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MATHEMATICAL CONTRIBUTIONS TO THE SCIENTIFIC UNDERSTANDING OF WAR†

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Abstract—During the past 50 years mathematics has provided a unique tool for the rational acquisition of new objective knowledge about war as a social phenomenon. Four discoveries are examined: the *hyporeliability effect* in deterrence systems aimed at war prevention; the *N-crisis problem* of war escalation (originally conjectured by Q. Wright); a *ledge of time* at the onset of war; and a phenomenon called *war dilation*, which explains the decreasing propensity of wars to terminate once they have begun. Though in different ways, each of these discoveries exemplifies significant new aspects of war which would not have been discovered without the medium of mathematics.

INTRODUCTION

War is a lethal social phenomenon presently dated from *ca* 8000 B.C. (Ferrill 1985). When viewed against the background of such a long-range timeline, war is thus a form of social interaction which has been reoccurring for $\approx 10^4$ years. However, until very recently the study of war—i.e. research to investigate its causes, mechanisms and the like—was conducted without the assistance of mathematics. Thucydides, Caesar, von Clausewitz, Machiavelli, Tolstoy and Sun-Tzu are well-known thinkers which exemplify the earlier, pre-scientific stage of the study of war.

In the first decades of this century the pioneering work of Lewis Fry Richardson—followed a few years later by that of Quincy Wright, Nicholas Rashevsky, Anatol Rapoport, Kenneth Boulding, Thomas Saaty and others more recently—began a scientific revolution in the study of war through the application of mathematics. To date, many different mathematical models have been applied to investigate aspects of war, such as patterns of onset, intensity, propagation and diffusion.‡ As in any other young field of science, many new avenues remain unexplored, but it is now clear to scientists working in this area that some new aspects of war have already been discovered as a direct result of mathematical modeling and analysis. Significantly, most of these discoveries on the nature of war would not have occurred without the assistance of mathematics.

The purpose here is to consider a small sample of realworld aspects of war which have been discovered through the medium of mathematics. Indeed, a key feature of the phenomena we shall examine is that the main discovery would not have taken place without the application of mathematical methods. We also show how, in some instances, results were obtained through the combined use of mathematics and empirical, statistical approaches.

In their chronologic order of occurrence in the realworld, the four war-related phenomena we shall consider pertain to the pre-war, onset and duration phases of war. The first case concerns a phenomenon called the *hyporeliability effect*, an inherent feature of deterrent systems aimed at preventing war. The second is about properties of crisis escalation and the probability of war, in a generic epoch of history containing N international crises (the so-called *N-crises problem*). The third case has to do with an intriguing time period which occurs prior to the onset of war, called the *war ledge*. The fourth case concerns a property called *war dilation*, which sheds new light on the nature of warfare and some unstable properties in the international system.

A word is in order about these four war-related phenomena. First, since the purpose here is to

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‡An extensive bibliography may be found in Cioffi-Revilla (1979). Recent books illustrating mathematical approaches to the study of war include the following: Allan (1983), Brams (1985), Brams and Kilgour (1988), Bueno de Mesquita (1981), Cioffi-Revilla, Merritt and Zinnes (1987), Dacey (1988), Kugler and Zagare (1987) and Luterbacher and Ward (1985), and others cited later in this paper.

illustrate how and why some uses of mathematics have made some significant contributions to our current scientific understanding of war, no attempt has been made to be all-inclusive. Second, the mathematical models of these four phenomena are based on applications of elementary probability; no attempt has been made to include numerous contributions made by other types of mathematical structures (e.g. dynamical systems, game theory etc.). This is because war is a type of social phenomenon naturally amenable to probabilistic treatment (deterministic models also exist). Also, while it is often though that probabilistic ideas are vague or imprecise, these cases demonstrate the opposite: remarkably precise knowledge about war can originate from the use of probability models. Viewing war as a random phenomenon, even in an objective sense (Suppes 1984), does not render it immune to rational scientific understanding; *au contraire*. A third feature is that each case illustrates the *cognitive* (as opposed to merely *descriptive*) use of mathematics in this area of science. Thus, here we are chiefly concerned with what many consider to be the most powerful contribution of mathematics to empirical science—for obtaining new knowledge and developing theoretical understanding—rather than with subsidiary functions, such as description, prediction or engineering control.† Fourthly, each case shows the progressive, expansive power of mathematics in generating new research paths, showing new aspects of war not accessible by intuition or historical observation alone. Paraphrasing M. Kline (1985, p. v), we shall be solely concerned with realworld aspects of war which have become accessible *only* through the medium of mathematics. Finally, each case has the methodological property of being accessible through elementary, rather than advanced mathematics—most ideas stop short of calculus. This demonstrates that new scientific knowledge obtained through mathematical means does not always require the application of advanced methods.

THE HYPORELIABILITY EFFECT

In the transition from peace to war, the first case of mathematical insight we shall consider is called the *hyporeliability effect*—an intriguing phenomenon inherent in all inter-nation security relations which are maintained by some type of *deterrence-type system*. To understand the phenomenon, we shall first briefly describe the character of a deterrent relation, for this is where the phenomenon occurs, and then focus on the hyporeliability effect itself.

Inter-nation Deterrent Relations

Deterrence is an ancient method for attempting to prevent war, and is still probably the most common type of inter-nation security relation found in the international system (Naroll, Bullough and Naroll 1974; Huth and Russett 1984, 1988). Though modern theory distinguishes several species of the *genus* deterrence (e.g. Kahn 1960; Schelling 1960; Morgan 1983), the fundamental logic on which all deterrence relations rest is the same: *given a pair of adversary nations, a deterrence relation exists whenever at least one nation of the two is dissuaded from attacking the other by fear that the other might retaliate* (Brodie 1946; Snyder 1959; Raser 1969). When both nations in a pair are related by deterrence, this forms a system of mutual assured destruction (MAD), as with the superpowers, though one-way deterrence relations are not uncommon (Israel and Arab nations; India and Pakistan; South Africa and regional neighbors). Weapons and organizations used for maintaining deterrence vary considerably (i.e. the actual politico-military hardware and organization of national deterrent systems). Also, weapons need not be nuclear [in fact, they need not even be weapons of mass destruction; see Mearsheimer (1983)]. What counts is maintaining an unacceptably high level of expected damage through retaliation (mathematically, an expected value). Currently, nuclear weapons provide a common, though not unique, way of implementing deterrence.

Given these general concepts, consider the deterrent system, Z, of a single nation in a deterrent relation with some other nation. For Z to work (i.e. maintain deterrence = dissuade an adversary's

†The aforementioned distinction between *cognitive* and *descriptive* uses of mathematics in empirical science is important (Bochner 1966; Kline 1985). The main descriptive function of mathematics in this field is for summarizing empirical war data into laws or statistical regularities.

attack) it must meet a set of requisites (tasks, functions, operational requirements) with credibly high overall reliability probability of success. This includes the political will to carry out post-attack retaliation(s). In a landmark paper, A. Wohlstetter (1959, p. 216) first proposed the following six requisites for a viable deterrence system Z:

> "Deterrent systems must have [1] a stable "steady-state" peacetime operation within feasible budgets.... They must also have the ability [2] to survive enemy attracks, [3] to make and communicate decisions to retaliate, [4] to reach enemy territory with fuel enough to complete their mission, [5] to penetrate enemy active defenses, ... and [6] to destroy the target in spite of any "passive" civil defense."

Wohlstetter's requirements are stated in terms of the military technology of his time (in the age of Star Wars referents would vary). Here interest lies in the logic of the multi-requisite structure of a deterrent system.

More generally, a deterrent system Z consists of two sets of coordinated entities: (a) a set $\mathbb{C}$ containing C operational component entities (decisionmakers, organizations, communications channels, weapons systems etc.), *organized to perform* (b) a set $\mathbb{Q}$ consisting of S necessary functions (requisites, missions, "logic blocks"). The elements of $\mathbb{Q}$ are jointly needed for retaliation (assured destruction). Thus, if $\mathbb{C}$ denotes the set of deterrent components, then C is the cardinality of $\mathbb{C}$, with $C = 1, 2, 3, \ldots$; and, similarly, if $\mathbb{Q}$ is the set of deterrence requisites, then S is the cardinality of $\mathbb{Q}$, with $S = 1, 2, 3, \ldots, n$. We note immediately that, since $S > 1$, the fundamental structure of Z is *serial*, not parallel (though any element of $\mathbb{Q}$ may have deeper parallel structures). Modeling and analyzing the overall reliability R_Z of a given deterrent system $Z(S, C)$—where R_Z is defined as the probability that Z will work—is of central interest, for empirically it is never the case that every required function $q \in \mathbb{Q}$ will obtain with certainty. Generally, $0 \leqslant R_Z < 1$.†

Let r_i denote the reliability probability of successful performance of the ith deterrence requisite of Z. (For example, in the Wohlstetter model r_3 is the reliability of command, control, communications and intelligence, or C^3I.) Using the standard general theory of systems reliability, and assuming independence across requirements in $\mathbb{Q}$, then (by the multiplication theorem for the probability of compound events) overall deterrent reliability R_Z is given by

$$R_Z = R(r_i; n) = r_1 r_2 r_3 \ldots r_n = R(r, S) = r^S, \qquad 0 < r < 1, \tag{1}$$

where r is an average reliability across elements in $\mathbb{Q}$ (deterrence requisites) and S is the serial order of Z (cardinality of $\mathbb{Q}$). Since weapons and organizations vary across national deterrent systems, S is best viewed as a discrete variable taking integer values $1, 2, 3, \ldots, n$, rather than as a constant. Though formally simple (and well-known to systems analysts and operations researchers), equation (1) has several interesting properties which shed light on how deterrence works in the realworld (and how it may fail).

Hyporeliability of Deterrent Systems

Using these ideas, the hyporeliability effect refers to the following phenomenon: *in all realworld cases (i.e. when $S > 1$, $0 < r < 1$) the overall reliability R_Z of a deterrent system Z is always strictly* ***lower*** *than the* ***lowest*** *in the set of lower-level reliabilities*; i.e.

$$R_Z < \min\{r_1, r_2, r_3, \ldots, r_n\}. \tag{2}$$

Using equation (1), the *intensity* of the hyporeliability effect may be expressed as

$$\Delta R = R - r = r^S - r. \tag{3}$$

So $\Delta R < 0$ (whence *hypo*reliability), since $S > 1$.

Though perhaps well-known to modelers in other fields, this debilitating phenomenon inherent in all deterrent systems ($\Delta R < 0$) is seldom recognized by conflict and war analysts. Note that, since

†Without loss of essential detail, here we may ignore both the role of $\mathbb{C}$ (because the hyporeliability effect is best detected against the background of $\mathbb{Q}$), as well as the nature of the mapping $\mathbb{C} \to \mathbb{Q}$ (the functional relations matrix from components to functions).

every deterrent system Z has a fundamentally serial structure (because $S > 1$), the hyporeliability effect is a universal phenomenon. In particular, it does not vanish with improvements in weapons, communications or organizational aspects of Z, nor does there exist a mapping $\mathbb{Q} \rightarrow \mathbb{C}$ which can make it disappear. It is a robust effect, in the strong sense that it stems fundamentally from the serial structure of Z—the inequality (3) is an inherent property of all multi-task, multi-functional systems.

A closer look at the hyporeliability effect is useful to appreciate what it means for deterrence. Consider a naive observer (e.g. a politician untrained in mathematical reliability theory) having to assess the overall reliability $R_Z(r, S)$ of a deterrent system Z with S requirements, as just described. Such an observer will subjectively assess R_Z using primarily two *heuristics*. The first is to examine the set $\{r_i\}$ (obtained in some fashion, such as from expert advice), and mentally computing the rough mean value. This yields an estimate of R_Z, expressed as

$$E_{\text{mean}}(R) = \left(\sum r_i\right)\Big/ n. \tag{4}$$

By inequality (2), however, the value of E_{mean} (R) is always greater than the objective reliability R_Z, so E_{mean} (R) always overestimates R_Z.

The second heuristic, used by "more sophisticated" subjects, is based on viewing $\mathbb{Q}$ as a "chain" (or set of "hurdles," as Wohlstetter called them), and then use the aphorism "a chain is only as strong as its weakest link". This estimate may be expressed as

$$E_{\text{chain}}(R) = \min\{r_i\}. \tag{5}$$

Though R is closer to E_{chain} than to E_{mean}, the hyporeliability effect says that E_{chain} also overrates, since R is (objectively) not *as* weak, but *weaker than* the weakest link. E_{chain} is incorrect (though not as much as E_{mean}) because it disregards the effect of the remaining $(n - 1)$ "links" making the entire chain even weaker than its weakest link. Thus,

$$R_Z < E_{\text{chain}} < E_{\text{mean}}. \tag{6}$$

Both heuristics overestimate, so both are objectively incorrect.

Though hyporeliability is not apparent from plain intuition alone, it objectively weakens efforts to strengthen deterrence, inducing a reductive effect quite the opposite of a multiplier. The effect is detectable only through mathematical analysis of the properties of reliability probability R_Z.

Hyporeliable Deterrence and Subjectivity

Beyond its intrinsic importance, the hyporeliability effect also raises questions about strategic decisionmaking and war. For instance, consider recent findings in experimental psychology and cognitive science, on how humans process information to form subjective probability estimates. (The "human factors" literature is also relevant here.) According to experiments by Khaneman and Tversky (1979, 1982), when confronted with outcomes in a decisionmaking situation (events, in a probabilistic sense), humans estimate the probability r of an outcome by assigning a likelihood weight, w, which generally differs from the objective value r, as shown in Fig. 1. In the *low* probability range (i.e. $r < r^*$ in Fig. 1), subjects overestimate r, and for medium and high values $(r > r^*)$ they underestimate. Thus, the probability r of a highly likely event (even when the event may be a deterrence-related disaster, such as failure in C^3I) is *underestimated*, $w < r$, for $r^* < r < 1$; whereas that of a very improbable event (such as a pre-emptive enemy strike) is *overestimated*, $w \gg r$, for $1 < r < r^*$. As shown in Fig. 1, the estimate for the latter case may be off by as much as a factor of 10.

The implications of these ideas for a proper understanding of hyporeliability in deterrence relations remain unexplored. In particularly, little is known about the link between such cognitive phenomena and the prevention of war (the chief purpose of deterrence). However, the objective existence of subjective probability weights readily suggests that the net practical effect of hyporeliability, ΔR, must vary, depending on the value of r with respect to r^*. Thus, inequality (3) should reflect this dependence, as $\Delta R = f(r - r^*)$. In particular, since decisionmakers overestimate in the range $0 < r < r^*$, ΔR is strongest in the interval $(0, r^*)$, and increases as $r \rightarrow 0^+$.

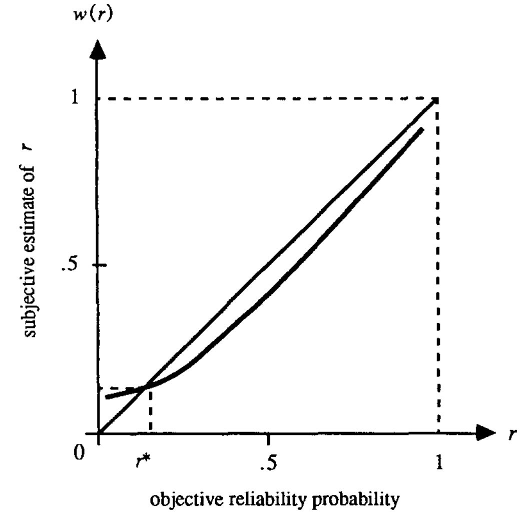

Fig. 1. Graph of a weight function $w(r)$ relating subjective to objective reliability probability r. The r^* value across individuals ($0 \lesssim r^* \lesssim 0.25$). *Source:* adapted from Kahneman and Tversky (1979, 1982, p. 168).

Similarly, the effect is least pronounced (and tends to vanish) in the upper range, where $r^* < r < 1$, since there the subjective underestimation of objectively high reliability tends to balanced the hyporeliability effect. The case when $1/2 \lesssim r \lesssim 3/4$ is interesting, for here subjective underestimation reaches its maximum, so the net result approximates the objective probability value. For a decisionmaker, therefore, the intensity of the hyporeliability effect, ΔR, varies depending on the objective value of r [and on the decisionmaker's personal weight function $w(r)$]. In any case, the joint occurrence of low and high probability events, in the war and peace decisions commonly associated with deterrence, may easily result in miscalculation.

THE N-CRISES PROBLEM

Our second case of discovery by mathematical means actually consists of two separate aspects of war within a more general puzzle called the *N-crises problem.* The two aspects of war are: (1) the rapidity with which the probability of war approaches certainty as crises (war opportunities) re-occur in the international system; and (2) the tradeoff (for avoiding war) between avoiding crises and lowering war escalation probability in each crisis. To understand both phenomena, we shall first describe the general puzzle within which they arise, the N-crises problem, and then examine each phenomena separately.

The N-crises Problem and Wright's Solution

The N-crises problem concerns the relation between crises, escalation and wars, and may be described as follows: *in a period of history containing N crises (say, in a decade), where each individual crisis may escalate to war with probability p, what is the overall probability of war P over the entire period?*

The first published solution to this problem (it may have been posed earlier, possibly in the eighteenth century) was conjectured some 30 years ago by Q. Wright (1942), and proven more recently (Cioffi-Revilla 1987a). Wright's classic solution states that

$$P = 1 - (1 - p)^N, \tag{7}$$

and is based on assuming the independence of crises occurrences.

Two other solutions to the N-crises problem have been derived more recently (Cioffi-Revilla and Dacey 1988). One is based on an analogue to Daniel Bernoulli's *St Petersburg Paradox*, the other on the interpretation of the natural language phrase "during N crises" as in the common conditional probability interpretation (i.e. as meaning "war occurs, given that N crises occur"). For simplicity, here we shall use Wright's classic solution, equation (7), since it is the current, dominant solution in the interpretation of the N-crises problem (Russett 1983; Deutsch 1978).

Two important questions arise in connection with the N-crises problem and Wright's solution, equation (7). The first is "how fast" does war probability P approaches certainty (the speed of the convergence $P \to 1$) as the number of crises (N) increases. The second regards the different sensitivity of $P(p, N)$ with respect to N and with respect to p. Answers to these realworld questions are not easily obtained—or not available at all—without mathematical analysis.

War Probability as Crises Recur

The rapidity with which $P \to 1$ as $N \to \infty$ is interesting for several reasons. First, though ordinary intuition correctly suggests that $P \to 1$ as $N \to \infty$, since the greater the opportunities for war (crises) the greater the likelihood of war, intuition alone tells us nothing about the *speed* with which convergence to certainty (or near certainty) actually occurs. Second, Wright used the idea of the limit "as $N \to \infty$" for normative purposes [he was issuing a warning about the risk of war if crises continued to occur; similar to Rusten and Stern (1987, p. 1) more recently]. Though this is proper in a mathematical sense, his discussion suggests that he actually viewed N as having to be some sense "extremely large" for $P \to 1$, otherwise he would have simply said "as N increases", rather than "as $N \to \infty$".

As it turns out, $P \to 1$ very quickly, as N increases, being virtually equal to 1 while N is still quite small, so long as p is anything but negligible. Thus, for $N = 5$ (which currently occurs in a period of only a few years),

$$0.67 \lesssim P \lesssim 0.97, \qquad \text{for } 0.2 \lesssim p \lesssim 0.5.$$

For a period twice as long (still less than a decade), when $N = 10$, we have

$$0.893 \lesssim P \lesssim 0.999, \qquad \text{for } 0.2 \lesssim p \lesssim 0.5.$$

Thus, contrary to the impression one gets from the formalism "as $N \to \infty$", analysis shows that for historical values of crisis escalation probability p, in the range $0.2 < p < 0.5$, war probability P is *virtually equal to 1*, even while N is *quite small.*

To illustrate the point empirically, consider that for crises involving *major powers* (meaning something like "powerful nations"), the historical value of escalation probability during a crisis has been $p \approx 0.13$ in the past century and a half (Singer 1981, p. 11). This means that, for major power wars in recent history, equation (7) may be expressed as the empirical equation

$$P(\text{major power war occurs}) = 1 - (1 - 0.13)^N = 1 - 0.87^N. \tag{8}$$

This has a value nearly equal to 1, even for small historical values of N, and as a baseline, $P = 1/2$ for $N = 5$ (see Table 1).

The main conclusion to be drawn from these mathematically-generated results about war in the realworld is that it does not take many crises at all to make the probability of war P very high. Further, though the current empirical probability of escalation during crises may not seem high (0.13), equation (8) explains why wars are not uncommon.

War Probability, Crises and Escalation

A second puzzle within the N-crises problem concerns the different sensitivity of P with respect to p and to N. To make such sensitivities comparable (p and N are not directly commensurate, since they are expressed in different units), assume the following normalizations based on first-order changes. The sensitivity of P with respect to p, denoted by σ_p, is equal to the ratio of a percentage change in P with respect to a percentage change in p. That is,

$$\sigma_p = [(\Delta P/P)100]/[(\Delta p/p)100] = (\Delta P/p)(P/\Delta p),$$

Table 1. Probability P of major power war in recent history (with escalation probability $p = 0.13$) for various numbers of crises N

N	5	10	15	20	25	30
P	0.50	0.75	0.88	0.94	0.97	0.98

and taking the limit as $\Delta p \to 0$,

$$\sigma_p = (\partial P/\partial p)(p/P) = [Np(1-p)^{N-1}]/P. \tag{9}$$

Similarly, the sensitivity of P with respect to N, or σ_N, is equal to the ratio of a percentage change in P with respect to a percentage change in N:

$$\sigma_N = [(\Delta P/P)100]/[(\Delta N/N)100] = (\Delta P/N)(P/\Delta N);$$

and using the first-order difference (since N takes only integer values),

$$\sigma_N = [P_{N+1} - P_N](N/P) = -[Np(1-p)^N]/P. \tag{10}$$

Thus, as may be easily seen by comparing equations (9) and (10), $\sigma_p > \sigma_N \forall P$. So, P is more sensitive to p than to N.

In practice, this finding has several implications. For instance, it offers a rational basis for choosing conflict resolution policies aimed at war avoidance. In particular, given a choice between reducing the incidence of crises (lowering N) or reducing the probability of war escalation during crises (lowering p), the latter policy has greater impact on decreasing P. This is a nice result, since from an engineering perspective N seems uncontrollable (because international problems which cause crises to arise seem themselves uncontrollable), while p may be controllable. Crises occur for reasons which are still largely unknown (no one has provided a scientific explanation as to why N has a given value per decade, as opposed to some other value), whereas lowering the probability of crisis escalation p seems to be more within the realm of political control (unreliable as this may be). For instance, the influence of mediators and peacemakers, the pressure to prevent war (exerted by public opinion, other nations or groups), the availability of information technology aimed at decreasing uncertainty and fear (e.g. data channels for rapid communication), are but a few of the possible ways, or policy instruments, which can be used to lower p. In reference to some empirical cases, the current Israeli–Egyptian security relation (since the 1973 war) yields a very low value of p for that particular dyad of nations, since if a crisis were to occur, the security systems presently deployed in the Sinai make it virtually impossible for either side to surprise the other—and without surprise neither side can likely contemplate victory. If so, p is very low for this dyad (Israel–Egypt), for it would take a very severe level of dissatisfaction with the status quo for either side to chose to go to war, knowing it would lose.

Another empirical example from the current international system consists of the U.S.–Soviet agreement, signed in May 1987, to establish "risk reduction" centers to replace the aging MOLINK Hotline (Blechman 1985; Ury 1985).† New superpower communications systems include, among other features, very high speed data channels for text and image processing, making it possible for decisionmakers to exchange key information that may reassure the other side in times of crisis. The purpose is to decrease fear of first strike in crisis conditions of severe uncertainty. Such measures are useful, since war probability (P) is more sensitive to escalation probability (p) than to the incidence of crises (N)—a mathematical insight with both theoretical and practical relevance.

THE WAR LEDGE

Thus far we have been mostly concerned with phenomena occurring prior to the actual initiation of war—deterrence failing and crises occurring. Our third case of mathematical discovery takes us to the brink of war, so to speak—in the mathematical neighborhood of the instant "when war breaks out"—to a phenomenon called the *war ledge*. As was the case for the previous two phenomena, this too has received little notice, at least relative to the size of the scientific literature on the onset of war.‡ However, as we shall see, empirical evidence hinting at the existence of the war ledge is abundant, though the phenomenon itself still awaits more extensive theoretical interpretation. The historic background for understanding the *war ledge* may be found in Lewis

†MOLINK consists mainly of two electromechanical teletype terminals, one in Roman, the other in Cyrillic.

‡For example, Moyal (1949), Singer and Small (1972), Small and Singer (1982), Levy (1983), Cioffi-Revilla (1985), Choucri and North (1974), Midlarsky (1986), Davis, Duncan and Siverson (1978), Eberwein (1981), Wilkinson (1980), Howeling and Kuné (1984) and Howeling and Siccama (1985).

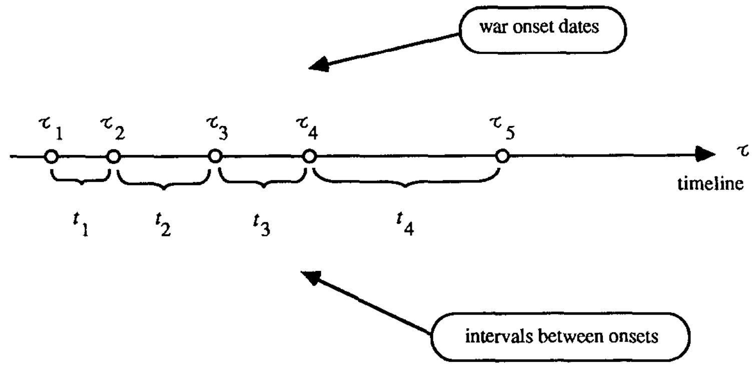

Fig. 2. The timeline τ along which wars break out on date τ_1, τ_2, τ_3, ... separated by intervals t_1, t_2, t_3, ... measured in time units.

Fry Richardson's empirical discovery in the early decades of this century [1941, 1945a, b; now collected in Richardson (1960)] that the onset of wars in history obeys a Poisson distribution. Though this finding set the stage, the war ledge came to light only recently, when war onset distributions were analyzed mathematically (Cioffi-Revilla 1985, 1987b).

The war ledge phenomenon may be described as follows: *on average, war breaks out at a mean time $\bar{t}$ which comes **after** the time φ when the probability of war occurring is 1/2 (the "even odds" point). The **war ledge** is the interval of time Λ given by the difference $\bar{t} - \varphi$ between these two points in time.*

War Onsets on the Timeline of History

To understand the war ledge phenomenon, let τ_1, τ_2, τ_3, ... denote historical calendar dates on which war breaks out on the timeline τ, as in Fig. 2. Also, let t_1, t_2, t_3, ... denote the time intervals (measured in some time unit such as days, month, years etc.) between consecutive onsets of war. Thus, $t_i = \tau_{i+1} - \tau_i$.

From a modeling perspective, the set $\{t\}$ of intervals between wars may be viewed as realizations of a continuous random variable (crv) $\mathscr{T}$. Accordingly, the onset of war at time t is determined (caused) by a large set of factors (domestic and international) acting through a complex mechanism which is not treated explicitly (as in a classic stochastic approach). The onset of war is—in this view—an indeterminate social phenomenon, rather than a deterministic process with identifiable mechanics. If $\mathscr{T}$ is a crv, then we define the cumulative density function (cdf) and the probability density function (pdf) of $\mathscr{T}$ as

$$\Phi(t) = \Pr(\mathscr{T} \leqslant t) \tag{11}$$

and

$$p(t) = \mathrm{d}\Phi(t)/\,\mathrm{d}t, \tag{12}$$

respectively.

In the first scientific papers published on the onset of wars, Richardson (1941, 1945a, b) showed that a simple Poisson model of $\mathscr{T}$ fits the onset of wars with intensity parameter $\lambda \approx 0.045$ onsets/month. More recent studies have confirmed this empirical finding for various types of severe international conflict events (classes of wars), as shown in Table 2.

Mathematical Discovery of the War Ledge

With the preceding ideas in mind, the war ledge phenomenon occurs as follows. If the crv $\mathscr{T}$ is Poisson-distributed with intensity parameter λ, as most studies have found, it then follows that the cdf (11) is of the form

$$\Phi(t) = 1 - \mathrm{e}^{-\lambda t}, \tag{13}$$

since $p(t)$ is the simple negative exponential pdf. Now, the function $\Phi(t)$, which describes the cumulative probability of war breaking out by time t, attains the even odds point at a time φ which

Table 2. Estimates of the intensity parameter λ in the Poisson model of war onsets in the international system (*ca* 1500–1965)

Type of war event	Force intensity,* $\hat{\lambda}$ (month^{-1})	Historical era, $\Delta\tau$	Database
Deadly quarrels,[a] $3.5 \leqslant \mu \leqslant 4.5$	0.045	1820–1929	Richardson (1960)
Wars of modern civilization[b]	0.050[c]	1500–1931	Wright (1942)
International wars	0.052[d]	1816–1965	Singer and Small (1972)
Interstate wars	0.028[d]	1816–1965	Singer and Small (1972)
Reciprocated military actions	0.032[e]	1815–1965	Siverson and Tennefoss (1982)

Source: adapted from Cioffi-Revilla (1985).
*All estimates are significant at the 0.05 level.
[a]$1\mu = \log_{10}$ (fatalities).
[b]V. Wright (1942, app. XX) and revised by Richardson (1960, p. 129).
[c]Wright's estimate calculated from data reported in Richardson (1960, p. 129).
[d]Calculated by U. Luterbacher, in Singer and Small (1972).
[e]Reported in Cioffi-Revilla (1985, Table 2).

satisfies the equation

$$1 - e^{-\lambda\varphi} = 1/2,$$

so therefore

$$\varphi = (\ln 2)/\lambda \approx 0.69/\lambda \text{ time units.} \tag{14}$$

(The length of time from 0 to φ may be interpreted as the half-life of peace.) However, *on average*, war breaks out at time $t = \bar{t}$, given by

$$\bar{t} = \langle \mathscr{T} \rangle = \int_0^\infty tp(t)\,dt = 1/\lambda. \tag{15}$$

From this last result it follows that, since $\bar{t} > \varphi$, on average war breaks out *after* the time φ when the value of the probability of outbreak has reached 1/2 (even odds point). The difference between the two time values, the interval of time $\Lambda = \bar{t} - \varphi$, is the war ledge, as illustrated in Fig. 3.

From the point of view of mathematical statistics (distribution theory) the existence of the war ledge is explained by the fact that the first moment of the distribution of $\mathscr{T}$, i.e. the expected value $\langle \mathscr{T} \rangle$, differs from the median value when $\mathscr{T}$ is Poisson- rather than normally-distributed. So if $\mathscr{T}$ were a normally-distributed crv, then there would be no war ledge, for in this case $\bar{t} = \varphi$ and, therefore, $\Lambda = 0$.

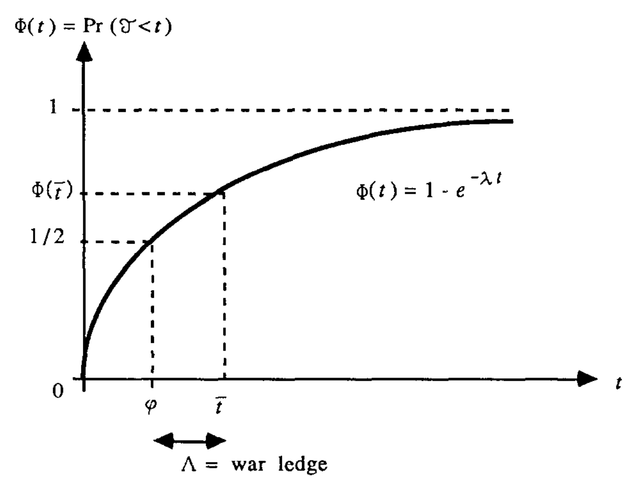

Fig. 3. The war ledge. On average, war breaks out after its probability of occurrence is strictly greater than even odds, $\Phi > 1/2$.

The war ledge phenomenon has not received much attention, though it is important for making scientific sense of propositions dealing with the idea of "when a war breaks out". Indeed, *when* does war break out? As we have seen, the mean ($\bar{t}$) and the median (φ) both provide alternative perspectives on the central tendency of the distribution of values of $\mathscr{T}$. The conceptual implications of this duality, for the theory of war, still await further interpretation. Presently, the war ledge may be accounted for on a purely formal basis, in terms of the conventional interpretations assigned to the mean and the median. Whereas the former refers to the "expected time of war onset" (in an average sense), the latter refers to a different qualitative view of the central tendency of the distribution of $\mathscr{T}$ and lacks a succinct common language expression: namely, the median is the value (realization) of $\mathscr{T}$ which divides in half the rank-ordered set of values of t, i.e. the inter-war duration which divides in two the ordered set $\langle t_1, t_2, t_3, \ldots \rangle$. Asking "when does war break out?" is like asking "which of the two statistics *is* the central tendency of $\mathscr{T}$"? The answer depends on one's perspective, since both $\bar{t}$ (as the mean) and φ (as the median) make sense as measures of central tendency. The war ledge begins with one (φ) and ends with the other ($\bar{t}$).†

WAR DILATION AND INTERNATIONAL INSTABILITY

Finally, our fourth case of mathematical insight—a phenomenon called *war dilation*—places us directly inside the timespan of war, within the period of "warfare-in-progress". War dilation ("prolongation", "protractedness" or "propagation") refers to a dynamic property of the stochastic pattern of *war duration*, and is of central importance to the question of "how long wars last" once they begin. As we shall see, beyond its intrinsic significance, the war dilation phenomenon also sheds new light on a previously unknown—but no less important—equilibrium property of the international system. Though war dilation originally stems from a statistical result (an isolated empirical finding first reported some 20 years ago), the very real properties of war to which it gives rise would have remained undetectable without the medium of mathematics.

In qualitative terms, war dilation is a phenomenon which may be described as follows: *war is a social interaction process which, once begun, has a* ***declining*** *propensity to terminate, in a probabilistic sense, as opposed to a constant, or a rising propensity. Alternatively, once underway, war has an increasing propensity to continue.* As we shall demonstrate, thanks to mathematical analysis this long-lived, protracted character of war now has a dynamic explanation—one based explicitly (an formally) on the social forces which account for warfare.

Horvath's Empirical Finding

The discovery of war dilation began about 25 years ago, with the pioneering work of H. K. Weiss (1963) and W. J. Horvath (1968) (both inspired by L. F. Richardson), and took place in two different stages—one statistical, the other mathematical. The first stage, which is only of background interest here, began with an empirical discovery by Horvath. Examining Richardson's list of 315 wars of magnitude 2.5 and greater for the historic era 1820–1949 (the Richardson war-magnitude is defined as the $\log_{10}$ of war fatalities; so, for example, the 1862–1867 American–Sioux War, with about 1000 fatalities, was of magnitude 3), Horvath found that the duration of wars (D) obeyed a Weibull probability law with cdf given by

$$\Phi(D) = 1 - \exp[-(D - D_0)^{\alpha}/\beta], \tag{16}$$

where α, β and D_0 are constants, and

$$\hat{\alpha} = 0.066, \quad \hat{\beta} = 0.608 \quad \text{and } \hat{D}_0 = 0.05,$$

expressed in years.

An instructive lesson may be learned from this first stage of the discovery of war dilation. After reporting his statistical findings (i.e. the above estimates of α, β and D_0), curiously enough Horvath

†The preceding discussion also attracts attention to the mode, or the value of $\mathscr{T}$ corresponding to the maximum value of $p(t)$ (onset time with maximum density). However, since here $p(t)$ is a simple negative exponential function, the maximum of $p(t)$ occurs when $t = 0$, so the modal value of $\mathscr{T}$ is degenerative and has no sensible substantive interpretation.

dedicated only minor discussion to the fact that he had found $\hat{D}_0 > 0$ (proving that wars "could not be ended in a shorter length of time" than about 18 days, for whatever reason), *making nothing of the fact that he had found* $\hat{\alpha} < 1$. As we shall see, he missed a key insight which may have led him to the discovery of war dilation some 20 years earlier.

The Mathematical Discovery of War Dilation

The discovery of war dilation—a phenomenon no less real than the fact that wars have a finite duration no shorter than about 18 days—came later, in a second stage. This took place when Horvath's empirical findings were placed in a broader theoretical context with the use of mathematics. To get from this initial finding, to the more theoretical idea of war dilation, we must examine more closely the probabilistic nature of warfare.

As was earlier the case for the time interval between war onsets (the crv $\mathscr{T}$), *war duration* (warfare) also may be viewed as a crv, denoted by $\mathscr{D}$. Accordingly, $\Phi(D) = \Pr(\mathscr{D} \leqslant D)$, and $p(D) = \Pr(\mathscr{D} = D)$, denote the cdf and the pdf of $\mathscr{D}$, respectively. In addition, we also define the hazard rate function (hrf) of war duration, $F(D)$, as

$$F(D) = p(D)/[1 - \Phi(D)]. \quad (17)$$

In substantive terms, the hrf $F(D)$ may be interpreted as a *war termination force* at time D, since it is defined by equation (17) as the ratio of two probabilities: the probability that war will terminate *at* time D, or the value of the pdf $p(D)$; relative to the probability that war does *not* terminate *by* time D, or the value of $1 - \Phi(D)$. Thus, F is a variable akin to an intensity, a propensity, or a tension—in brief, a social force—for war to terminate, and $F(D)$ describes the value of such a force at time D.

Now when $\mathscr{D}$ obeys a Weibull distribution it may be shown that $F(D)$ is a power function of the form

$$F(D) = \kappa\alpha\,(D - D_0)^{\alpha - 1}, \quad (18)$$

where $\kappa = 1/\beta$. Since Horvath had earlier found that $\hat{\alpha} \approx 0.07 < 1$, from equation (18) it follows immediately that $\mathrm{d}F(D)/\mathrm{d}D < 0$. Therefore, $F(D)$ is strictly decreasing, meaning that wars have a decreasing force to end. (As a baseline, not that when $\mathscr{D}$ is Poisson-distributed with intensity λ, then $\alpha = 1$, $\kappa = \lambda$ and $F(D) = \lambda$, meaning a constant force for war to end.)

Horvath's empirical finding—that $\mathscr{D}$ obeys a Weibull distribution with $\alpha < 1$—was initially reported simply as a statistical property of the historical distribution of war durations, and aimed at improving an earlier Markov model that had been proposed by Weiss. But when the mathematical properties of the Weibull distribution are examined more closely—particularly, when the force function (18) is mathematically defined by equation (17), and modeled in a meaningful fashion—what was initially an isolated statistical finding can then be used as a new starting point. Using mathematics, this new information (that $\alpha < 1$) can be analyzed more rigorously to shed new light on a previously unknown aspect of war: namely, that since $F(t)$ is *empirically* a strictly decreasing function of war duration D, since $\alpha < 1$, this carries the necessary implication that wars have a declining propensity (force) to terminate, once they have begun (at least for wars of the era 1820–1949). War dilation is simply a shorthand for describing the decreasing character of the force of war termination F; and the effect this has on the nature of warfare.

Beyond its intrinsic interest, war dilation has several theoretical and practical implications:

1. The *mortality* of war (Ω), when defined as the probability of war terminating by time D, is given by

$$\Omega(D) = \Pr(\mathscr{D} < D) = \Phi(D) = 1 - \exp[-(D - D_0)^{\alpha}/\beta]. \quad (19)$$

2. The *mean* duration of warfare is

$$\langle\mathscr{D}\rangle = D_0 + \kappa^{-1/\alpha}\,\Gamma(1 + 1/\alpha), \quad (20)$$

where $\Gamma(x)$ is the value of the gamma function evaluated at x, and (when $D_0 = 0$) its variance is

$$\mathrm{var}(\mathscr{D}) = \kappa^{2/\alpha}\{\Gamma(1 + 2/\alpha) = [\Gamma(1 + 1/\alpha)]^2\}. \quad (21)$$

3. Warfare has a *half-life* φ, defined as the value of D such that $\Phi(\varphi) = 1/2$, and is given by
$$\varphi = D_0 + (\beta \ln 2)^{1/\alpha}. \quad (22)$$
Using Horvath's estimates, $\varphi = 18.25 + 0.7 \times 10^{-3}$ days ≈ 18 days $\approx D_0$. So, since the second term in the r.h.s. of equation (22) is very small, φ occurs practically at the same time as D_0. As a consequence, as soon as war has run its minimal duration D_0 the probability of it ending (war mortality $\Omega = \Phi$) surpasses the even odds point (1/2), and tends to 1 as $D \to \infty$. However, since $\alpha < 1$, the value of $\Phi(D \geqslant D_0)$ approaches 1 *slower* than when $\alpha \geqslant 1$. Indeed, $\Phi(D \geqslant D_0)$ has a long asymptotic tail which accounts for the high variability of war durations, and for the existence of many wars of extreme duration.
4. Since $\alpha < 1$, it can also be shown that $\varphi \ll \langle \mathscr{D} \rangle < \text{var}(\mathscr{D})$, so the distribution of $\mathscr{D}$ is hypoexponential. [See also Rood (1978) for a discussion of deviations from the exponential distribution in political analysis.]

As some of these mathematical ideas suggest, a phenomenon similar to the war ledge, though not identical, also occurs in the context of war duration: namely, since $\alpha < 1$, it can be shown that $\varphi < \langle \mathscr{D} \rangle$, and therefore $\Lambda = \langle \mathscr{D} \rangle - \varphi > 0$. This means that something like a"peace ledge" occurs for war duration: *war has a propensity to be protracted, or long-lived* (***war dilation***), *and on average war termination does not occur until* ***after*** *its probability of ending is greater than even odds* (*i.e. after the half-life* φ, *when* $\Phi(\langle \mathscr{D} \rangle) > 1/2$). Also, the peace ledge Λ rapidly increases as α decreases. These and other properties of war duration in general, and insights like war dilation and the war/peace ledges in particular, bring to light a well-known consequence of using mathematical methods in empirical science: one discovery often leads to another (synergistically), and quite frequently such a progression of knowledge also makes possible the establishment of new, unsuspected connections between previously unconnected phenomena. In the case of war dilation, not only do we now know that wars have a decreasing propensity to terminate (war dilation); we also know that, since $\Lambda > 0$ when $\alpha < 1$, this establishes a new intriguing link between the onset and termination of war: *on average, both events occur after their cumulative probability of occurring is strictly greater than even odds.* Since $\Phi > 1/2$ in both cases, both exhibit a ledge. Not every random process exhibits this characteristic.

CONCLUSIONS

The phenomena discussed in this paper provide evidence of progress in our scientific understanding of why, when and how war occurs. Significantly, none of these phenomena may have been known—discovered—without the medium of mathematics, for neither common intuition, nor statistical approaches alone, could have brought the underlying ideas to light. In reference to each of the four phenomena:

1. The hyporeliability effect not only runs counter to common intuition—which says that a deterrence system aimed at preventing war will be *as* strong, or as reliable, as the weakest-link-in-the-chain. It also tells us that the popular chain-of-links aphorism is actually an overly optimistic heuristic—one which is scientifically incorrect—for assessing the objective reliability of a deterrent system. Discovery of the hyporeliability effect in deterrence has also been instrumental in focusing research attention on the cognitive puzzle of how decisionmakers, analysts and others involved in the realworld use of deterrence systems perceive various levels of reliability probabilities. An avenue of future inquiry leads to consideration of findings from cognitive science and experimental psychology—to prospect theory, as developed by Kahneman, Tversky and others. The implications of these ideas for a better understanding of the hyporeliability effect in deterrence (e.g. fuzzy sets and possibility theory), are yet to be explored.†
2. The N-crises problem also brought to light some realworld phenomena not easily grasped without the use of mathematics. On the speed with which $P_{\text{war}} \to 1$ as $N \to \infty$, Wright's original

†In passing, much the same can be said in reference to the study of political coalitions—where the hyporeliability effect is no less real. In this paper the focus is confined to the study of war.

account had (incorrectly) suggested a much slower convergence, expressed as the extreme limit "as N goes to infinity". As we demonstrated, if the probability of crisis escalation p is anything but negligible, then the global (long-term) probability of war P increases to values which are virtually 1, even for small values of N. Fast convergence is not apparent without mathematical analysis.

Also within the N-crises problem, the tradeoff between avoiding crises and reducing war escalation probability is another aspect which is anything but clear without the use of mathematics. That nature favors lowering the probability of war escalation, rather than lowering the number of crises, may seem intuitive on a *post hoc* basis (hindsight is always 20/20!), but without formal analysis such a result would have no scientific basis. On this, as in other questions in social science, there is no possibility of experimentation, so even experience cannot determine which of the two is favored by nature. Mathematical analysis alone accounts for the discovery of this result.

3. The war ledge phenomenon was unknown prior to the application of mathematical methods to the earlier empirical finding that war obeys a Poisson law of onsets to an acceptably good degree of fit. This case of mathematical discovery also typifies the power of mathematics when used to derive more information from a previously verified empirical regularity—the Poisson law in this instance. That something like the war ledge actually exists in the realworld is by no means apparent from statistical analysis alone (Richardson's extensive work did not report it), nor does it become apparent from the study of military history. The ledge is of interest to students of war because it points to a time interval preceding the average onset of war, the interval Λ, which may contain (no one has yet determined this) particularly relevant processes or events for explaining why, how or when wars occur at a given time. For instance, in reference to the traditional distinction between *immediate* and *remote causes of war*, Λ may offer an appropriate criterion for distinguishing between the two. Immediate causes may be said to refer to those events which arise within the ledge Λ, whereas remote causes operate in the more remote past (i.e. for $t < \varphi = [\langle \mathscr{T} \rangle - \Lambda]$). Empirically, Λ is of the order of 6 months. Why this particular length of time, as opposed to some other, no one has thus far explained.
4. The mathematical discovery of war dilation—that wars tend to be protracted rather than brief—is also insightful. Rightly so, war is often viewed as a "vortex", involving more and more belligerence, as in a runaway reaction. Mathematical analysis, grounded on an empirical pattern, provides a basis for this belief. Unlike other social processes, wars in general do not tend to terminate after they have started. As was earlier the case for the war ledge, the decreasing propensity of wars to terminate once they have begun is also something which would not have been discovered from the empirical analysis of war durations alone, without the assistance of mathematics, let alone from the erudite study of military history—no matter how carefully conducted. War dilation also sheds new light on the equilibrium properties of the international system, making the task of peacekeepers more difficult than it was previously imagined. War dilation suggests that, when perturbed by the initiation of a war, the international security system tends *not* to restore equilibrium (peace), by making it less and less likely (as opposed to more and more likely) for an initiated war to terminate—as when a perturbation to the state of a system cannot be easily controlled by a stabilizing regulator attempting to restore a goal state (peace). Further, the existence of war dilation perhaps suggests that international organizations, such as the U.N., should be more productively employed in war prevention, rather than in war containment, since once war is underway it exhibits a decreasing propensity to terminate. Another plausible inference may be that the protracted propensity of wars is related to the learning capacity of belligerents to sustain war once they have started it, so once war begins their mobilization rate matches or exceeds the rate at which war consumes their resources. (If belligerents could not mobilize war-waging resources at a rate higher than the rate of resource consumption, then wars would tend to be short-lived, with $\alpha > 1$.)

In each of the four cases we examined, as in others beyond the limited scope of this paper, the basic discoveries could not have been made, and new knowledge could not have been obtained, without the application of mathematical analysis to theoretical concepts, to empirical findings or to a combination of both. The heuristic power of mathematical analysis has allowed us to discover these realworld, objective properties of war.

Where do these and other findings lead? Which new research directions do they suggest? As the scientific community addresses questions such as these in the years ahead, it would prove wise to assess where we stand today—some 50 years since mathematics were first applied to the study of war by L. F. Richardson. We should consider the many aspects of war, and examine the various mathematical tools (models) which have been used for each. At a minimum, such an inventory would determine major areas of concentration, along with those areas where gaps currently exist.

When viewed against the long-range timeline of war in human history, the application of mathematics to understand war has just begun. However, the last 50 years provide convincingly evidence that mathematical modeling approaches have begun their unique and powerful contribution to the study of conflicts and war. While moving toward the end of this century, it is any scientist's guess as to what may come next by way of new discoveries in this field. In spite of this uncertainty it is clear that in this young science much exciting new knowledge will be growing at an unprecedented rate. An intriguing feature of this progress—as we have attempted to show in this paper—is that progress is occurring with the application of basic and intermediate mathematical tools, not with highly advanced methods, much as was the case in the early history of physical science. Only a rational instrument as powerful as mathematics is capable of this feat—allowing us to learn much about something as complex as war, while the tool itself remains simple.

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