. However, resolving such new issues would only introduce complexity, and would not make the calculations more reliable, since in real battles formations are seldom continuous—to say nothing about the influence of artillery, envelopments, and other factors. Therefore, nothing will be said at first regarding its complete and exact resolution, and we will base the theory on the original postulates, which are sufficiently plausible for moderate or small differences in strength; then testing the theory on sample battles will provide an empirical evaluation of it. Then the theory's validity or lack of it, whichever is the case, provides a basic measure for evaluating the relation of casualties to numerical strengths, and after deciding on the best formula we can add appropriate more realistic conditions.

Therefore, continuing to develop the previous theory, we shall introduce now a new factor—unequal skill of riflemen or quality of armaments and the resultant inequality of hits inflicted by the individual riflemen on each of the engaged sides. To illustrate, we solve a problem of this type:

EXAMPLE 3: Taking opponents $A = 1000$ riflemen and $B = 800$ riflemen, hits by side A equal $\alpha = 0.05$, and hits by side B equal $\beta = 0.04$ per time unit. What will be the remaining strengths A' and B' after four units of time in the battle?

We begin the resolution of this issue with the method of step-by-step iteration of the strengths and casualties (see Example 1).

No. of time units	Numbers A	Casualties $a = 0.04B$	Numbers B	Casualties $b = 0.05A$
1	1000	32	800	50
2	968	30	750	48
3	938	28	702	47
4	910	26	655	46
Final	$A' = 884$		$B' = 609$	

The general case can be solved, and fully and accurately at that, with the help of formulas similar to Eq. (1). The derivation of this formula is not presented here, since in general outline it is similar to the one previously cited. Here are the formulas:

$$\sqrt{\alpha}A' = \sqrt{\alpha}A \cosh(t\sqrt{\alpha\beta}) - \sqrt{\beta}B \sinh(t\sqrt{\alpha\beta}),$$

$$\sqrt{\beta}B' = \sqrt{\beta}B \cosh(t\sqrt{\alpha\beta}) - \sqrt{\alpha}A \sinh(t\sqrt{\alpha\beta}). \tag{5}$$

Here α and β are the hits caused by the riflemen on sides A and B in one unit of time. In applying this formula to the resolution of the previously solved problem we take: $A = 1000$, $B = 800$, $\alpha = 0.05$, $\beta = 0.04$, and $t = 4$ time units. Then $t\sqrt{\alpha\beta} = 0.1799$, $\sqrt{\beta/\alpha} = 0.8944$, $\sqrt{\alpha/\beta} = 1.180$, $\cosh(0.1789) = 1.0160$, $\sinh(0.1789) = 0.1789$, and hence $A' = 887$ and $B' = 612$.

By doing these computations, it is easy to convince oneself that the preparation of tables of casualties as a function of time, as was done above, is far easier and will give results differing but little from the precise ones (884 and 609 versus 887 and 612).

Formula (5) can be simplified and put in the form of formula (1): For this it is necessary to transform, observing that formulas (5) are in the form of formula (4), as we see when we substitute αt for $t\sqrt{\alpha\beta}$, A for $A\sqrt{\alpha}$, B for $B\sqrt{\beta}$, A' for $A'\sqrt{\alpha}$, and B' for $B'\sqrt{\beta}$; therefore, the latter more complicated expression translates into formula (4), which would give remaining numerical strengths if the hits were $\sqrt{\alpha\beta}$ (in place of the α in formula (4)); hence, replacing in formula (1) the numerical strengths and survivors A, B, A', B', with their new expressions, we get

$$\alpha(A^2 - A'^2) = \beta(B^2 - B'^2), \text{ or}$$

$$A^2 - A'^2 = \frac{\beta}{\alpha}(B^2 - B'^2). \tag{6}$$

Applying this formula to the preceding problem, taking $A = 1000$, $B = 800$, $B' = 612$, $\alpha = 0.05$, and $\beta = 0.04$, we find that

$$A'^2 = A^2 - \frac{\beta}{\alpha}(B^2 - B'^2),$$

from which we get $A' = 887$, that is, completely in agreement with the results of formula (5).

Section 4. Formulas for Determining Casualties Taking into Account Artillery and Machine Guns

In Section 3 we assumed that the combatants consisted solely of riflemen and did not take artillery, which causes the combatants much greater casualties, into consideration. In order to take artillery into consideration, we have to solve the following problem: We have two combatants, one with A riflemen and M cannons, and the other with B riflemen and N cannons. Hits will be assumed in one unit of time to equal α for each rifleman and β for each cannon. Furthermore, we still assume that casualties befall only riflemen but not cannon, the number of which then remain constant with respect to time into the battle (for cannon crews see Section 5). The question is: How many riflemen remain on each side at the end of t units of time into the battle?

The derivation of formulas for this, although complex, will basically be similar to the derivation of formulas (2)–(4) and like them starts by deriving the surviving numbers after the passage of infinitesimally small times dt, $2\ dt$, $3\ dt$, . . . and so forth. Therefore, a demonstration will not be presented, but the resulting formula for determining the remaining strengths A' and B' of the sides after the passage of t units of time into the battle is

$$\left(A' + \frac{\beta}{\alpha}M\right) = \left(A + \frac{\beta}{\alpha}M\right)\cosh(\alpha t) - \left(B + \frac{\beta}{\alpha}N\right)\sinh(\alpha t),$$

$$\left(B' + \frac{\beta}{\alpha}N\right) = \left(B + \frac{\beta}{\alpha}N\right)\cosh(\alpha t) - \left(A + \frac{\beta}{\alpha}M\right)\sinh(\alpha t). \tag{7}$$

Comparing this formula with (4), we see that they are identical, except that the quantities A, A' and B, B' are increased by the amounts $M\beta/\alpha$ and $N\beta/\alpha$. These latter terms represent the hypothetical number of riflemen which could replace the effects of M and N cannon. Thus, if we want to take into account the effects of artillery, then we can use the earlier formula, except that the number of riflemen must be increased by the number of artillery equivalents.

If, in addition to artillery (M and N cannon with hit rates β), the combatants also have P and Q machine guns with hit rate γ, then the numbers on the sides must be reckoned as

$$A + M\beta/\alpha + P\gamma/\alpha$$

and

$$B + N\beta/\alpha + Q\gamma/\alpha,$$

where γ/α is the coefficient for conversion of machine guns into riflemen equivalents.

Completely analogously to our derivation in Section 3 of formula (1) from formula (4), we can derive from formula (7) the following:

$$\left(A + \frac{\beta}{\alpha}M\right)^2 - \left(A' + \frac{\beta}{\alpha}M\right)^2 = \left(B + \frac{\beta}{\alpha}N\right)^2 - \left(B' + \frac{\beta}{\alpha}N\right)^2. \tag{8}$$

With this formula one can determine either A' from a knowledge of A, B, M, N, B', and β/α; or the numerical coefficient β/α from a knowledge of A, B, M, N, and the casualties to each side a and b. For the latter purpose, it is necessary to expand formula (8) and to cancel common terms, after putting $A' = A - a$ and $B' = B - b$, whereupon we will have

$$(A^2 - A'^2) = (B^2 - B'^2) - 2\frac{\beta}{\alpha}(aM - bN). \tag{9}$$

EXAMPLE 4: Determine the coefficient for converting artillery cannon into riflemen equivalents for the sample battles Borodino, Lutzen, and Waterloo, for which we have the data given below. The solution of this problem depends on the assumed law of casualties, and therefore we present three solutions of it.

	Sides with the most artillery			Sides with the least artillery		
Battle name	A (thousands)	a (thousands)	M (cannon)	B (thousands)	b (thousands)	N (cannon)
Borodino	100	40	640	130	35	590
Lutzen	92	12	650	157	15	350
Waterloo	72	32	240	100	22	200
Total	264	84	1,530	387	72	1,140

SOLUTION 1: If it is assumed that the law of casualties is as in formula (1), then the coefficient $x = \beta/\alpha$ will be determined by formula (9), in which one must substitute the numerical strengths, casualties, number remaining, and the number of cannon, taking all in units of 1000. Thus we will have $A' = 264 - 84 = 180$, $B' = 387 - 72 = 315$, $M = 1.53$, and $N = 1.14$. Substituting these values into formula (9) yields $x_1 = \beta/\alpha = 143$, or 150 in round numbers. This means that one cannon is nearly as effective as a company of riflemen.

SOLUTION 2: Below we will see (see Section 3 and Comment 3 of the present section) that the casualty ratio can be calculated using the formula (1-bis), that is, $Aa = Bb$. Taking here $A = 264 + 1.53x$, $B = 387 + 1.14x$, $a = 84$, and $b = 72$, we get $(264 + 1.53x)/(387 + 1.14x) = 72/84$, and hence $x_2 = 123$.

SOLUTION 3: The observations for 38 battles of Section 1 indicates (see Section 6) that the law of casualties is given by formula (12-bis); that is, $a\sqrt{A} = b\sqrt{B}$, considerably better than formula (1) or (1-bis). From this we get

$$(264 + 1.53x)/(387 + 1.14x) = 72^2/84^2,$$

from which it follows that $x_3 = 59$ [We get 29.35—Tr.].

From this we see that similar determinations of the coefficient $x = \beta/\alpha$ are unreliable. Note that this results partly from our inexact law of casualties, partly from the fact that this law is subject to large variations for individual battles, partly because the number of riflemen relative to guns and cannon are not in accord with the reported values, partly because the organizations of armies for combat are so nearly all the same that the solutions to our equations are always close to the form 0/0, and so forth.

A more practical determination of the coefficient β/α would be to take the ratios: (1) number hit by artillery to the number hit by rifle fire, and (2) the number of cannon shots to the number of rifle shots by each combatant side, and then to divide the former by the latter.

There is yet another interesting implication of formulas (7) and (8). Let us suppose the stronger side presents A riflemen on the front and C on a turned flank, while the weaker presents only B riflemen on the front. Since combating an enveloping maneuver is difficult, then for simplicity it can be assumed that the flanking element C is not exposed to B's fires, but itself—as in the case of artillery cannon—merely causes hits $\beta = m\alpha$, that is, m times the strength of the frontal fires. Then from the moment the flank is turned the number of survivors A' and B' on the sides is determined by the formulas:

$$A' + mC = (A + mC)\cosh(\alpha t) - B\sinh(\alpha t),$$

$$B' = B\cosh(\alpha t) - (A + mC)\sinh(\alpha t),$$

$$(A + mC)^2 - (A' + mC)^2 = B^2 - B'^2. \qquad (10)$$

We derived these formulas in order to show that there is an increase in the number of survivors A' and a decrease in B' for the case where m is greater than 1, i.e., when the flank is turned, than in battles with a frontal attack by $A + C$ riflemen with B riflemen, for which m equals 1.

More generally, we may suppose that the two combatants have (1) A riflemen with hits α, M cannon with hits γ, P machine guns with hits ϵ (per unit time) and (2) B riflemen with hits β, N cannon with hits δ, and Q machine guns with hits ζ.

Then for the numbers A' and B' left after the passage of t units of time into the battle we will have the following formulas:

$$\sqrt{\alpha}\left(A' + \frac{\gamma}{\alpha} M + \frac{\epsilon}{\alpha} P\right) = \sqrt{\alpha}\left(A + \frac{\gamma}{\alpha} M + \frac{\epsilon}{\alpha} P\right) \cosh(t\sqrt{\alpha\beta})$$

$$- \sqrt{\beta}\left(B + \frac{\delta}{\beta} N + \frac{\zeta}{\beta} Q\right) \sinh(t\sqrt{\alpha\beta}),$$

$$\sqrt{\beta}\left(B' + \frac{\delta}{\beta} N + \frac{\zeta}{\beta} Q\right) = \sqrt{\beta}\left(B + \frac{\delta}{\beta} N + \frac{\zeta}{\beta} Q\right) \cosh(t\sqrt{\alpha\beta})$$

$$- \sqrt{\alpha}\left(A + \frac{\gamma}{\alpha} M + \frac{\epsilon}{\alpha} P\right) \sinh(t\sqrt{\alpha\beta}),$$

$$\left(A' + \frac{\gamma}{\alpha} M + \frac{\epsilon}{\alpha} P\right)^2 = \left(A + \frac{\gamma}{\alpha} M + \frac{\epsilon}{\alpha} P\right)^2$$

$$- \frac{\beta}{\alpha}\left[\left(B + \frac{\delta}{\beta} N + \frac{\zeta}{\beta} Q\right)^2 - \left(B' + \frac{\delta}{\beta} N + \frac{\zeta}{\beta} Q\right)^2\right]. \tag{11}$$

We conclude this collection of formulas, relating to the theory of casualties, with a few comments.

COMMENT 1: All of the derived formulas are corollaries of explicit assumptions. These include an assumption that in an infinitely short span of time, casualties to the sides will be inversely proportional to their numerical strength, that is, that $dA \times A = dB \times B$. If we were to base the theory of casualties on the assumption that in an infinitely short time casualties are inversely proportional to the square root of the numerical strengths, that is, that $dA\sqrt{A} = dB\sqrt{B}$, then in place of formula (1), we would have instead

$$A^{3/2} - A'^{3/2} = B^{3/2} - B'^{3/2}. \tag{12}$$

With other hypotheses we would get yet other formulas.

COMMENT 2: The composition of formulas (4) and (5), into which enter the rarely encountered hyperbolic cosines and sines, were deduced by us to complete our research by giving exact solutions for cases with prescribed initial conditions. For approximate solutions it will suffice to compile the following tables:

(1) in place of formula (4)—a table similar to that used in Example 1.
(2) in place of formula (5)—a table similar to that used in Example 3.

COMMENT 3: In the same way, in place of formula (1), in which the squares of large numbers enter, we can usually use the following formula

$$Aa = Bb, \tag{1-bis}$$

although, if casualties to the weaker side are higher than 20 to 25%, it is better to use the exact formula (1). The derivation of formula (1-bis) is based on the exact formula

(1); that is, $A^2 - A'^2 = B^2 - B'^2$, which can be written in the form $(A - A')(A + A') = (B - B')(B + B')$, where $A - A' = a$ and $B - B' = b$, and where $A + A'$ and $B + B'$ for moderate values of a and b will be nearly proportional to the quantities A and B. This is also the basis for the following simple alternate to formula (6):

$$\alpha A a = \beta B b. \tag{6-bis}$$

Also, we observe that we can get for formula (12) an alternate, and can use simply

$$a\sqrt{A} = b\sqrt{B}. \tag{12-bis}$$

Section 5. Taking Reserves into Account and Substituting the Roster Number for the Number of Actives

Up to this point we have assumed that battles are carried out only by riflemen ranks and artillery, while reserves have not been mentioned so far, although they too take casualties. Moreover, in formulas (7) and (8) mentioned above we used the numbers of active fighters A and B and the numbers M and N of effective emplaced cannon, although most historical accounts give only the side's roster numbers C and D and their casualties, a and b, and also sometimes the roster numbers M and N of cannon. This raises the question: What is the practical significance of the derived formulas if in place of the generally unknown *active numbers* $A + M\beta/\alpha$ and $B + N\beta/\alpha$ in (7) and (8) we simply use the *roster numbers* C and D? Will this replacement not cause gross errors? To resolve this issue we assume some idealized conditions, namely that the organizations, doctrines, and tactics of the opponents are the same, and consequently [we will have the proportion—Tr.] $C{:}A{:}M = D{:}B{:}N$, and the only difference will be in the total numerical size of the force. Deviations from these conditions will of course alter the dependence of the casualties a and b on C and D.

Part II

Part II appeared in Voenniy Sbornik, Issue No. 7, 25–36 (July 1915). First of all it is necessary to derive a relation between the numbers of actives $A + \beta M/\alpha$ and $B + \beta N/\alpha$ and the roster numbers C and D.

A and B are the numbers of riflemen in the engaged ranks. At the start of the battle they are equal to approximately $\frac{1}{5}$ of the total riflemen, which, aside from noncombatants and other arms, equals about $\frac{2}{3}$ of the total numbers C and D; by the end of the battle almost all [of the riflemen—Tr.] have taken part, therefore on the average $A = (0.2 + 1.0)/2 = 0.6$ of all the riflemen, or $A = (0.6 \times \frac{2}{3})C = 0.4C$. Analogously, $B = 0.4D$.

The quantities M and N are about equal to 0.004 times the number of riflemen, or $M = 0.004 \times \frac{2}{3}C$, that is, roughly $M = 0.003C$, and $N = 0.003D$. The coefficient β/α for converting cannon into riflemen equivalents is generally not known, but according to the data of Example 4 of Section 4 above it can be calculated as bounded between 60 and 150, consequently $(\beta/\alpha)M$ equals $0.18C$ to $0.45C$; in exactly the same way $(\beta/\alpha)N$ is bounded between $0.18D$ and $0.45D$. Hence we see that the numbers of actives $A + \beta M/\alpha$ and $B + \beta N/\alpha$ are bounded between $0.58C$ and $0.85C$ and $0.58D$ to $0.85D$.

If we took into account such active numbers, then from formula (1) we would get just a relation of casualties a_1 and b_1 for riflemen ranks, but casualties a_2 and b_2 to reserves, artillery, and so forth would not yet be dealt with. In order to do that, we can assume that casualties a_2 and b_2 result from the operation of the hypothesized active fractions kC and kD mentioned earlier, where k is a moderate fraction, since casualties outside of the ranks are appreciably less than those in the ranks. Thus for use in formula (1) the active numbers must be reckoned as from $(0.58 + k)$ to $(0.85 + k)$ of the roster numbers. We will reckon them as from 0.6 to 1.3 of the roster number of the whole which may be expressed as mC and mD, where m is equal to from 0.6 to 1.3. Then the relation of total casualties $a = a_1 + a_2$ and $b = b_1 + b_2$ can be calculated from formula (1). Thus, letting casualties b be given, we get

$$a = mC - \sqrt{m^2C^2 - (2mD - b)b}. \tag{13}$$

But since we generally do not know m, in examples from history it is necessary in place of the aggregate numbers mC and mD to use simply the quantities C and D, that is, to take $m = 1$, in which case the casualties to side C will be

$$x = C - \sqrt{C^2 - (2D - b)b}. \tag{14}$$

From the outcomes of 38 battles, it can be seen that on the average $D/C = \frac{3}{4}$, $b/D = \frac{1}{5}$, and therefore $b/C = \frac{3}{20}$.

Therefore, assuming b to be known and the same in formulas (13) and (14), we can calculate that $C = \frac{20}{3}b$ and $D = 5b$. Replacing C and D in formula (13) by these new quantities, we get

$$a = \frac{20}{3}mb\left[1 - \sqrt{1 - \frac{9(10m - 1)}{400m^2}}\right].$$

Giving m various values in this formula, we get finally the quantities a and $x - a$:

$m = 0.6$	$a = 0.68b$	$x - a = +0.03b$
$= 0.8$	$a = 0.70b$	$= +0.01b$
$= \underline{1.0}$	$\underline{x = a = 0.71b}$	$\underline{= +0.00b}$
$= 1.2$	$a = 0.72b$	$= -0.01b$
$= 1.3$	$a = 0.72b$	$= -0.01b$

So, for average conditions, casualties to a side can be reckoned by formula (1) from roster values of the numerical strengths C and D in place of the actives mC and mD; moreover, the errors from this will not exceed 3% of the casualties.

If we use formula (1-bis) instead of formula (1), then the entire deduction is superfluous, since for $A:M:C = B:N:D$ the equations $Aa = Bb$, $Ca = Db$, and $mCa = mDb$ follow from one another.

These results are based on the uniformity of organization and doctrine of the combatants, which, strictly speaking, is applicable only to the total numbers from many battles rather than to individual battles, as individual variations tend to cancel out in the totals, and so they are closer to the theory and less variable than an individual battle could be.

Thus, when all conditions are similar except for the numbers per se, a law of casualties that applies to the ranks can be extended to the entire force, including casualties of reserves, of artillery, and so forth, with errors not exceeding 3% of the casualties.

The theory has now been presented in sufficient detail. Without such a presentation, we could not begin to check the theory against examples from military history, because every little discrepancy that came up would involve the issue of the applicability of the theory to history. Now we can proceed with that verification, bearing in mind that theoretical discrepancies should be small only for applications of the theory to the totals for the battles in our list, rather than to individual battles.

Section 6. Tests of the Theory Using Examples from Military History

A basic premise of our theory then, is that in a battle casualties to the stronger side should be less than those of the weaker. Let us see how well this compares with military history. In Section 1 the list of 38 battles includes 14 in which the stronger side suffered more casualties than the weaker side, 3 battles where the casualties were equal for both sides, and 21 where the stronger side suffered fewer casualties than the weaker side. This amounts to 37% clearly opposed to the theory, 8% inconclusive results, and only 55% favorable to the theory and even then in a qualitative rather than a quantitative sense. Thus we are compelled to raise the question: Are there not some gross errors in the theory? A closer examination of individual battles would show that casualties depend not only on the numerical strengths of the sides, but also on many other factors (morale, artful leadership, training of the troops, weapons, terrain, and so forth). In the theory, we assumed that these factors were equal for both sides, but this is never possible to achieve. Therefore, it is not possible to require the theory to agree with every individual battle—it need be correct only for certain idealized average conditions. It remains yet to verify the theory for the sum total of numerical strengths and casualties. Since we take a large enough number of fairly well-known battles, we anticipate that the total casualties will be independent of factors which alter casualties in particular battles randomly, for the stronger as well as for the weaker (left and right sides of the list in Section 1), and so the total sum will throw into high relief the influence of numerical strength according to which we have arranged the list of battles. It is possible, too, that along with the numerical strength of the sides, there are other factors which affect casualties for just one of the weaker or the stronger side. Then tests of the theory using examples from history will indicate what changes in the assumed conditions are necessary to arrive at the best formulas.

Below is given a table of the 38 battles from Section 1, ordered not chronologically but instead by the numerical strengths A of the stronger side [see Table 3—Tr.]. The columns under the letters A, a, B, b contain the same numbers as in the list of Section 1, that is, the numerical strengths and casualties of the sides. Then we give the remainder after the battle for the weaker side, $B' = B - b$, which is needed to compute the casualties $(a)_1$ to the stronger side, according to formula (1). We use the letter (a), included in parentheses, to denote the casualties to the stronger side A, calculated by some formula or other. In our table we tested three formulas for calculating (a), namely: First of all we take the basic formula (1) and compute

$$(a)_1 = A - \sqrt{A^2 - B^2 + B'^2}.$$

Table 3.

Battle name	A	a	B	b	B'	Formula (1) $(a)_1$	v_1	v_1^2	Formula (1-bis) $(a)_2$	v_2	v_2^2	Formula (12-bis) $(a)_3$	v_3	v_3^2
Craonne	30	8	18	5	13	3	−5	25	3	−5	25	4	−4	16
Kulm	46	9	35	10	25	7	−2	4	8	−1	1	9	0	0
Auerstadt	48	8	30	7	23	4	−4	16	4	−4	16	6	−2	4
Magenta	58	10	54	5	49	5	−5	25	5	−5	25	5	−5	25
Chernaya River	60	2	56	8	48	7	5	25	7	5	25	8	6	36
Aladja	60	2	36	15	21	8	6	36	9	7	49	12	10	100
Alma	62	3	34	6	28	3	0	0	3	0	0	4	1	1
Custozza	70	8	51	8	43	6	−2	4	6	−2	4	7	−1	1
Dennewitz	70	9	57	9	48	7	−2	4	7	−2	4	8	−1	1
Grochow	72	9	56	12	44	9	0	0	9	0	0	11	2	4
Subtotal	576	68	427	85	342	59	−9	139	61	−7	149	74	6	188
Jena	74	4	43	12	31	6	2	4	7	3	9	9	5	25
Berezhina	75	6	45	15	30	8	2	4	9	3	9	12	6	36
Hanau	75	15	50	9	41	6	−9	81	6	−9	81	7	−8	64
Katzbach	75	3	65	12	53	10	7	49	10	7	49	11	8	64
Aspern	75	25	70	35	35	31	6	36	33	8	64	34	9	81
Eylau Investment	80	25	64	26	38	19	−6	36	21	−4	16	23	−2	4
Austerlitz	83	27	75	12	63	11	−16	256	11	−16	256	11	−16	256
Friedland	85	12	60	15	45	10	−2	4	11	−1	1	13	1	1
Inkerman	90	12	63	6	57	4	−8	64	4	−8	64	5	−7	49
Subtotal	712	129	535	142	393	105	−24	534	112	−17	549	125	−4	580
Laon	100	2	45	6	39	3	1	1	3	1	1	4	2	4
Waterloo	100	22	72	32	40	20	−2	4	23	1	1	27	5	25
Worth	100	10	45	5	40	2	−8	64	2	−8	64	3	−7	49
Ligny	120	11	85	18	67	12	1	1	13	2	4	15	4	16
Mars-la-Tour	125	16	65	16	49	8	−8	64	8	−8	64	12	−4	16
Borodino	130	35	103	40	63	29	−6	36	32	−3	9	36	1	1
Liaoyang	150	18	120	24	96	18	0	0	19	1	1	21	3	9
Lutzen	157	15	92	12	80	7	−8	64	7	−8	64	9	−6	36
Dresden	160	20	125	15	110	11	−9	81	12	−8	64	13	−7	49
Wagram	160	25	124	25	99	18	−7	49	19	−6	36	22	−3	9
Subtotal	1302	174	876	193	683	128	−46	364	138	−36	308	162	−12	214
Bautzen	163	18	96	12	84	7	−11	121	7	−11	121	9	−9	81
Solferino	170	20	150	18	132	16	−4	16	16	−4	16	17	−3	9
Metz	200	6	173	20	153	17	11	121	17	11	121	19	13	169
Sha-Ho	212	40	157	20	137	14	−26	676	15	−25	625	17	−23	529
Gravelotte	220	20	130	12	118	7	−13	169	7	−13	169	9	−11	121
Koennigratz	222	10	215	43	172	41	31	961	41	31	961	42	32	1024
Sedan	245	9	124	17	107	8	−1	1	9	0	0	12	3	9
Leipzig	300	50	200	60	140	36	−14	196	40	−10	100	49	−1	1
Mukden	330	59	280	70	210	57	−2	4	60	1	1	64	5	25
Subtotal	2062	232	1525	272	1253	203	−29	2265	212	−20	2114	238	6	1968
Grand total	4652	603	3363	692	2671	495	−108	3302	523	−80	3120	599	−4	2950
Sum of errors as percent of sum of losses:							−22%			−15%			−0.7%	
Number of errors greater than 0:							10			13			18	
Number of errors equal to 0:							3			3			1	
Number of errors less than 0:							25			22			19	
Probable error in the grand total of the calculated casualties:							39			38			37	

Then we take the approximate formula (1-bis), that is,

$$(a)_2 = b\,\frac{B}{A}.$$

Finally, we take the formula (12-bis), that is,

$$(a)_3 = b\sqrt{\frac{B}{A}}.$$

For inferences about the merit of some formula, we take the difference $v = (a) - a$, representing its error, and the square of this error, that is, v^2. We tally these figures of merit for each group of 9 or 10 battles. For example, for the battle of Craonne (1814) the numerical strengths and casualties of the sides in thousands were $A = 30$, $a = 8$, $B = 18$, and $b = 5$, the remainder $B' = 18 - 5 = 13$ thousand. Then formula (1) gives $(a)_1 = 30 - \sqrt{30^2 - 18^2 + 13^2} = 3$, the error $v_1 = (a)_1 - a = 3 - 8 = -5$, and $v_1^2 = 25$. Calculating with formula (1-bis) gives $(a)_2 = bB/A = 3$, $v_2 = 3 - 8 = -5$, $v_2^2 = 25$. Finally, (12-bis) gives $(a)_3 = b\sqrt{B/A} = 4$, $v_3 = 4 - 8 = -4$, $v_3^2 = 16$.

In Table 3, the grand totals in columns 2, 3, 4, 5, and 6 represent what would be the total numbers, casualties, and remaining from 38 episodes of one large battle, in which $A = 4652$, $B = 3363$, $a = 603$, $b = 692$, and $B' = 2671$. Here the casualties as a percentage of the related quantities can be calculated with much less random fluctuation than in individual battles because of the mutual cancellation of random errors in individual battles.

From the total of columns 7, 8, and 9, we see that calculations with formula (1) gave $(a)_1 = 495$ thousand in place of 603, which was the case in reality; hence this calculation is in error by -108 or by -22% of the calculated value of 495 thousand. In order to judge how probable such errors are, it is necessary to calculate the probable error of the sum 495. The standard error ϵ_1 of this sum equals[4] the square root of the sum v_1^2; that is, $\epsilon_1 = \sqrt{3302} = 57.46$, so the probable error $\rho_1 = 0.67449\epsilon_1 = 38.7$. The error of the sum 495 equals $-108/38.7 = -2.8$ probable errors. Hence, the probability that an error as extreme as -108 would occur is (see normal probability distribution tables) less than 3%, and 97% on the other hand that this error is systematic, that is, that it is due to formula (1) being incorrect. The very same conclusion results from working with the 38 errors v_1. If these errors were peculiar to the individual battle, then the number of them that are + and − would be approximately equal, but among those 38 errors v_1 we find that 25 or 66 percent are negative, 10 or 26% are positive, and 3 or 8% are equal to zero.

Formula (1-bis) gives somewhat better, though nevertheless inadequate, results. Here the sum of casualties turns out to be 523, that is, by 80 or 15% less than the real figure 603. The probable error of the obtained sum 523 amounts to $\rho_2 = 0.67449\sqrt{3120} = 37.6$; consequently, the error is equal to -2.11 probable errors, and therefore in 92 cases out of 100 this large an error will not occur by chance, so it is systematic. The distribution of signs here is also a little better than before, and specifically: positive errors amount to 13, negative 22, and 3 equal zero.

Formula (12-bis), that is, $(a)_3 = b\sqrt{B/A}$ conforms incomparably better to reality. Here the calculated sum of casualties amounts to 599 or only 4 less than the actual 603.

The probable error of the sum equals $0.67449\sqrt{2950} = 36.6$, the resultant error obtained by summing the v_3 errors is equal to -4, and is clearly random (specifically in

[4][But see Section C-4 of Appendix C—Tr.]

Table 4. Values in thousands.

Battle Nos.	A	a	B	b	$(a)_1$	v_1	v_1^2	$(a)_2$	v_2	v_2^2	$(a)_3$	v_3	v_3^2
1–10	576	68	427	85	59	−9	139	61	−7	149	74	+6	188
11–19	712	129	535	142	105	−24	534	112	−17	549	125	−4	580
20–29	1302	174	876	193	128	−46	364	138	−36	308	162	−12	214
30–38	2062	232	1525	272	203	−29	2265	212	−20	2114	238	+6	1968
Total	4652	603	3363	692	495	−108	3302	523	−80	3120	599	−4	2950

940 cases out of 1000 this would have been the case, so it is not systematic). These good results also show up in the signs of v_3: Here a total of 18 are positive, 19 negative, and 1 equals zero.

But these tests still are not convincing, since the favorable results for formula (12-bis) could be merely accidental. However, the following verification of all battles by groups confirms again the results we just got from the grand totals. Table 4 represents a summary of all battles by groups of 9 or 10 battles, extracted from Table 3.

From Table 4 we see that the first two hypotheses for $(a)_1$ and $(a)_2$ gave numbers systematically less than actuality, as seen by the fact that their errors v_1 and v_2 were all negative, while the third hypothesis gave for v_3 both positive and negative quantities, that is, the distribution of signs also was satisfactory only for the third hypothesis.

Finally, for completeness, we present also Table 5, in which $(a)_1$, $(a)_2$, and (a_3) are computed from the totals of the groups, and in the last line from the grand total for all battles. In Table 3 the new values are $(a)_1 = 576 - \sqrt{576^2 - 427^2 + 342^2} = 60$, $v_1 = 60 - 68 = -8$. Similarly, $(a)_2 = 85 \times 427/576 = 63$, $(a)_3 = 85\sqrt{427/576} = 73$, $v_2 = 63 - 68 = -5$, $v_3 = 73 - 68 = +5$, and so forth.

Hence, again we see that errors v_1 and v_2 are all negative, while v_3's signs are distributed evenly. In addition the errors v_3 are not large and in total do not exceed 2.6 percent (equal 15), and for the groups amounts to 2–16%. The errors v_1 for the sum of the group values is 130 or 27% of the computed casualties, and in the groups 8–53%.

Using cases considered in the tables, we make two comments.

COMMENT 1: Comparing the 38 quantities $(a)_1$ and $(a)_2$ of Table 3, computed by formulas (1) and (1-bis), we see that Example 3 in Section 4 is correct, since in most cases $(a)_1$ and $(a)_2$ are either equal or differ only by one.

Table 5. Values in thousands.

Battle Nos.	A	a	B	b	$(a)_1$	v_1	v_1^2	$(a)_2$	v_2	v_2^2	$(a)_3$	v_3	v_3^2
1–10	576	68	427	85	60	−8	64	63	−5	25	73	+5	25
11–19	712	129	535	142	100	−29	841	107	−22	484	123	−6	36
20–29	1302	174	876	193	121	−53	2809	130	−44	1936	158	−16	256
30–38	2062	232	1525	272	192	−40	1600	201	−31	961	234	+2	4
Total	4652	603	3363	692	473	−130	5314	501	−102	3406	588	−15	321
Calculated from grand totals:													
	4652	603	3363	692	473	−130	16900	500	−103	10609	588	−15	225

COMMENT 2: Totals for the groups are given (see Table 5, row "Total", with numbers 473, 501, and 588) for the quantities $(a)_1$, $(a)_2$, and $(a)_3$, as well as figures for total casualties computed using the grand total (see Table 5, last row, numbers 473, 500, and 588). The comparable total casualties in Table 4 do not differ sharply from those of Table 5 (495, 523, and 599 in place of 473, 501, and 588), and specifically not by more than 5%. Hence, we can conclude that, when given a few episodes of a single battle, then instead of more accurately calculating the total casualties (a) from the casualties and numerical strengths of individual episodes, we can at once compute the total casualties (a) from the sum of casualties b and from the sums of the numerical strengths of the combatants for all of the episodes. These numerical strengths can be far from being equal and can even be out of proportion to the reported numerical strengths [literally, those on the roster—Tr.]. An example of this is afforded by the case of the battle of Austerlitz (1806)—see Section 7 below, subsection on artful leadership.

Thus, using in succession formulas (1), (1-bis), and (12-bis) has shown

(1) That there are systematic errors in formulas (1) and (1-bis).
(2) That formula (12-bis) explains the dependence of casualties on numerical strength much better than the previous ones, and without systematic errors.

It follows that the list of formulas (1) to (11), since they are not in agreement with reality, are superfluous. However, we left them in for the following reasons:

FIRST: For moderate numerical strengths (not over 75,000) of the combatants formulas (1) and (1-bis) agree with reality, and can be better than (12-bis), as can be seen from Table 6's totals for battles 1 through 15, taken from Table 3.

In Table 6 the total casualties, taken from Table 3, turn out to be $(a)_1 = 120$ and $(a)_2 = 126$, and their errors are $v_1 = -1$ and $v_2 = +5$, whereas $(a)_3 = 147$ and $v_3 = +26$. If we had not computed casualties by summing up the quantities (a), but instead obtained them directly from the totals A, B and b, then we would have gotten:

$$(a)_1 = 950 - \sqrt{950^2 - 700^2 + 532^2} = 116,$$

$$v_1 = 116 - 121 = -5,$$

$$(a)_2 = 168\,\frac{700}{950} = 124,\ v_2 = +3,$$

$$(a)_3 = 168\sqrt{\frac{700}{950}} = 144, \qquad \text{and } v_3 = +23.$$

Table 6. Values in thousands.

Battle Nos.	A	a	B	b	$(a)_1$	v_1	v_1^2	$(a)_2$	v_2	v_2^2	$(a)_3$	v_3	v_3^2
1–15	950	121	700	168	120	−1	313	126	+5	361	147	+26	458

Thus, here formulas (1) and (1-bis) explain casualties better than (12-bis). This phenomena of the dependence of the error on the numerical size of the combatants is due to the increase in firing range and to the intermittent nature of the fighting by the ranks (see Section 7 below, subsection on the numerical size of the combatants). Therefore, we can accept the conclusion that in the future formula (1) might turn out to be better than (12-bis), even for numerical strengths greater than 75,000.

SECOND: The derivation of formula (1) is clear, and it is more convenient than formula (12-bis) for investigating the causes of errors in different circumstances, since the latter formula's derivation is empirical and thus less clear.

In conclusion of this section we must admit that historical examples cannot give a conclusive demonstration of the theory, since the figures given by different authors are far from being in agreement,[5] and it is not possible to investigate all possible versions of formulas analogous to (1) and (12). Therefore, the theses in Section 9 regarding the inverse dependence of numerical strengths and casualties are given only in general terms, rather than in the form of mathematical formulas.

Section 7. Causes of the Incorrectness of Formulas (1) and (1-bis)

Thus, testing formulas (1) and (1-bis) on historical examples has revealed that, while they may correspond to reality when the numerical size of the combatants is small (see Table 6), generally speaking, they are incorrect and formulas (12-bis) is in far better agreement with reality. Thus, clearly we wish to find out the cause of this.

The basic cause of errors in the theory is that there is a lack of agreement between the idealized conditions for which the formulas were derived and those realized in actual circumstances. Thus, use of formulas (1) and (4) requires that the two opponents be equal in all respects except for numerical strength; if it is to be assumed that one of them shoots or has better weapons than the other, then formulas (5) and (6) must be used; if artillery and machine guns are to be taken into consideration, then formulas (7) and (8) must be used; if both of the above are to be taken into consideration, then it is necessary to use formula (11), and so forth. But in addition to these indicated conditions of battle, there are a multiplicity of others; to enumerate, much less to take them all into consideration, would be impossible; therefore, for individual battles, no theory could predict the corresponding casualties. Thus, the influence of certain conditions of battle on the casualties to the combatants can only be examined using the aggregate of many battles. The method of such a study can be the following: Compile a list of a large

[5]For example, if we sum the casualties in killed and wounded of the stronger and the weaker sides for the 55 battles (from Marengo through Sedan) from the book "Die Zahl im Kriege," by Captain O. Berndt of the Austrian General Staff, then we get casualties $a = 472.1$ thousand and $b = 489.4$ thousand. Here figures on casualties are taken strictly according to the book, except that in their arrangement according to the strength of the sides, notwithstanding the author, we considered the Austrians to be the stronger side in the battle of Aspern (1809), rather than the French. Other corrections that would have been favorable to the theory we did not use, but it would have been impossible to go against the hypothesis in any case, since for the battles most favorable to the theory (which according to Berndt were Koennigratz, Borodino and Sedan), the stronger side was correctly given.

Thus, the new figures afford a solid basis for asserting the theory that casualties of the stronger should be less than those of the weaker, subject to the exception that casualties to the sides appeared very similar to each other for numbers A around 4,780,000 and B around 3,270,000.

number of carefully studied battles and tabulate the numerical strengths of the sides, casualties, artillery strengths, tactics used, and several other conditions that have an influence on casualties. If we want to investigate, for example, the influence of the use of offensive or defensive tactics on the casualties of the sides, it is necessary to group the battles into two columns—left for offensive and right for defensive (or vice versa) and to compute the total; then the ratio of casualties will depend principally on the kind of tactic used by the sides. This method we have already employed in testing formulas (1), (1-bis), and (12-bis), and also in Example 4 for determining the factor for converting artillery cannon [into riflemen equivalents—Tr.].

Part III

Part III appeared in Voenniy Sbornik, Issue No. 8, 31–40 (August 1915). As is customary, we divide the theoretical errors into two categories: random and systematic. In individual battles the former sometimes affect casualties quite a bit, but in the aggregate of many battles their influence is small, since these errors favor the weaker as often as the stronger. The latter, that is, systematic errors, although masked in individual battles by random errors, stand out in high relief in the aggregate, since with increased numbers of battles they accumulate rather than cancel one another out, as do the random errors. Studying the systematic errors, rather than the random errors, is more important for us, since this allows us to adjust our formulas to the conditions of the battle and to derive more accurate formulas for determining casualties.

Proceeding now to the study of errors, we have an obligation to caution that in view of the novelty of the issues and of the urgency evoked by current events, we limited ourselves to just a sketchy survey of the causes of a few of the more important errors, with the aim of explaining the acceptability of several general propositions concerning casualties and numerical strengths of the combatants.

Causes of Random Errrors

1. The Art of Leadership. This consists in knowing how to advance on the battlefield and how to bring to bear the greatest number of active troops, maintain their morale, execute the proper maneuvers, and generally in taking advantage of every circumstance.

In order to have a clear understanding about how much artful leadership can diminish casualties, we introduce the following calculation:

Adjust the list of 38 battles in Section 1 by rearranging it so that on the left are exclusively the victorious sides, while on the right are only the losers. To do this, it is only necessary to interchange the places of the sides for those 10 battles in which the winner had the smaller number of troops, making a note on the right by underlining it. By doing this we get, as shown in the last row of List No. 1 of Section 1, the following new totals: winners $A = 4392$, $a = 566$, losers $B = 3623$, $b = 729$. If with these data for A, B, and b we calculate casualties $(a)_3$ by formula (12-bis) we get $(a)_3 = 729\sqrt{3623/4392} = 662$, from which $v_3 = (a)_3 - a = 662 - 566 = +96$, which amounts to 15 percent of $(a)_3$. Since formula (12-bis) expresses casualties $(a)_3$ for equality of all conditions except numbers, then the conclusion is that casualties have been decreased by 15% from what formerly appeared in this row, by something contributing to economy of casualties. This cause is principally the result of skilled leadership, since other causes are equiprobable for both sides.

Another example, which at first seems completely opposed to the theory, but later proves to be sufficiently in accord with it, is the battle of Austerlitz (1805). In it the Allies had $A = 83$, $a = 27$, and the French had $B = 75$ and $b = 12$. If we calculate casualties of the Allies (a) from $b = 12$ and from the numbers A and B, then for all formulas we get $(a) = 11$ (see Table 3), and it is concluded that the calculated casualties are almost 2.5 times [specifically, 27/11—Tr.] the actual data, that is, very strongly opposed to the theory. However, if one reads the description of the battle, then it turns out that this increase in casualties of the enemy is due to Napoleon's skill, who guessed the Allies' intentions and met them one by one and each time exceeded them in numbers. In place of one battle of 83,000 Allies against 75,000 French, there were at least three battles:

(1) 20,000 Allies (Kutusov) against 40,000 French,
(2) 15,000 Allies (Bagration) against 20,000 French,
(3) 30,000 Allies (Buxhowen) against 40,000 French.

On the basis of the results of Section 6, for calculating casualties (a) it is possible to include all of these three episodes in a single battle and then we get:

$$(A) = 20 + 15 + 30 = 65$$
$$(B) = 40 + 20 + 40 = 100$$
$$(B') = B - b = 100 - 12 = 88.$$

Since the battle forces here are not very large detachments (less than 75,000), then it follows that it is possible to apply formula (1) and then we get

$$(a)_1 = 65 - \sqrt{65^2 - 100^2 + 88^2} = 21.$$

Now the calculated casualties agree so much better with the actual casualties that the errors of the formulas have become altogether $v_1 = (a)_1 - a = 21 - 27 = -6$, that is, they do not differ sharply from those of the other errors in Table 3.

The art of leadership does not depend on the number of soldiers, and therefore it can give rise only to random errors, which are as likely to favor the stronger as the weaker.

2. *Morale.* The side inspired with a desire for battle without needing the urging of its commander is the side that will have the greatest number of active soldiers. The side that is not animated, on the other hand, suffers not only the physical casualties (dead and wounded), but also suffers considerably greater moral losses (straggling, ineffectives, malingering, retreating, surrendering, and so forth). Bringing these losses under some law is difficult, since it is difficult to treat or to express them quantitatively, but such losses occur predominantly to the side that is taking a beating,[6] and therefore the ratio of moral losses is determined not by formula (1) but rather by (6), where the coefficient α/β, that is, ratio of the hits by the winners to the losers, must be significantly greater than 1. For example, losses due to prisoners in the total listed in Section 1 shows $a = 54$ and $b = 374$, while numerical strengths $A = 4652$, $B = 3363$. From this, on applying (6-bis), we get $\alpha/\beta = Bb/Aa = 5/1$. This relation can be applied to measure the morale of the sides A and B; it is a rather simple corollary to the law of casualties. The principal harm of weakened morale is not just in the prisoners taken, since they are fewer than

[6]This is why their fire weakens.

the killed and wounded, but rather in the number of soldiers who, while physically participating in the battle lose confidence in its successful outcome to the point where they are not what one might call active fighters. Napoleon's remark that victory in war is 3/4 dependent on the morale of the soldiers aims precisely at this kind of loss. The same impact of fear manifests itself in the eastern saying that, in time of cholera, for every death from sickness there are three others from fear. As these formulas testify, in military affairs fear of death or wounding is three times more destructive than the actual casualties in wounded and killed. From the point of view of the theory of casualties, better morale is equivalent to a large increase in numerical strength, and therefore it increases the enemy's casualties and decreases ours. Every ineffective soldier commits not only the crime of eliminating himself from the number of actives, but is especially to blame for the death of many of his comrades (see Problem 3 in Section 8). Morale is itself one of the most important causes producing large random errors in the theory of casualties, but in the aggregate it probably turns out to be better on the side with the greater number.

3. Relative Numerical Strength of Reserves. In Section 5 it was pointed out that formula (1), requiring knowledge of the number of active soldiers, can be applied also to the roster numbers, provided the number of active soldiers on each side are in the same proportion to their numerical strengths. Although this holds when organization and doctrine, and also armaments, are the same for both sides, in actuality these conditions are never ever satisfied. If one of the opponents puts a greater percent of its strength into the ranks, then it more quickly attrites a given percent of the strength of its opponent, and with less casualties to itself (see problems 1 and 2 below). If at first few are committed to action, then their ranks will be relatively weak, while the reserves will have many passive participants in the battle, the significance of which from the point of view of causing casualties to the enemy is zero. But, on the other hand, it is impossible to be without any reserves at all. So this raises the question of the best possible ratio of ranks and reserves. According to our doctrine, increasing the reserves is left to the judgmental discretion of the commander, although stronger reserves are recommended in case of uncertain intelligence of the opponent or for activity on the flanks; but conversely, when it is necessary to increase the volume of fire, the ranks are strengthened. The balance of these two opposing requirements from the point of view of the theory of casualties would be that for optimal use of our forces, it is necessary to have in reserve just so many soldiers that at the end of the battle almost all of the men will have taken part in it. If at the conclusion of the battle there remain unengaged reserves, then this indicates that we were somewhat stingy in ammunition usage (or that we decided to limit our commitment to the battle) and therefore wasted more time and suffered heavier casualties for achievement of the whole than would have been necessary with a fuller utilization of our strength.

If we try to use too long a line of fire [that is, too wide a front—Tr.], then the flanks will have no one to shoot at while at the same time we risk having no reserves left at the end of the battle and subject ourselves to the risk of loss of control of the battle at the decisive moment, and in general risk various sorts of accidents. Therefore, if in a series of earlier battles it was observed that at the end there were left uncommitted reserves, then this would serve to indicate in the future it is necessary to strengthen the ranks at the expense of the reserves (lengthening the line of fire from the very first moment of the battle), and vice versa. In general, only superiority in numbers of active soldiers will increase enemy casualties and diminish our own. Therefore, it is necessary to strive for the least possible number of inactive troops. However, if we are defending and experiencing a lack of ammunition, then naturally the final decision may be otherwise.

The relative number of active fighters as well as the length of the line of contact impact equally upon the casualties of both the stronger and the weaker side. This means, in other words, that those factors account only for random errors in calculating casualties. However, for a series of battles within the same war, they might account for a systematic error. Therefore, we included battles from several wars in our list of 38 battles.

We will speak later of the fact that when the numerical sizes of the sides are great the weaker side can make [relatively—Tr.] better use of its forces.

4. *Relative Numerical Strength of Artillery.* This is 4 to 5 cannons per 1000 riflemen. If it is the same for both sides, then casualties to the sides will be in accord with formula (1) or (12), because the number of cannons is proportional to the numerical strength.

If one side has a relatively greater number of artillery, then that would be equivalent to raising its numerical strength. And for the opposite case, vice versa. The influence of artillery is taken into account in formula (8), although example 4 of Section 4 determined that the coefficient for converting cannon into riflemen equivalents was uncertain. The cause of this was indicated in connection with example 4.

The influence of machine guns is like the influence of artillery cannon, although the coefficient for converting them into riflemen equivalents is not known.

Superiority with respect to relative numbers of cannons and machine guns is as likely to lie with the stronger as with the weaker: Therefore, in general, its influence will be a random effect.

5. *Quality of the Armaments.* This sometimes exerts a very large influence on the casualties of a side (for example, Koennigratz). The same role is played by training [literally, instruction—Tr.], organization, and doctrine. A good demonstration of the importance of this is afforded by the common recognition of the need for a professional army, which almost always comes out on top when opposed by masses of irregular forces, which was the reason why the list of battles did not include battles where one side consisted of irregular forces. This avoided systematic errors in the total of casualties.

6. *Modernity of Implements for Shielding from Hits and Kills.* Finally, there is yet the influence on casualties of new means of defense, neutralization, and destruction, but the initial influence of these devices usually is negligible, due to their novelty.

With the random errors must also be mentioned errors in the casualties and numerical strengths, which depend on the inaccuracy of figures taken almost entirely from G.L. Leyer's "Encyclopedia of Military and Naval Science" (see footnote 4 to the List No. 1). About the selection of figures, therefore, nothing more can be said.

Causes of Systematic Errors

7. *Position, Concentration and Type of Operation of Forces.* In the theory we have presented the combatants were assumed to be equal; disposed on identical terrain, and using identical tactics, that is, engaging each other with fire, making bayonet charges, sword fighting, and so forth. But, in reality, the weaker side usually defends and the stronger attacks. Therefore, if the stronger side does not execute an envelopment or a breakthrough, or does not complete its victory by an energetic pursuit, then it will bear

Table 7. Values in thousands.

Battle name	Attacker A	Attacker a	Defender B	Defender b
Worth	100	10	45	5
Mars-la-Tour	125	16	65	16
Gravelotte	220	20	130	12
Liaoyang	120	24	150	18
Sha-Ho	212	40	157	20
Mukden	280	70	330	59
Total	1057	180	877	130

$$A = 1057 \quad a = 180$$
$$B = 877 \quad b = 130$$
$$(a)_3 = b\sqrt{B}/\sqrt{A} = 118$$
$$(a)_3 - a = -62, \text{ or } 52\% \text{ of } (a)_3$$

casualties far larger than those which follow from the theory. The cause of this is clear: defenders more frequently take up advantageous positions, fortify them, and await the attacker's blow, being well covered, while their opponent, attacking, occupies inconvenient positions: By moving forward they are not shooting, and they expose themselves from head to foot. In former times, rifle fire was too weak to strongly influence the distribution of casualties, but now it has vastly increased power, range, rapidity of fire, and accuracy. In this connection, we present in Table 7 numbers and casualties of the attacker and defender for six large battles that occurred relatively recently. Metz and Sedan are not included in Table 5, since they are not actually of the attack and defense type, but correspond rather to investments or sieges with unsuccessful sallies.

From the calculations at the bottom of Table 5, it is seen that the attacking side lost on the average 52% more than it should by formula (12-bis), that is, for average conditions.

It is possible to think that numerical strength, fortifications, and tactics more often are encountered in the form of two combinations: (1) lower numerical strength, suitable positions, fortifications and defensive operations; or (2) greater numerical strength, unsuitable terrain, and offensive operations. Apparently, this is one of the reasons why total casualties of the stronger in 38 battles was 22% larger than indicated by formula (1).

8. *Larger Numerical Size of the Engaged Forces.* In the theory, it has been assumed that all shooters in the combat ranks of the opponents can attack each other with the same facility, but that condition may be assumed only for forces that are not especially large: For armies as vast as the modern ones, for which the front line reaches hundreds of versts [1 verst = 3500 feet = 1066.8 meters—Tr.], additional strength of a side extends the front by 10 versts, and nobody in this additional force has any targets. In former times, additional strength could be assigned to envelopments, but nowadays, because of very long front lines and flanks possibly anchored on trustworthy natural obstacles, envelopment has lost its value. Thus, for long front lines casualties to the sides must deviate from formulas (1) and (1-bis) and approach equality, since overall and almost equally for both sides lengthening the line of combat does not make it possible for the side with greater strength to introduce into battle numbers of active warriors out

Table 8. Values in thousands.

Battle nos.	A	a	B	b	$(a)_1$	v_1	$v_1\%$	$(a)_2$	v_2	$v_2\%$	$(a)_3$	v_3	$v_3\%$
Early	2718	405	1967	442	309	−96	−31	330	−75	−23	381	−24	−6
Late	1934	198	1396	250	186	−12	−6	193	−5	−3	218	+20	+9

of proportion to its own strength. Nevertheless, the stronger side retains certain advantages, which weaken the not very great value of extending the front. These are

(1) Superiority in the absolute number of artillery cannon which can shoot over the heads of the riflemen ranks.[7]
(2) The historical increase with time of the range of artillery and gunfire and the intermittent nature of the line of battle, which allows the stronger to make better use of its superiority by introducing larger number of active troops into the battle.

These considerations do not conflict with the numerical data which can be taken from the list of 38 battles.

We have already seen in Table 6 (see Section 6) that for numerical strength not over 75,000 casualties to the sides satisfy formula (1) better than (12-bis). On the other hand, if we take the last 13 battles of Table 3 where the numerical strength of the stronger side was not less than 150,000, then we get for the total of battles 26 through 38 inclusive: $A = 2689$, $a = 310$, $B = 1986$ and $b = 348$.

An application of formula (1) gives

$$(a)_1 = 2689 - \sqrt{2689^2 - 1986^2 + 1638^2} = 246 \quad \text{and} \quad v_1 = -64,$$

that is, 26% of $(a)_1$. Formula (1-bis) gives

$$(a)_2 = bB/A = 257, \qquad v_2 = -53,$$

that is, about 21% of $(a)_2$. Formula (12-bis) gives

$$(a)_3 = b\sqrt{B/A} = 299, \qquad v_3 = -11,$$

that is, less than 4% of $(a)_3$. We conclude that the error of formula (12-bis) is almost seven times smaller than the error of formula (1).

The influence of increased ranges and intermittent fronts on the application of formulas (1) and (1-bis) is seen from Table 8. In this table, battles of the early group were fought between 1805 and 1859; battles of the late group were fought between 1866 and 1905.

[7]Riflemen on a 1-verst front [1 verst = 3500 feet = 1066.8 meters—Tr.] have a hit rate or active force not over 750 to 1000 rifles. Artillery, or the other hand, in current organizations, is 20 cannons per verst, but nevertheless can rise to 50 cannons, located to the rear of the riflemen in one line at closely spaced intervals. If the hits of one cannon are equivalent to the hits of 50 rifles, then this shows that the artillery almost doubles the hits of the rifles and other weapons on a front 1-verst wide. A similar role is played by increased numbers of machine guns. From this it is clear that the increase in riflemen ranks gained by extending the front in no way offsets a substantial superiority in artillery and machine guns.

The numerical strengths and casualties in Table 8 provide the totals for those battles, taken in chronological order (that is, as in List No. 1 of Section 1, rather than as in Table 3 of Section 6) in the form of two groups. The early group includes battles from Austerlitz to Solferino inclusive, and the second includes battles from Custozza to Mukden, that is, battles after 1866, when firearms affording appreciably greater range and reliability were perfected. Then we give the casualties (a) calculated from A and B and the casualties b, just as in Table 5 of Section 6, except that in place of v^2 in Table 8 we substitute the percent which v is of (a).

The average numerical strength of side A for the 27 battles of the early group here is equal to about 100 thousand men (2718/27), and for the 11 battles of the late group equals about 180 thousand men (1934/11). In spite of the fact that the average number A for the late group nearly doubled, the errors for formulas (1) and (1-bis) actually decreased in percent of their size, by much more than half (specifically, in place of -31% and -23% we have here -6% and -3%, respectively). Very likely the cause of this is that increased ranges and intermittent battle lines allowed the stronger side to make better use of its superiority.

So increasing the numerical strengths of the combatants makes casualties deviate from the law of casualties given by formulas (1) and (2), in the direction of equalizing the casualties of the sides, and causes their ratio to approach formula (12-bis).

9. Density of the Line of Troops. Our usual density in skirmish lines is one rifleman for every two paces along the front. If we were to dispose the riflemen more densely than this, we would increase the number of active troops, that is, increase the hits on the enemy, but at the expense of increasing our casualties as well, since hitting denser ranks is easier. On the other hand crowding of the ranks above doctrinal norms inconveniences the shooters. Therefore it may be that concentrating ranks above doctrinal norms would hardly be beneficial.

We must note here that formulas (12) and (12-bis) can be obtained by considering denser or more compact dispositions of the forces. If, for example, we let A, B, a, and b be the numerical strengths and casualties of the sides, α the hits caused by one shooter in one unit of time for some nominal density of the battle formation (that is, the number of men per unit area), and m and n the actual densities of the sides, then the casualties dA and dB during a short time dt will be given by $dA = \alpha mB\,dt$ and $dB = \alpha nA\,dt$. Eliminating $\alpha\,dt$, we get

$$nA\,dA = mB\,dB.$$

Casualties to the sides in battle happen mainly in the last decisive moment, when both sides collide. If at this moment both sides have troop densities related as:

$$m/n = \sqrt{A}/\sqrt{B}$$

then substituting this into the previous equation yields:

$$\sqrt{B} \times A\,dA = \sqrt{A} \times B\,dB, \quad \text{or} \quad \sqrt{A}\,dA = \sqrt{B}\,dB.$$

This is precisely how we obtained the formulas (12) and (12-bis) given in the comments of Section 4.

10. Encirclements and Envelopments. The three previously mentioned causes (numbers 7, 8, and 9) tend to equalize casualties for the sides, as computed from formulas (1) and (1-bis).

But it is possible to indicate some things that show a diametrically opposite influence, that is, that amplify differences in casualties. For example, *encirclements and envelop-*

ments are more often peculiar to the side with the most numbers, since it is easier for it to allot a certain portion of its troops for this. As seen from formula (10), the enveloping force causes somewhat greater casualties and suffers fewer of them itself than would follow from formulas (1) and (1-bis), that is, for average conditions. Also, the possible influence which envelopments produce *strengthens morale* which, in general, must be higher on the side with the superior numbers, since victory falls to that side more often than on the other.

Part IV

Part IV appeared in Voenniy Sbornik, Issue No. 9, 25–37 (September 1915). Besides the indicated causes of systematic errors, there are probably still others, but in view of the novelty of the issues, we limit ourselves only to those enumerated.

So terrain, fortifications, tactics, extended fronts, denser formations on the stronger side—all these factors tend to equalize the casualties of the sides, compared to those calculated from formulas (1) and (1-bis), while encirclements, envelopments, and perhaps morale, on the contrary, increase the difference in these casualties. In aggregate, the former factors overpower the latter, whose influence is very weak. That is why the total casualties for the numerically stronger side in the list of battles is found to be quite a bit closer to the total casualties for the weaker than would follow from formulas (1) and (1-bis), and why the ratio of casualties is determined far better by formula (12-bis).

Section 8. Some Consequence of the Law of Casualties

Reviewing all that has been presented so far, we see that at first, beginning with conditions that seemed to us not very probable, we derived formulas establishing the dependence of casualties on the numerical strengths of the sides. Later, we occupied ourselves with verifying these formulas with historical examples and convinced ourselves that the formulas obtained could be regarded as correct only for moderate numerical strengths of the sides, but, generally speaking, formulas (1) through (11) do not agree with reality and give errors around 25 percent of the calculated casualties and much better agreement with reality is provided by formulas (12) and (12-bis). Finally, in Section 7 above were indicated briefly:

(1) The causes of the random errors, by which casualties to the sides in individual battles can deviate widely from the average or theoretical value, and
(2) The cause of the systematic errors, which influence not only individual battles but in the same way also the totals of many battles and which are the principal source of the errors in formulas (1) through (11).

All this is quite interesting from the scientific point of view, but only those conclusions which can be adapted to applications are of practical importance. Therefore we display some inferences or implications of the law of casualties, some of which are interesting for their novelty, while others—being familiar principles of military art—give fresh indirect confirmation of the validity of our basic thesis, that casualties to the numerically stronger must be less than those of the weaker.

These conclusions can be obtained from formulas (1) and (1-bis), or from (12) and (12-bis), but if we do not need to determine mathematical quantities, then the three general theses which are given in Section 9 below are sufficient. For ease of presentation the conclusions are given in the form of solutions to various problems, where for such solution formulas (1) and (1-bis) are used more frequently than (12) and (12-bis), since the former are simpler and more easily solved, and because our practical conclusions—since they consist of general expressions and we do not pursue arithmetical precision—remain the same in either case.

PROBLEM 1: Given opponents A and B. Find their casualties after six units of time into the battle if the hits caused by one rifleman in a unit of time is equal to $\alpha = 0.04$, and the numerical strengths of A and B are as follows:

(1) $A = 1000$, $B = 800$.
(2) $A = 2000$, $B = 800$.

We solve this problem by using formula (4) with $\alpha t = 0.04 \times 6 = 0.24$. Then, by the table in Section 3, we find $\cosh(0.24) = 1.0289$, $\sinh(0.24) =' 0.2423$.

CASE 1: $A = 1000$, $B = 800$. In this case $A' = 1029 - 194 = 835$, $B' = 823 - 242 = 581$. So $\underline{a = A - A' = 165, b = B - B' = 219}$.

CASE 2: $A = 2000$, $B = 800$. In this case $A' = 2{,}058 - 194 = 1864$, $B' = 823 - 485 = 338$. So $\underline{a = 136, b = 462}$.

Thus, doubling the strength A while keeping the duration of the battle the same, the casualties a are slightly decreased (165 versus 136), while casualties b are more than doubled (219 versus 462). Consequently, *by increasing our own numerical strength, we cause the enemy higher casualties and at the same time endure somewhat lower casualties.* This consequence of the theory, which nowhere in military science has ever before been clearly stated, is implied by its basic assumptions. But it is obtained for conditions that the battle last for the same length of time in each of the cases considered. However, victory depends not on the duration of the battle, but principally on the ability of the sides to endure casualties; therefore, it is correct to reckon that battles last until the loss to one side has achieved some definite percentage.

That percentage on the average can be calculated as 20%, since the total casualties to the victor in 38 battles is equal to 729,000, which amounts to about 20% of the total of the numerical strengths of 3,623,000. In the following problem, the solution also satisfies this new condition.

PROBLEM 2: Given opponents A and B. Let the latter be the weaker and be able to sustain the battle so long as its casualties do not exceed 20% of its initial strength. What will be the casualties of the sides if:

(1) $A = 1000$ and $B = 800$,
(2) $A = 2000$ and $B = 800$?

This problem can be solved with formula (1) in which the knowns are A, B, and $B' = B - b = B - 0.2 \times B = 0.8B$, while A' and $a = A - A'$ are to be found.

CASE 1: $A = 1000$, $B = 800$. In this case $b = 160$, $B' = 640$. $A' = \sqrt{1000^2 - 800^2 + 640^2} = 877$, $\underline{a_1 = 123}$.

CASE 2: $A = 2000$, $B = 800$. In this case $b = 160$, $B' = 640$. $A' = \sqrt{2000^2 - 800^2 + 640^2} = 1942$ $a_2 = 58$.

For an approximate solution of the problem, it is possible to use formula (1-bis); that is, $Aa = Bb$, which gives:

$$a_1 = 800 \times 160/1000 = 128 \quad \text{and} \quad a_2 = 800 \times 160/2000 = 64,$$

which are close enough to the original solution.

Finally, for comparison, we also solve the problem using formulas (12) and (12-bis). This gives for (12) $a_1 = 141$ and $a_2 = 97$, and for (12-bis) $a_1 = 143$ and $a_2 = 101$.

Thus, for all formulas, increasing the stronger side's strength implies a reduction in its casualties while maintaining the casualties of the weaker. From the solutions of the first and second problems, it follows that *with superior strength to send people into battle in the greatest possible numbers does not mean to sacrifice them uselessly, but rather this is intended to save them and gain time for attaining the main objective.* But we should not forget that victory is far from depending solely on our own numerical strength, but depends also on a multiplicity of other factors, and that even if the theory explains the employment of numerical strength, it at the same time requires the observation of all the rules of military art, as can be seen from the analysis of errors in Section 7. For example, the influence of morale can be seen from the following problem.

PROBLEM 3: Given opponents $A = 1000$ and $B = 800$. What is the impact on casualties to a side that loses through ineffectives 25% of its force so that effectively its riflemen are only 75%, while the other 25% riflemen are ineffective? The battle lasts until one side loses 20% of its numerical strength.

For simplicity, formula (1-bis) is used. When there are no ineffectives, then by the previous problem, $b = 160$, $a = 128$.

(1) Losses from ineffectives are assigned to the stronger side:

Number of active troops	$A = 750$	$B = 800$
Number ineffective	250	*none*

casualties $b = 160$, so $a = 160 \times 800/750 = 171$. It follows that casualties are increased by $171 - 128$ men, or by 4%.

(2) Losses from ineffectives are assigned to the weaker side:

Number of active troops	$A = 1000$	$B = 600$
Number ineffective	none	200

casualties $b = 160$, $a = 160 \times 600/1000 = 96$.

It follows that casualties a are decreased by $128 - 96 = 32$ men, or by 3%.

Thus, when the stronger side wastes forces through ineffectives, it bears unnecessary casualties; therefore, anyone who is ineffective is guilty of causing some of his comrades to sacrifice themselves needlessly. When the weaker side has ineffectives, its enemy saves some number of troops. *Consequently,* in either case, *wastes of formations through ineffectives is tantamount to giving aid to the enemy.* This is why cowardice is always equated to betrayal.

For ineffectives amounting to 50% for A we get for $a = 200$ (that is, 20% of the formation A), $b = 125$ (that is, 16% of the formation B). We also conclude that, in this case, victory falls to the weaker side. [Since A has taken 20% casualties while B has yet to reach that level of casualties—Tr.]

These deductions give the distribution of casualties for open battles. But if we are numerically weaker than our opponent and cannot avoid battle, then obviously we will bear fewer casualties if we are shielded by fortifications. An example of this, when fortifications shield only the defender and increase the attacker's casualties, is addressed in the following two problems.

PROBLEM 4. On the 18th of June at Plevna the strength of the Turks (B) was 18,000 with 60 guns, and of the Russians 24,000 with 100 guns. Casualties to the former were $b = 1200$ men and to the latter $a = 7500$ men. The difference in casualties is explained by the fact that the Turks were protected by good fortifications. Deduce from this the ratio β/α of hits by Turkish and Russian riflemen, setting the coefficient for converting cannon into riflemen equal to 100. The latter number is taken as roughly approximating the average of 150 and 60 (see Example 4 of Section 4).

$$\text{Active strength } B = 18{,}000 + 60 \times 100,\ b = 1200,\ B' = 22{,}800$$

$$\text{Active strength } A = 24{,}000 + 100 \times 100,\ a = 7{,}500,\ A' = 26{,}500$$

The following solutions will be distinguished by their dependence on the formulas which they use. For example if we take formula (6) then we get

$$\beta/\alpha = (A^2 - A'^2)/(B^2 - B'^2) = (34^2 - 26.5^2)/(24^2 - 22.8^2) = \underline{8.08}.$$

If instead we use formula (6-bis), that is, $\alpha \times Aa = \beta \times Bb$, then we get

$$\beta/\alpha = Aa/Bb = 37 \times 7.5/(24 \times 1.2) = \underline{8.85}.$$

So, thanks to fortifications, the Turkish troops were 8 to 9 times more successful than the Russians. From this we see that the strength of the Russians was insufficient, and therefore unwaveringly the question arises: Was it not possible to increase the strength of the Russians by some multiple, to a level that their casualties would have been equal to those of the Turks? This question is the subject of the following problem.

PROBLEM 5: How large must force A be in the previous problem to make the casualties a and b equal to each other?

We again use formula (6), taking in it $\beta/\alpha = 8.08$, $a = b = 1200$ men, $B = 24{,}000$, $B' = 22{,}800$. Here A is an unknown quantity, but $A' = A - a = A - 1.2$ thousand. From this we get the equation:

$$A^2 - (A - 1.2)^2 = (\beta/\alpha) \times (B^2 - B'^2), \text{ whence}$$

$2.4A - 1.44 = 8.08 \times (576 - 519.84) = 453.75$ thousand and $A = 190{,}000$.

Thus, the unknown quantity is equal to 190,000 men. However, this number could not have simultaneously taken part in the battle. At the very most, it would only have been possible to have engaged in the siege of a fortress triple the number of Turkish troops, that is, 54,000, while it would have been necessary to convert arithmetically the other 190,000 − 54,000 = 136,000 into cannon, which at the rate of one cannon per 100 men would amount to 1,360 [field—Tr.] cannon (fewer siege cannon would have been required). This theoretical deduction about the importance of field artillery for attacking fortifications is fully tested by the examples of Mount Dubnyak and Telishu in the year 1877.

Problems 4 and 5 indicate:

(1) That from the point of view of casualties fortifications have immense significance for the defense.
(2) That to attack field fortifications with exposed forces is possible only with a significant superiority in strength and especially in artillery.
(3) That for a variety of purposes it is very useful to have good statistical materials, allowing accurate determination of the coefficients for converting artillery cannon, machine guns, and so forth into riflemen equivalents as well as for determining the degree of protection afforded by fortifications, and so forth.

PROBLEM 6: Take two opponents with A being the stronger and B the weaker. Each can advance various numbers of active troops; therefore the ratio of their casualties will vary. Compare these ratios for various cases, assuming that casualties to the weaker side are always equal to $b = 600$ men, while the numbers A and B are as indicated below.

Although, generally speaking, these are but modifications of Problems 1–3, we treat them separately in view of their importance. For their solution we use formula (1-bis).

CASE 1: $A = 2500, B = 2000, b = 600, a = 600 \times 2000/2500 = 480.$

CASE 2: $A = 3000, B = 2000, b = 600, a = 600 \times 2000/3000 = 400.$

CASE 3: $A = 3000, B = 2500, b = 600, a = 600 \times 2500/3000 = 500.$

From the battle casualties a in Cases 1 and 2, where B remains constant, we see that *for the stronger side it is always advantageous to increase its active numbers and thereby reduce its own casualties* (from 480 to 400). On the other hand, from the battle casualties a and b of Cases 2 and 3, where A remains constant, it is apparent that for *the weaker side, too, it is advantageous to increase its active strength*, not in order to diminish its own casualties, but *in order to increase its opponent's casualties and in addition to ease its own moral stress*. These conclusions follow from the findings that casualties b in Cases 1 and 2 comprised 30% of B's numerical strength, while in Case 3 they were only 24%. Thus, for *both the stronger and the weaker it is advantageous to throw into action the greatest possible active strengths*. This entirely *agrees* with the commonly known *principle* of military art *to commence and to conduct military operations with the full strength of the entire force.*

PROBLEM 7: Opponents A and B have 3000 riflemen each. Compare their casualties for the following conditions:

(1) When both sides engage all of their riflemen in battle throughout six units of battle time.
(2) When A engages all 3000 of his riflemen, while B sends initially 1000, after two units of time another 1000, and again after two units of time (that is, after four units of time have expired) the final 1000.

After six units of time have expired, the battle ends. The hits caused by one rifleman in one unit of time are $\alpha = 0.04$. For the solution of this problem, we take formula (4) and then we get the following

(1) For the first case:
$t = 6$, $\alpha = 0.04$, $\alpha \times t = 0.24$, $\cosh(0.24) = 1.0289$, $\sinh(0.24) = 0.2423$. Then by formula (4) $A' = B' = 2360$, $a = b = 3000 - 2360 = 640$, that is, the casualties to the sides are equal to 21.3%.

(2) For the second case:
$t = 2$, $\alpha = 0.04$, $\alpha t = 0.08$, $\cosh(0.08) = 1.0032$, $\sinh(0.08) = 0.801$.

Next, applying formula (4) to determine the results of the prescribed three phases of the battle, we get at the end of t units of time:

$t = 2$,	$A_1 = 3{,}000$,	$B_1 = 1{,}000$,	$A'_1 = 2{,}930$,	$B'_1 = 763$,	$a_1 = 70$,	$b_1 = 237$
$t = 4$,	$A_2 = 2{,}930$,	$B_2 = 1{,}763$,	$A'_2 = 2{,}798$,	$B'_2 = 1{,}534$,	$a_2 = 132$,	$b_2 = 229$
$t = 6$,	$A_3 = 2{,}798$,	$B_3 = 2{,}534$,	$A'_3 = 2{,}604$,	$B'_3 = 2{,}318$,	$a_3 = 194$,	$b_3 = 216$
				Total	$a = 396$,	$b = 682$

That is, the aggregate casualties amount to 13.2% for A and 22.7% for B.

So by sending reinforcements into the line step by step instead of simultaneously engaging all possible numbers of riflemen, we have caused a small increase in our own casualties (by $682 - 640 = 42$ men or by $22.7\% - 21.3\% = 1.4\%$) and at the same time observed a reduction in our enemy's casualties (by $640 - 396 = 244$ men or by $21.3\% - 13.2\% = 8.1\%$). This means that *gradual reinforcement of our ranks, instead of an initially strong force, is not advantageous to us, but rather to an opponent who engages a strong force from the very beginning.*

This problem considered in relation to the previous one indicates how one should manage the number of troops in battle, specifically:

(1) From the battle casualties a and b in Case 2, we conclude that, if we intend to give the opponent a decisive repulse or defeat, then we must from the very beginning send into the ranks as many riflemen as possible.

(2) From the battle casualties b in Cases 1 and 2, which are quite close to each other (the difference in their totals is only $22.7\% - 21.3\% = 1.4\%$), it follows that, if we do not have enough information about the enemy but desire to conceal our own strength or avoid battle, then we might limit ourselves to a weaker rank without thereby sustaining unnecessary casualties. However, such a period of uncertainty should not be dragged out; otherwise we could allow the opponent impunity to inflict on us greater casualties than in the same time he suffers from us. These latter remarks do not have meaning if we have special objectives, for example, guarding the flanks. *Thus, the conclusions of the theory and doctrinal principles about the ratio of ranks to reserves agree with each other rather well.*

PROBLEM 8: Side A with 3000 riflemen takes part in three battles with side B, whose strength is also 3000 riflemen, but divided into three components of 1000 men each. With each component of B, side A fights until B has lost 20 percent of his strength. What will be the casualties to the sides at the conclusion of each of these component battles?

For the solution of this problem, formula (1-bis) is used; that is, $a = bB/A$, where B is always 1000 men, and casualties $b = 200$ men (20% of 1000), while A is equal to

3000 men in the first battle, but in the ensuing it equals those remaining from the previous engagements.

Then we get

First battle:	$A = 3000$,	$B = 1{,}000$,	$b = 200$,	$a = 200 \times 1000/3000 = 67$
Second battle:	$A = 2933$,	$B = 1{,}000$,	$b = 200$,	$a = 200 \times 1000/2933 = 68$
Third battle:	$A = 2865$,	$B = 1{,}000$,	$b = 200$,	$a = 200 \times 1000/2865 = 70$
	Total casualties:		$b = 600$	$a = 205$
			20%	6.8%

Comparing the aggregate casualties of the sides, we see immense advantages in committing all of one's strength against components of the opponent's strength, that is, *this is a good example of the confirmed principle of "defeating the opponent piecemeal."* However, this rule is easily gotten from any theory in which the casualties to a side increase by an amount that depends on an increase in its opponent's strength according to any arbitrary rule. On the other hand, since our military art teaches us to defeat our opponents piecemeal, it follows that our casualties will be diminished by whatever diminishes the numerical strength of our opponents. Thus the *principle "defeat the opponent piecemeal," unquestionably confirms the basic thesis of our theory, that casualties to the numerically stronger must be less than those of the weaker.*

PROBLEM 9: Suppose that we and our opponent each have 4000 troops. The opponent has divided his force into two equal components. Which way is more advantageous for us to conduct battle with him: divide our force also into two equal components or divide it unequally?

For the solution of this problem it is necessary to compare two cases:

(1) When $A_1 = A_2 = 2000$.
(2) When $A_1 > A_2$, for example, $A_1 = 3500$ and $A_2 = 500$.

In both cases, B_1 and B_2 are each equal to 2000 and are engaged, respectively, with A_1 and A_2. Each battle lasts until the weaker side loses 20% of its initial strength.

CASE 1: $A_1 = A_2 = 2000$ and $B_1 = B_2 = 2000$. In this case casualties to the sides in each component of the battle will be equal to 400 men, so all told sides A and B lose 800 men each. In view of the equality of casualties these battles will be drawn [literally, indecisive—Tr.].

CASE 2: In this case:

	$A_1 = 3500$,	$B_1 = 2000$,	$b_1 = 2000 \times 0.2 = 400$,	$a_1 = b_1 \times B_1/A_1 = 229$,
	$A_2 = 500$,	$B_2 = 2000$,	$b_2 = a_2 \times A_2/B_2 = 25$,	$a_2 = 500 \times 0.2 = 100$.
Totals:	$A = 4000$,	$B = 4000$,	$b = 425$,	$a = 329$.

So in the second case the results of the two battles are that side A loses 329 men and B loses 425 men, that is, A's chances were improved thanks to lower casualties. They may be lower still if the weaker component A_2 fortifies itself well while A_1 finishes with B_1. From this we see that the general *principle—defeat the opponent piecemeal, is* simply *a special case of this problem*, specifically that instead of dividing one's force into unequal components, we unite them all into one group, or, in terms of the present problem, put $A_1 = 4000$ and $A_2 = 0$. Although to win piecemeal is more advantageous than dividing into components, nevertheless sometimes, for bait, it is necessary to have an A_2. This problem illustrates well the commonly known principle of military art: *Do not separate your forces, but be strong in one place*, and of course *at the most important place for the given conditions.*

PROBLEM 10: What distribution of the strength of two engaged sides is advantageous from the point of view of minimizing the sum of the casualties of the two engaged sides: equal strength or unequal?

We assume that the sum of the strength of the two Sides equals $A + B = 8000$. The battle lasts so long as the weaker side has not lost 20% of its initial strength.,

CASE 1: $A = B = 4000$, $a = b = 800$, $a + b = 1600$, and, as a result, the outcome will be drawn [literally, indecisive—Tr.].

CASE 2: $A = 5000$, $B = 3000$, $b = 3000 \times 0.2 = 600$, and $a = bB/A = 360$ men. Consequently, $a + b = 960$, and the battle must be considered as resolved in favor of A.

So in the case of equal strengths of the opponents, aggregate casualties equal 1600 men, and the battle is indecisive; but when the strengths of the sides were unequal, the battle was decisive with total casualties of 960 men. Consequently, *battles with equally strong forces must be exceptionally bloody and indecisive.*

Finishing with these deductions of some well-known consequences of military art, we remind you that the same deductions could have been obtained not only on the basis of formulas (1), (1-bis), (12), or (12-bis), but just as well from any other, as long as it assumes that casualties of the numerically stronger side are less than those of the weaker and would decrease still further if its superiority were increased. Since the principles of military art are based on battle experiences throughout history, the logical connection between the theory of casualties and the principles mentioned above provides an additional indirect proof of the validity of the theory insofar as its essential, basic assumptions are concerned.

Section 9. Conclusions

On the basis of all that has been said, we can see that there is a dependence of casualties on the strength of the opposing sides, but testing theoretical deductions by examples of individual battles would be inappropriate, because casualties depend—in addition to strengths—on many other conditions (see Section 7), the influence of which in most cases cannot be expressed quantitatively. Only the aggregate result of many battles provides averages for testing the theory.

When it is not necessary to state the laws of casualties in mathematical form, then they can be expressed in the following theses:

(1) The side strongest in numbers bears the absolutely smaller casualties, rather than the weaker side.

(2) If superiority in strength is on our side, then by increasing the number of our active troops we gain time and diminish our casualties. If superiority of strength is on our opponent's side, then by increasing the number of our active troops, we increase our enemy's casualties and ease the moral stress on our own troops, while our casualties remain at the same level.
(3) By the numerical strength of a side we mean the number of troops actively wielding military rifles, artillery, guns, machine guns, sabers, and so forth, rather than the number on the roster, and do not include the numbers of unengaged reserves. For valid comparisons, other weapons should be converted into rifle equivalents.

The validity of these theses is confirmed by the theoretical arguments and the numerical data on 38 battles as well as by a great many basic principles of military art, as is clear from the problems of Section 8. It would be absurd to claim that the three theses mentioned above represent for military affairs something completely new, since they have always been implicit in the fundamental principles of tactics and strategy, but at the same time, it seems that these theses have nowhere been explicitly formulated. This is why knowledge of them might be considered useful for practical application to military affairs. These basic theses of the theory specify in the first place to assemble for battle the greatest possible roster of soldiers, and in the second place to conduct the actual combat with the greatest number of active troops possible under the circumstances. Not every commanding officer can expand an initial roster of effectives, but it is always possible to divide the force into active and reserve components, since doctrine in no way restricts such division. Therefore, the chief use of the laws of casualties are, apparently, to the division of forces into active components (rifles in ranks, artillery in position, and so forth) and into reserves, or—which is the same thing—to determining the best strength [literally, length—Tr.] of the line of battle. We will not concern ourselves with such issues; instead they must be resolved by each commanding officer on the basis of his own experience in the most recent battles, since each commander is authorized to determine the strength [literally, length—Tr.] of the line of battle, that is, the relative sizes of the active forces and reserves. In regiments, for example, the normal strength [literally, length—Tr.] of the battle line of riflemen is composed of $\frac{1}{4}$ to $\frac{1}{5}$ of the total number of riflemen. Naturally, the thought arises: Is not this active component too weak? If, in all preceding major battles, there always remained some uncommitted reserve units, then this would serve to indicate that the battle line should be strengthened [literally, lengthened—Tr.]. However, it is possible that experience would show that there is a shortage of reserves at the end of the battle, in which case it is necessary to weaken [literally, shorten—Tr.] the line and strengthen the reserves. The theory of casualties directs attention to the fact that a correct solution of this question is very important, because it would allow a reduction of our casualties and a more certain achievement of our battle objectives.

Moreover, this knowledge of the laws of casualties can be of use in other situations as well, facilitating our resolution of various military problems. However, one cannot ever forget that the inverse dependence of casualties and numerical strength can only hold for battles where both sides are in comparable conditions.

Therefore, if we overlook obvious differences between our circumstances and those of our enemies, we cannot depend on numerical superiority [alone—Tr.] to give us an advantage, although it will still have a very strong influence on success in battle. Thus, the theory of casualties does not reject any military doctrine or principles, but on the contrary requires their fulfillment, reminding us that any neglect in this respect will alter the average, valid ratio of casualties to the advantage of our enemy, that is, it will involve excessive casualties to us, which is to be avoided if at all possible.

Only the practical application of this theory of casualties to a more conscientious management of the numerical strengths of troops will reduce our casualties and increase those of our opponents.

ADDENDUM

Part V

Part V appeared in Voenniy Sbornik, Issue No. 10, 93–96 (October 1915). Since in recent newspapers the opinion has been stated that casualties are usually equal for both sides, and therefore the theory I expounded was erroneous, I—in this letter—make bold to say a few words in defense of my own claims.

In view of the haste with which the article was written and since I am not a specialist in either military history or practical military affairs, some of the things I said in the article may be incorrect or prone to misinterpretation. Thus, for example, the note on page 35 in No. 7 *Voenniy Sbornik* [footnote 5 of this translation—Tr.] gives almost equal total casualties in 55 battles (472,000 and 489,000) for respective strengths (4,780,000 and 3,270,000) that are far from being the same. These are taken from O. Berndt's book *Die Zahl im Kriege*. The list of 38 battles I provided in the article was taken from G.L. Leyer's Encyclopedia, since at the time of preparing the article I had not yet seen Berndt's book. Leyer's figures on casualties have some errors (for example, casualties in killed are not separated from casualties in wounded), but I took all of them without modification so as not to be reproached for having picked the figures myself. The latter's [that is, List No. 1's—Tr.] errors I classify as two types: (1) errors in the numerical strengths A and B; and (2) errors in the casualties a and b. The most dangerous I consider to be the former, rather than the latter, even though it might seem otherwise at first sight. I will explain this using the example of Aspern (1809). Taking Berndt's figures: French $A = 90{,}000$, $a = 42{,}000$; Austrians $B = 75{,}000$, $b = 22{,}000$. If we dispute the numbers for casualties, then we can hardly reduce their difference of $a - b = 42 - 22 = 20$ thousand to zero, that is, to alter it by 20,000. From history it is well known that the battle of Aspern lasted 2 days. On the first day the French put into action 35,000 against the whole Austrian force, but on the second day the French did not involve any of the corps of Davout or Parke. It is obvious that the less active side was the French, who therefore also bore much larger casualties. If one orders a list of battles by using the roster strengths, then the strongest side would be the French, and $a - b = 42 - 22 = 20$ thousand. But if one considers the French to be the weaker side, they would have to be placed under column B (on the right-hand side of the list), and then the difference of the total casualties for Aspern would become $a - b = 22 - 42 = -20$ thousand. Hence we see that incorrectly estimating the stronger side's strength changes the difference in total casualties by $20 - (-20) = 40$ thousand, that is, by twice the amount of the maximum error in the number of casualties. This is why it is so necessary to take the active forces on a side, rather than those on the roster, in determining the ratio of casualties. In the list of 38 battles, I estimated the French force at Aspern as $(35 + 100)/2 = 67.5$ thousand or, with rounding off, at most 70,000, but certainly not 90,000. In general, the theory involves inverse relations between casualties and the numbers of *actives*, rather than the numbers on the roster (that is, the numbers of riflemen, gunners, and artillerymen or machine gunners, translated into riflemen equivalents). When testing the theory, we were compelled to use the roster numbers, because the number of actives are not given by history and would have had to be computed from other information, and for this I had no time. I think that even in peacetime it would be a very complex task. In Section 5 it was indicated that active numbers can be replaced by the numbers on the roster, provided these numbers are proportional. In the list of 38 battles, we tended to take the bloodiest, because for them nearly all of those present are forced to participate in the action, whether they wish to or not, and in such conditions we do not expect major discrepancies between the number of actives and those listed on the roster. In his list of 55 battles, O. Berndt has 22 for which the casualties to a side did not exceed 10%, and there are some with numerical strengths of 40–60 thousand where total casualties amounted to no more than 200–800 men. It is obvious that in such battles the participants were merely a small part of the total force. Therefore, the list of 38 battles given in the article, although having some errors in its figures for casualties, nevertheless is more suitable for testing the theory than Berndt's list. From this we see that a historical test of the theory nevertheless cannot be entirely definitive, since it shows only that casualties are inversely related to the number on the *roster*, rather than to the number of *actives*, although the latter is what is required by the theory (see point 3, Section 9). That is why I believe that *military principles* confirm the theory more convincingly than examples from history. On the other hand, do not expect miracles. If in one unit of time we take 1000 casualties, while we are inflicting only 500, then naturally (all other things being equal) we must suffer more casualties, and so we do not like this situation. It suits us better to resort to the defensive, and to yield territory step by step to our enemy. When we ran out of shells, then it became clear to everyone that casualties are in inverse relation to the fighting forces. But the fighting forces are in my terminology just the active numbers. Theory only sheds more light on this issue. If we take the formula, involving the number of actives rather than the roster numbers,

$$A + M\beta/\alpha + P\gamma/\alpha,$$

then from this formula we see that in addition to having ample numbers of shells and bullets, it is also important to have sufficient numbers of cannons M and machine guns P, because otherwise the fires delivered in one unit of time would not reach the proper quantity (it would be like a wise man who is unable to display his immense intellect because his mouth is too small). Likewise, this theoretical formula reminds us that what is important is not the numbers of shells and bullets fired, but the number of hits inflicted; consequently, it is necessary to strive for the greatest possible accuracy, not sparing the ammunition reserves, although that sometimes happens in battles. The position I have presented is, I believe, not only of theoretical interest, but also of practical interest, if it is possible to have available more reliable statistical data on various coefficients of hits and protective defenses. For example, we Russians could have had a sufficiently important practical result of my theory if we had paid attention not only to increased means of hitting, but also to improved means of protection. In addition to fortifications I here include, for example, shields on left arms, which are too often hit, helmets to protect against shrapnel shot and small shell fragments, the possible use of butt straps or other means of protection in bayonet (hand-to-hand) fighting, and so forth. By these means we more quickly neutralize German superiority in the production of artillery and machine guns, not only after we surpass them in this respect, but also if they in turn adopt these protective measures; in the latter case, battles will become more stubborn and we will defend our position for a longer time against the enemy. But in general, showing practical applications of the theory is difficult for me, since I have never been in a war and do not know the difficulties which have to be overcome by commanders.

In order to more fully elaborate on numerical superiority, I will venture to add the following example, which actually represents a version of the idea already expressed in the note on page 38 of *Voenniy Sbornik* No. 8 [footnote 7 of this translation—Tr.]. If the Germans somewhere had superiority by 200 cannons, then we could balance this enemy superiority by advancing ranks numbering $200 \times 50 = 10{,}000$ men, counting one artillery hit as equivalent to the hits of 50 rifles. Such an excess without encircling the flanks or without strongly concentrated ranks will be too cramped to maneuver effectively. This implies that superiority even by 200 cannons would have great significance and to neutralize it we must either multiply our artillery, or strengthen our defenses, or accept heavy casualties, or exploit some special circumstances in the situation (for example, multiple ranks firing in mountain passes), and so forth.

Return to the opinion stated in the newspapers that casualties commonly are equal to each side. I will avail myself of this opportunity to point out that this is a consequence of my hypotheses—in the first place. Second, this equality of casualties, if it were indeed a reality, would serve to prove that errors in the theory could not imply by themselves an increase in casualties; that is, the errors are of a mutually offsetting nature. Hence, errors in the theory are not very dangerous. If in addition we consider that the Germans nearly always act as if they were acquainted with the theory, then it follows that publishing the theory would not be particularly useful to them. For these reasons I did not consider publication of my work to be objectionable.

I consider it my duty to present this statement to the public in order to overcome their fear that the theory is actually correct. Notwithstanding the lack of reliable figures given by history, the three principles put forward in Section 9 are nevertheless correct. Of course, all this is aimed strictly at clarifying possible misinterpretations.

APPENDIX A: LIST OF SYMBOLS, ABBREVIATIONS, AND FORMULAS

A-1. Symbols Used

A, B = names of the two opposed sides. Normally, A is used for the numerically stronger side and B for the weaker.

A, B = rifleman strengths of sides A and B.

a, b = losses to sides A and B, respectively.

A', B' = remaining strengths of sides A and B, that is, $A' = A - a$ and $B' = B - b$.

t = elapsed time from the start of the engagement.

α = hits inflicted by each infantryman per unit time for each side, when this is the same for both sides, as in Eqs. (2)–(4) and (7)–(9).

α, β = hits inflicted by each infantryman per unit time for side A or B, respectively, when different for the two sides, as in Eqs. (5), (6), (6-bis), and (11).

β = hits inflicted by each artillery piece per unit time in Eqs. (7)–(9).

M, N = number of cannons on sides A and B in Eqs. (7)–(9) and (11).

γ, δ = hits inflicted by each artillery piece per unit time for sides A and B, respectively, in Eq. (11).

ϵ, ζ = hits inflicted by each machine gun per unit time for sides A and B, respectively, in Eq. (11).

P, Q = number of machine guns on sides A and B, respectively, in Eq. (11).

P = number of prisoners of war in List No. 1.

C = number of infantry turning the flank of the weaker side B in Eq. (10).

C, D = roster numbers for sides A and B in Eq. (13).

$m = \beta/\alpha$, ratio of flanking fire hit rate to that of frontal fire in Eq. (10).

m = effective ratio of active numbers to roster numbers in Eq. (13).

A-2. Formulas

$$A'^2 - B'^2 = A^2 - B^2, \tag{1}$$

$$Aa = Bb, \tag{1-bis}$$

$$A' = A - B(\alpha t) + \frac{A}{1\cdot 2}(\alpha t)^2 - \frac{B}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{A}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 - \cdots,$$

$$B' = B - A(\alpha t) + \frac{B}{1\cdot 2}(\alpha t)^2 - \frac{A}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{B}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 - \cdots, \tag{2}$$

$$1 + (\alpha t) + \frac{1}{1\cdot 2}(\alpha t)^2 + \frac{1}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 + \cdots,$$

$$1 - (\alpha t) + \frac{1}{1\cdot 2}(\alpha t)^2 - \frac{1}{1\cdot 2\cdot 3}(\alpha t)^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4}(\alpha t)^4 - \cdots, \tag{2-bis}$$

$$A' = A\,\frac{e^{\alpha t} + e^{-\alpha t}}{2} - B\,\frac{e^{\alpha t} - e^{-\alpha t}}{2},$$

$$B' = B\,\frac{e^{\alpha t} + e^{-\alpha t}}{2} - A\,\frac{e^{\alpha t} - e^{-\alpha t}}{2}, \tag{3}$$

$$A' = A\cosh(\alpha t) - B\sinh(\alpha t),$$

$$B' = B\cosh(\alpha t) - A\sinh(\alpha t), \tag{4}$$

$$\sqrt{\alpha}A' = \sqrt{\alpha}A\cosh(t\sqrt{\alpha\beta}) - \sqrt{\beta}B\sinh(t\sqrt{\alpha\beta}),$$

$$\sqrt{\beta}B' = \sqrt{\beta}B\cosh(t\sqrt{\alpha\beta}) - \sqrt{\alpha}A\sinh(t\sqrt{\alpha\beta}), \tag{5}$$

$$\alpha(A^2 - A'^2) = \beta(B^2 - B'^2) \quad \text{or} \quad A^2 - A'^2 = \frac{\beta}{\alpha}(B^2 - B'^2) \tag{6}$$

$$\left(A' + \frac{\beta}{\alpha}M\right) = \left(A + \frac{\beta}{\alpha}M\right)\cosh(\alpha t) - \left(B + \frac{\beta}{\alpha}N\right)\sinh(\alpha t),$$

$$\left(B' + \frac{\beta}{\alpha}N\right) = \left(B + \frac{\beta}{\alpha}N\right)\cosh(\alpha t) - \left(A + \frac{\beta}{\alpha}M\right)\sinh(\alpha t), \tag{7}$$

$$\left(A + \frac{\beta}{\alpha}M\right)^2 - \left(A' + \frac{\beta}{\alpha}M\right)^2 = \left(B + \frac{\beta}{\alpha}N\right)^2 - \left(B' + \frac{\beta}{\alpha}N\right)^2 \tag{8}$$

$$(A^2 - A'^2) = (B^2 - B'^2) - 2\frac{\beta}{\alpha}(aM - bN), \tag{9}$$

$$A' + mC = (A + mC)\cosh(\alpha t) - B\sinh(\alpha t),$$

$$B' = B\cosh(\alpha t) - (A + mC)\sinh(\alpha t),$$

$$(A + mC)^2 - (A' + mC)^2 = B^2 - B'^2, \tag{10}$$

$$\sqrt{\alpha}\left(A' + \frac{\gamma}{\alpha}M + \frac{\epsilon}{\alpha}P\right) = \sqrt{\alpha}\left(A + \frac{\gamma}{\alpha}M + \frac{\epsilon}{\alpha}P\right)\cosh(t\sqrt{\alpha\beta}) - \sqrt{\beta}\left(B + \frac{\delta}{\beta}N + \frac{\zeta}{\beta}Q\right)\sinh(t\sqrt{\alpha\beta}),$$

$$\sqrt{\beta}\left(B' + \frac{\delta}{\beta}N + \frac{\zeta}{\beta}Q\right) = \sqrt{\beta}\left(B + \frac{\delta}{\beta}N + \frac{\zeta}{\beta}Q\right)\cosh(t\sqrt{\alpha\beta}) - \sqrt{\alpha}\left(A + \frac{\gamma}{\alpha}M + \frac{\epsilon}{\alpha}P\right)\sinh(t\sqrt{\alpha\beta}),$$

$$\left(A' + \frac{\gamma}{\alpha}M + \frac{\epsilon}{\alpha}P\right)^2 = \left(A + \frac{\gamma}{\alpha}M + \frac{\epsilon}{\alpha}P\right)^2 - \frac{\beta}{\alpha}\left[\left(B + \frac{\delta}{\beta}N + \frac{\zeta}{\beta}Q\right)^2 - \left(B' + \frac{\delta}{\beta}N + \frac{\zeta}{\beta}Q\right)^2\right], \tag{11}$$

$$A^{3/2} - A'^{3/2} = B^{3/2} - B'^{3/2}, \tag{12}$$

$$Aa = Bb, \tag{1-bis}$$

$$\alpha Aa = \beta Bb, \tag{6-bis}$$

$$a\sqrt{A} = b\sqrt{B}, \tag{12-bis}$$

$$a = mC - \sqrt{m^2C^2 - (2mD - b)b}. \tag{13}$$

APPENDIX B: SCIENTIFIC INFORMATION REPORT: THE MATHEMATICAL MODELING OF MILITARY ENGAGEMENTS (EXTRACTS)

[*Note*: This appendix is a translation of paragraphs 10–28 of [12]—Tr.].

In the summer of the year 1914, the first World War started. Beginning with the very bloody battles of 1914 to the start of 1915, the Russian Army suffered many casualties. In these conditions, great significance was attached to the questions of determining the number of troops and of analyzing the likely casualties from conducting operations. The investigation our comrade, M. Osipov, devoted to this question resulted in his article

[7], "The Influence of the Numerical Strength of Engaged Sides on Their Casualties," and in his Addendum to it [8].

Osipov analyzed 38 battles of the 19th and 20th centuries and concluded that in general the distribution of casualties is related to the numerical strength of the sides in such a way that the numerically stronger side suffers fewer casualties than the weaker. He also advanced two hypotheses about the manner in which casualties depend on the general numbers of troops. According to his first conjecture, for smaller armies (no larger than 75,000 men), casualties to the sides are inversely proportional to their strengths. The second he put forth states that casualties to the sides are inversely proportional to the square roots of their numerical strengths.

Considering the first hypothesis, M. Osipov obtained a model of combat operations in the form of the following differential equations:

$$dx/dt = -\beta y, \qquad dy/dt = -\alpha x, \tag{1}$$

connecting the rates of decrease of the number of combatants (combat elements)

$$dx/dt \quad \text{and} \quad dy/dt$$

with the current numerical strengths of the sides x and y and their corresponding casualty-producing intensities α, β.

In deducing Eqs. (1), it was assumed that each side has a definite number of similar combat elements (firers), and each element in the current battle may be either in a combat effective state or a casualty.

If one integrates (solves) the Eqs. (1) with regard to the initial numbers x_0, y_0, then one finds

$$x_0^2 - x^2 = \frac{\beta}{\alpha}(y_0^2 - y^2), \tag{2}$$

from which when $\alpha = \beta$ follows the square law of casualties: In every phase of the battle the difference between the squares of the numbers of effectives on the engaged sides remains the same.

In elaborating his model, the author introduced heterogeneous causes of casualties: rifles, machine guns, and artillery weapons. In this connection, he introduced conversion factors for relating the loss intensities of one combat weapon to another. Analogous relations were demonstrated by M. Osipov to hold for his second hypothesis.

The author tested the adequacy of his models against the outcomes of several battles of the 19th and 20th centuries. He analyzed the causes [literally, origins—Tr.] of the random and the systematic errors. Among the former, for example, he cited those which arise from the art of leadership, the moral fiber of the troops, the relative numbers of reserves, the relative proportions of artillery and machine guns to riflemen, doctrine, the organization and training of the force, and novel means of defense and attack. The second type of error arises from variations in the local conditions, fortifications, the form of tactical operations, numbers of active fighters, density of the skirmish line, and the possibility of maneuvers, encirclements, and envelopments.

The results of his mathematical models and specific examples permitted M. Osipov to formulate a series of conclusions and results synthesized from his theory of war:

1. "By increasing our numerical strength, we cause our enemy greater casualties and at the same time reduce our own casualties."
2. "With superior numbers sending men into battle in the greatest number is not to sacrifice them uselessly, but on the contrary, this conserves them and gains time to attain the primary objective."
3. "Ineffectiveness in war is an ally of the enemy. That is why cowardice is always equated to betrayal."
4. "From the point of view of casualties, fortifications have immense value for the defender. Even field fortifications can be attacked by exposed forces only with considerable superiority in strength and especially in artillery."
5. "The weaker side, as well as the stronger one, benefits from advancing the greatest possible number of actives. It is completely in accord with well-known principles of military art to initiate and conduct military operations by fielding as many troops as possible."
6. "Rather than gradually thickening the skirmish line, advancing at the outset a strong line is advantageous rather than costly when our opponent engages all of his combat forces at the outset."
7. "The principle of defeating the enemy in detail unquestionably confirms the basic thesis of our theory, that casualties to the more numerous force are less than for the smaller." "Strength of itself does not crush, but strength at one point, and of course at the most important [point—Tr.] under the circumstances [7].

M. Osipov well understood that mathematical methods are not of themselves a substitute for a well-grounded theory of military art, but are the prerequisites for improving those arts and the competence and validity of their application. He said: "The only practical aim of the theory of casualties consists of a more conscientious management of the number of troops in order to reduce their casualties and increase those of the opponent" [7].

It seems to us that the work of M. Osipov is valuable for current studies of military questions by mathematical methods. Its methodological approach is excellent. Its scientific standard is high. This publication serves as an example of a thorough and complete approach to actual solutions for problems of military art, a model of the masterly application of mathematical statistics, differential calculus, and algebra. The author does not merely work out a mathematical model of attrition, but tests it by a detailed investigation of its adequacy.

In speaking about the value of this work to our comrades, we would like to emphasize three things.

First, M. Osipov's results are the simplest model of military operations in the form of differential equations (1), which later on became the source of numerous investigations.

Second, this scientist introduced conversion coefficients for converting one weapon into another. Such an approach is used to this day to develop certain quantitative-qualitative methods for estimating the correlation of forces and weapons of the sides.

Third, Osipov justified his list of random and systematic errors in modeling military operations and developed some methods for analyzing and accounting for them.

Unfortunately, the authorship and priority of M. Osipov in this field has been practically unrecognized. Equations (1) obtained by him are called Lanchester's, and the relation (2) discovered by this Russian scientist are similarly well known as the square law of the English scientist.

F.W. Lanchester published his own equations in 1916 in his book *Aircraft in Warfare* [5]. The whole of his research was quite different from that of Osipov. After beginning at first with the sea warfare of Great Britain, which was motivated by the necessity for checking its validity, he applied it numerically from the perspective of developing the military applications of aviation. It was F.W. Lanchester's special interest to investigate this problem. He analyzed the military possibilities of aviation, dwelled upon the problems of its armament, considered a series of solutions to its military problems, etc.

Starting from the same premises as M. Osipov, Lanchester in an analogous way obtained equations for two-sided military operations in the form (1). Carrying out the analysis of this model, he, too, arrived at conclusions about the importance of concentrating forces in battles, about the essential role of active forces, and the influence of the loss intensities. Lanchester explained to what extent the model is suitable for evaluating sea battles and analyzing the military applications of aviation.

Thus, judging from the publications, M. Osipov in the year before F.W. Lanchester's work appeared, published in the journal *Voenniy Sbornik* (*Military Collection*) the world's first practical dynamic model of military operations. It may be that both military authors obtained their dynamic equations independently of the other. Therefore, when using these equations, both their writings should be cited by referring to them as the Osipov-Lanchester equations and to the square law of Osipov-Lanchester.

The equations derived by the Russian and English scientists demonstrably had a strong influence on the subsequent modeling of military operations. In the opinion on N.N. Moiseyev, they "laid the foundations of the mathematical analysis of military operations" [6].

APPENDIX C: CORRECTIONS TO OSIPOV'S TABLES

C-1. We can distinguish the following possible sources of errors in tables such as those presented by Osipov, roughly in order of increasing severity. First are the inaccuracies tolerated for computational convenience, such as carrying only a limited number of decimal places through a series of computations or using other computational shortcuts. Second are the typographical errors introduced by slips of the pen or by typesetters. Third are the arithmetical slips or mistakes. Fourth are the conceptual missteps. In the following, we present corrected versions of some of Osipov's tables, and occasionally guess at the reasons for the discrepancies between Osipov's versions and ours. But we should make it clear at the outset that none of the discrepancies we found are serious, and none alter Osipov's major conclusions. Tables not mentioned below are either correct in the original, or else had only minor typographical errors that we corrected on translation without individually identifying them.

C-2. Osipov's List No. 1 gave the correction to strengths of the sides due to crossover of the victorious side as −309. Our computations give −260 for the crossover. This makes the correct values for the "Total for the victors" 4392 for the strength of the victorious sides (in column A), and 3623 for the strength of the losing sides (in column B). *These corrections have already been made to the version of List No. 1 provided in this translation.*

C-3. We found no errors in the left-hand section of Osipov's Table 3, that is, in the columns A, a, B, and b. In other words, those values were correctly transcribed from List No. 1. We discuss the remaining values in Table 3 under several categories.

a. In Table 3, Osipov rounded all the calculated (a) values to whole numbers and carried the rest of the computations forward using only whole numbers. It is clear that

Osipov adopted this expedient for computational convenience. Our first observation is that, if we use Osipov's rounded (a) values, then (apart from a few minor typographical errors) the rest of the numbers in Osipov's Table 3 are correct (except for the computation of probable errors, as discussed in paragraph C-4 below). The version of Table 3 given in Section 6 of this translation does this; that is, it uses Osipov's version of the rounded (a) values and computes the rest of the values correctly (except that it uses Osipov's probable error values). Table 9 reproduces that table for ready reference here.

b. However, in two or three cases, Osipov's rounded (a) values are a shade off. Changing them to the correct values makes very little difference in the results. This is shown in Table 10, where we first rounded the (a) values correctly and then did the rest of the computations correctly (except that Osipov's method for computing the probable error is used—see paragraph C-4). Changes from Table 9 are boxed.

c. It is also interesting to see what happens if *all* the calculations are carried to the maximum precision attainable with a personal computer spread-sheet program. That is, we want to see what happens when *unrounded* (a) values are used in the rest of the calculations. Table 11 shows the results, where the final values have been rounded correctly to the number of places shown, even though many more significant figures were carried through the computations that led to them. Naturally, none of its error values v are *exactly* zero. Again Osipov's method for computing the probable error is used—see paragraph C-4.

d. A comparison of Tables 9–11 shows that Osipov's values are sufficiently accurate to support his conclusions.

C-4. In his Table 3, Osipov computed the probable error of the sum of the v's by taking the square root of the sum of their squares, that is, by taking

$$\text{P.E.} = 0.67449 \times E,$$

where P.E. is the probable error, 0.67449 is the factor for converting standard errors to probable errors, and

$$E = \sqrt{\Sigma v^2}$$

is Osipov's value for the standard error. Osipov then compared these probable errors to the algebraic sum of the v's,

$$S = \Sigma v,$$

in order to estimate how often chance alone would produce a more extreme value of that sum. Conceptually, Osipov makes the comparison by treating the ratio

$$t_{\text{Osipov}} = \frac{S}{E}, \tag{C-1}$$

as though it followed the standard normal distribution. This is analogous to what is nowadays known as a t test, based on Student's t ratio,

$$t_{\text{Student}} = \frac{S - n\mu}{\sqrt{n/(n-1)}\sqrt{E^2 - S^2/n}}, \tag{C-2}$$

Table 9. Corrected Table 3, Using Osipov's Rounded (*a*) Values.

Battle name	A	a	B	b	B'	Formula (1) $(a)_1$	v_1	v_1^2	Formula (1-bis) $(a)_2$	v_2	v_2^2	Formula (12-bis) $(a)_3$	v_3	v_3^2
Craonne	30	8	18	5	13	3	−5	25	3	−5	25	4	−4	16
Kulm	46	9	35	10	25	7	−2	4	8	−1	1	9	0	0
Auerstadt	48	8	30	7	23	4	−4	16	4	−4	16	6	−2	4
Magenta	58	10	54	5	49	5	−5	25	5	−5	25	5	−5	25
Chernaya River	60	2	56	8	48	7	5	25	7	5	25	8	6	36
Aladja	60	2	36	15	21	8	6	36	9	7	49	12	10	100
Alma	62	3	34	6	28	3	0	0	3	0	0	4	1	1
Custozza	70	8	51	8	43	6	−2	4	6	−2	4	7	−1	1
Dennewitz	70	9	57	9	48	7	−2	4	7	−2	4	8	−1	1
Grochow	72	9	56	12	44	9	0	0	9	0	0	11	2	4
Subtotal	576	68	427	85	342	59	−9	139	61	−7	149	74	6	188
Jena	74	4	43	12	31	6	2	4	7	3	9	9	5	25
Berezhina	75	6	45	15	30	8	2	4	9	3	9	12	6	36
Hanau	75	15	50	9	41	6	−9	81	6	−9	81	7	−8	64
Katzbach	75	3	65	12	53	10	7	49	10	7	49	11	8	64
Aspern	75	25	70	35	35	31	6	36	33	8	64	34	9	81
Eylau Investment	80	25	64	26	38	19	−6	36	21	−4	16	23	−2	4
Austerlitz	83	27	75	12	63	11	−16	256	11	−16	256	11	−16	256
Friedland	85	12	60	15	45	10	−2	4	11	−1	1	13	1	1
Inkerman	90	12	63	6	57	4	−8	64	4	−8	64	5	−7	49
Subtotal	712	129	535	142	393	105	−24	534	112	−17	549	125	−4	580
Laon	100	2	45	6	39	3	1	1	3	1	1	4	2	4
Waterloo	100	22	72	32	40	20	−2	4	23	1	1	27	5	25
Worth	100	10	45	5	40	2	−8	64	2	−8	64	3	−7	49
Ligny	120	11	85	18	67	12	1	1	13	2	4	15	4	16
Mars-la-Tour	125	16	65	16	49	8	−8	64	8	−8	64	12	−4	16
Borodino	130	35	103	40	63	29	−6	36	32	−3	9	36	1	1
Liaoyang	150	18	120	24	96	18	0	0	19	1	1	21	3	9
Lutzen	157	15	92	12	80	7	−8	64	7	−8	64	9	−6	36
Dresden	160	20	125	15	110	11	−9	81	12	−8	64	13	−7	49
Wagram	160	25	124	25	99	18	−7	49	19	−6	36	22	−3	9
Subtotal	1302	174	876	193	683	128	−46	364	138	−36	308	162	−12	214
Bautzen	163	18	96	12	84	7	−11	121	7	−11	121	9	−9	81
Solferino	170	20	150	18	132	16	−4	16	16	−4	16	17	−3	9
Metz	200	6	173	20	153	17	11	121	17	11	121	19	13	169
Sha-Ho	212	40	157	20	137	14	−26	676	15	−25	625	17	−23	529
Gravelotte	220	20	130	12	118	7	−13	169	7	−13	169	9	−11	121
Koennigratz	222	10	215	43	172	41	31	961	41	31	961	42	32	1024
Sedan	245	9	124	17	107	8	−1	1	9	0	0	12	3	9
Leipzig	300	50	200	60	140	36	−14	196	40	−10	100	49	−1	1
Mukden	330	59	280	70	210	57	−2	4	60	1	1	64	5	25
Subtotal	2062	232	1525	272	1253	203	−29	2265	212	−20	2114	233	6	1968
Grand total	4652	603	3363	692	2671	495	−108	3302	523	−80	3120	599	−4	2950
Sum of errors as percent of sum of losses:							−22%			−15%			−0.7%	
Number of errors greater than 0							10			13			18	
Number of errors equal to 0							3			3			1	
Number of errors less than 0							25			22			19	
Probable error in the grand total of the calculated casualties:							39			38			37	

which is distributed as Student's t distribution with $n - 1$ degrees of freedom, where n is the number of v's (also called the sample size), and μ is the value ascribed to the true mean value of the observations. In Osipov's case, μ is always zero, since he is seeking formulas that fit the observations with no systematic bias (that is, have an average error of zero). When (i) $\mu \equiv 0$, (ii) $|S|$ is sufficiently small compared to E, and (iii) $n \geq 38$,

Table 10. Corrected Table 1, Using Accurately Rounded (a) Values.

Battle name	A	a	B	b	B'	Formula (1) $(a)_1$	v_1	v_1^2	Formula (1-bis) $(a)_2$	v_2	v_2^2	Formula (12-bis) $(a)_3$	v_3	v_3^2
Craonne	30	8	18	5	13	3	−5	25	3	−5	25	4	−4	16
Kulm	46	9	35	10	25	7	−2	4	8	−1	1	9	0	0
Auerstadt	48	8	30	7	23	4	−4	16	4	−4	16	6	−2	4
Magenta	58	10	54	5	49	5	−5	25	5	−5	25	5	−5	25
Chernaya River	60	2	56	8	48	7	5	25	7	5	25	8	6	36
Aladja	60	2	36	15	21	8	6	36	9	7	49	12	10	100
Alma	62	3	34	6	28	3	0	0	3	0	0	4	1	1
Custozza	70	8	51	8	43	6	−2	4	6	−2	4	7	−1	1
Dennewitz	70	9	57	9	48	7	−2	4	7	−2	4	8	−1	1
Grochow	72	9	56	12	44	9	0	0	9	0	0	11	2	4
Subtotal	576	68	427	85	342	59	−9	139	61	−7	149	74	6	188
Jena	74	4	43	12	31	6	2	4	7	3	9	9	5	25
Berezhina	75	6	45	15	30	8	2	4	9	3	9	12	6	36
Hanau	75	15	50	9	41	6	−9	81	6	−9	81	7	−8	64
Katzbach	75	3	65	12	53	10	7	49	10	7	49	11	8	64
Aspern	75	25	70	35	35	31	6	36	33	8	64	34	9	81
Eylau Investment	80	25	64	26	38	19	−6	36	21	−4	16	23	−2	4
Austerlitz	83	27	75	12	63	11	−16	256	11	−16	256	11	−16	256
Friedland	85	12	60	15	45	10	−2	4	11	−1	1	13	1	1
Inkerman	90	12	63	6	57	4	−8	64	4	−8	64	5	−7	49
Subtotal	712	129	535	142	393	105	−24	534	112	−17	549	125	−4	580
Laon	100	2	45	6	39	3	1	1	3	1	1	4	2	4
Waterloo	100	22	72	32	40	20	−2	4	23	1	1	27	5	25
Worth	100	10	45	5	40	2	−8	64	2	−8	64	3	−7	49
Ligny	120	11	85	18	67	12	1	1	13	2	4	15	4	16
Mars-la-Tour	125	16	65	16	49	8	−8	64	8	−8	64	12	−4	16
Borodino	130	35	103	40	63	29	−6	36	32	−3	9	36	1	1
Liaoyang	150	18	120	24	96	18	0	0	19	1	1	21	3	9
Lutzen	157	15	92	12	80	7	−8	64	7	−8	64	9	−6	36
Dresden	160	20	125	15	110	11	−9	81	12	−8	64	13	−7	49
Wagram	160	25	124	25	99	18	−7	49	19	−6	36	22	−3	9
Subtotal	1302	174	876	193	683	128	−46	364	138	−36	308	162	−12	214
Bautzen	163	18	96	12	84	7	−11	121	7	−11	121	9	−9	81
Solferino	170	20	150	18	132	16	−4	16	16	−4	16	17	−3	9
Metz	200	6	173	20	153	17	11	121	17	11	121	19	13	169
Sha-Ho	212	40	157	20	137	14	−26	676	15	−25	625	17	−23	529
Gravelotte	220	20	130	12	118	7	−13	169	7	−13	169	9	−11	121
Koennigratz	222	10	215	43	172	41	31	961	42	32	1024	42	32	1024
Sedan	245	9	124	17	107	8	−1	1	9	0	0	12	3	9
Leipzig	300	50	200	60	140	36	−14	196	40	−10	100	49	−1	1
Mukden	330	59	280	70	210	57	−2	4	59	0	0	64	5	25
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Grand total	4652	603	3363	692	2671	495	−108	3302	523	−80	3182	599	−4	2950
Sum of errors as percent of sum of losses:							−22%			−15%			−0.7%	
Number of errors greater than 0							10			12			18	
Number of errors equal to 0							3			4			1	
Number of errors less than 0							25			22			19	
Probable error in the grand total of the calculated casualties:							39			38			37	

which apply to most of the situations considered by Osipov, then the two t ratios are not much different; that is,

$$t_{\text{Osipov}} \doteq t_{\text{Student}}.$$

Furthermore, both t_{Osipov} and t_{Student} approximately follow the standard normal distribution when $n \geq 38$. Hence, Osipov's calculations of how often chance alone would

Table 11. Corrected Table 3, Using Unrounded (*a*) Values.

Battle name	A	a	B	b	B'	Formula (1) $(a)_1$	v_1	v_1^2	Formula (1-bis) $(a)_2$	v_2	v_2^2	Formula (12-bis) $(a)_3$	v_3	v_3^2
Craonne	30	8	18	5	13	3	−5	28	3	−5	25	4	−4	17
Kulm	46	9	35	10	25	7	−2	4	8	−1	2	9	−0	0
Auerstadt	48	8	30	7	23	4	−4	16	4	−4	13	6	−2	6
Magenta	58	10	54	5	49	5	−5	29	5	−5	29	5	−5	27
Chernaya River	60	2	56	8	48	7	5	29	7	5	30	8	6	33
Aladja	60	2	36	15	21	8	6	31	9	7	49	12	10	93
Alma	62	3	34	6	28	3	0	0	3	0	0	4	1	2
Custozza	70	8	51	8	43	6	−2	6	6	−2	5	7	−1	1
Dennewitz	70	9	57	9	48	7	−2	4	7	−2	3	8	−1	1
Grochow	72	9	56	12	44	9	−0	0	9	0	0	11	2	3
Subtotal	576	68	427	85	342	58	−10	146	62	−6	155	72	4	182
Jena	74	4	43	12	31	6	2	5	7	3	9	9	5	26
Berezhina	75	6	45	15	30	8	2	4	9	3	9	12	6	32
Hanau	75	15	50	9	41	6	−9	87	6	−9	81	7	−8	59
Katzbach	75	3	65	12	53	10	7	51	10	7	55	11	8	67
Aspern	75	25	70	35	35	31	6	34	33	8	59	34	9	78
Eylau Investment	80	25	64	26	38	19	−6	39	21	−4	18	23	−2	3
Austerlitz	83	27	75	12	63	11	−16	267	11	−16	261	11	−16	243
Friedland	85	12	60	15	45	10	−2	5	11	−1	2	13	1	0
Inkerman	90	12	63	6	57	4	−8	63	4	−8	61	5	−7	49
Subtotal	712	129	535	142	393	104	−25	554	111	−18	554	125	−4	556
Laon	100	2	45	6	39	3	1	0	3	1	0	4	2	4
Waterloo	100	22	72	32	40	20	−2	4	23	1	1	27	5	27
Worth	100	10	45	5	40	2	−8	62	2	−8	60	3	−7	44
Ligny	120	11	85	18	67	12	1	1	13	2	3	15	4	17
Mars-la-Tour	125	16	65	16	49	8	−8	72	8	−8	59	12	−4	20
Borodino	130	35	103	40	63	29	−6	40	32	−3	11	36	1	0
Liaoyang	150	18	120	24	96	18	0	0	19	1	1	21	3	12
Lutzen	157	15	92	12	80	7	−8	69	7	−8	63	9	−6	34
Dresden	160	20	125	15	110	11	−9	74	12	−8	69	13	−7	45
Wagram	160	25	124	25	99	18	−7	42	19	−6	32	22	−3	9
Subtotal	1302	174	876	193	683	128	−46	364	138	−36	300	163	−11	213
Bautzen	163	18	96	12	84	7	−11	126	7	−11	120	9	−9	77
Solferino	170	20	150	18	132	16	−4	19	16	−4	17	17	−3	10
Metz	200	6	173	20	153	17	11	122	17	11	128	19	13	159
Sha-Ho	212	40	157	20	137	14	−26	658	15	−25	634	17	−23	519
Gravelotte	220	20	130	12	118	7	−13	172	7	−13	167	9	−11	116
Koenniggratz	222	10	215	43	172	41	31	981	42	32	1001	42	32	1044
Sedan	245	9	124	17	107	8	−1	1	9	−0	0	12	3	10
Leipzig	300	50	200	60	140	36	−14	191	40	−10	100	49	−1	1
Mukden	330	59	280	70	210	57	−2	5	59	0	0	64	5	30
Subtotal	2062	232	1525	272	1253	203	−29	2274	212	−20	2167	239	7	1966
Grand total	4652	603	3363	692	2671	493	−110	3338	523	−80	3176	599	−4	2917
Sum of errors as percent of sum of losses:							−22%			−15%			−0.7%	
Number of errors greater than 0:							12			15			18	
Number of errors equal to 0:							0			0			0	
Number of errors less than 0:							26			23			20	
Probable error in the grand total of the calculated casualties:							39			38			36	

produce a more extreme value of S are approximately correct. Student's famous article appeared in 1908, about 7 years before Osipov's was published. Fisher's definitive proof of its theoretical correctness appeared in 1925, 10 years after Osipov's. Either Osipov was familiar with Student's article and took some shortcuts for computational convenience, or else (which we think more likely) Osipov used an approach that would be

Table 12.

Formula	Student's t ratio	Lower-tail probability: By the t distribution	By the normal distribution	According to Osipov
(1)	−1.947	0.029	0.026	<0.03
(1-bis)	−1.453	0.077	0.073	0.08
(12-bis)	−0.073	0.471	0.471	0.47[a]

[a]Actually, Osipov's original gives a value equivalent to 0.498, which we changed in our translation to the value shown here, on the assumption that Osipov erred in reading the normal probability table.

accepted unquestioningly by the overwhelming majority of his contemporaries, despite today's view that the t test is more appropriate.

Our calculations afford a comparison with Osipov's chances of more extreme values; see Table 12.

C-5. The corrections made to Table 3 affect Table 6. The values previously given in Section 6 of this translation are shown in Table 13. These are based on the values given in Table 3 in Section 6 of this translation. If the unrounded values given in Table 12 of this appendix are used, then we get the values shown in Table 14. The differences are not large enough to make any difference to Osipov's main conclusions.

C-6. Osipov's original version of Table 8 in Section 7 appears to contain several errors. The values actually given in Osipov's article are shown in Table 15. The values previously given in Section 7 of this translation are shown in Table 16. Table 16 was obtained from this translation's version of Table 3 (that is, it is based on Osipov's rounded (a) values). It appears that Osipov made several arithmetical mistakes in his version.

Table 13. Section 6 version of Table 6. Values in thousands.

Battle nos.	A	a	B	b	$(a)_1$	v_1	v_1^2	$(a)_2$	v_2	v_2^2	$(a)_3$	v_3	v_3^2
1–15	950	121	700	168	120	−1	313	126	+5	361	147	+26	458

Table 14. Unrounded version of Table 4. Values in thousands.

Battle nos.	A	a	B	b	$(a)_1$	v_1	v_1^2	$(a)_2$	v_2	v_2^2	$(a)_3$	v_3	v_3^2
1–15	950	121	700	168	118.9	−2.09	326.9	126.9	+5.9	367.6	145.4	+24.4	443.1

Table 15. Osipov's original version of Table 8. Values in thousands.

Battle group	A	a	B	b	$(a)_1$	v_1	$v_1\%$	$(a)_2$	v_2	$v_2\%$	$(a)_3$	v_3	$v_3\%$
Early	2688	407	1967	442	305	−102	33	323	−84	26	378	−29	8
Late	1964	196	1396	250	169	−27	16	177	−19	11	211	+15	7

Table 16. Section 7 version of Table 8. Values in thousands.

Battle group	A	a	B	b	$(a)_1$	v_1	$v_1\%$	$(a)_2$	v_2	$v_2\%$	$(a)_3$	v_3	$v_3\%$
Early	2718	405	1967	442	309	−96	−31	330	−75	−23	381	−24	−6
Late	1934	198	1396	250	186	−12	−6	193	−5	−3	218	+20	+9

ACKNOWLEDGMENTS

This translation owes much to many people. The work was initiated in the early 1980s by Dr. Allan Rehm, who first called the world's attention to Osipov's article and who supplied Helmbold a copy of the Russian originals. Helmbold then made a very rough draft translation into English. Dr. Jakob Kipp and Dr. Maia Kipp graciously reviewed this rough draft. Jakob at the time was a Senior Analyst at the U.S. Army's Soviet Army Studies Office (SASO) at Fort Leavenworth, and Maia a native Russian speaker and Professor of Slavic and East European Literature at the University of Kansas. They corrected countless horrendous blunders and howlers, for which we are most grateful. Dr. Rehm, a well-known expert on Soviet military operations research and now with the MITRE Corporation, not only made an entirely independent translation from the Russian original, but labored along with Helmbold to add many crucial refinements to the translation and to improve its conversion into proper English. Without his participation, this translation would never have seen the light of day. We were also able to consult a rough draft translation done in the mid-1980s by Deborah Coulter-Harris of SASO. The final version is a consolidated and edited version based on the various English translations. We are also indebted to Professor James Taylor of the U.S. Naval Postgraduate School, who pointed out to us the article in the Military-Historical Journal from which Appendix B was translated. The translators, of course, are fully responsible for any inadequacies or shortcomings that may remain in this translation.

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Manuscript received October 16, 1991
Accepted August 12, 1993