TOPIC IV

Conic Sections

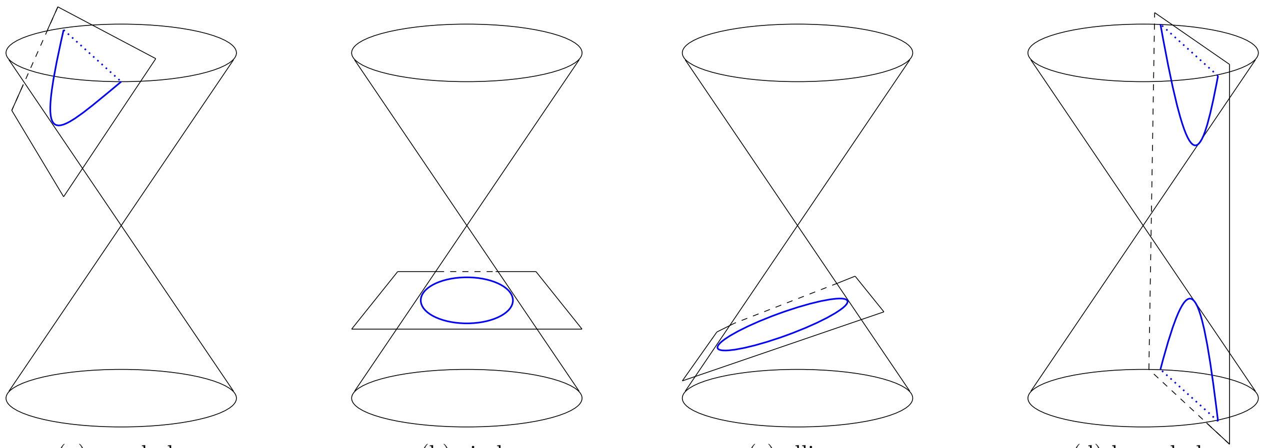
Conics, an abbreviation for conic sections, are cross sections that result from the intersection of a right circular cone and plane.

Parabolas are when the plane is tilted so that it is parallel to one generator.

Circles are when the plane is perpendicular to the axis of the cone when it intersects.

Ellipses are when the plane is tilted slightly when it intersects the cone.

Hyperbolas are when the plane intersects both parts (called nappes) of the cone.



(a) parabola (b) circle (c) ellipse (d) hyperbola

Some Applications of Conic Sections

- Parabola: used for searchlights, satellite dishes, and telescopes

- Ellipse: model orbits of planets and whispering chambers

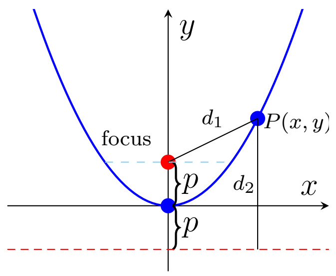
- Hyperbola: used to locate lightning strikes and design nuclear cooling towers

IV.1. The Parabola (ALEKS 10.3)

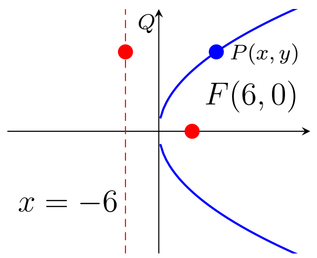
We learned in Section I.1 the graph of the equation $y = ax^2 + bx + c$ $(a \neq 0)$ is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. Now we extend our study of parabolas to include those that open left or right. To do so we define a parabola geometrically.

Definition IV.1.1: Conic Sections: The Parabola

A parabola is the set of all points in a plane that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus**.



Example IV.1.1. *The parabola in the figure below has vertex* $(0, 0)$ *and focus* $F(6, 0)$*. Its directrix is the vertical line that passes through* $(-6, 0)$*.*
The point $P(x, y)$ *is a point on the parabola that is above and to the right of the focus. The point* Q *lies on the directrix, and* $\overline{PQ}$ *is perpendicular to the directrix.*



(A) What are the coordinates for point Q?

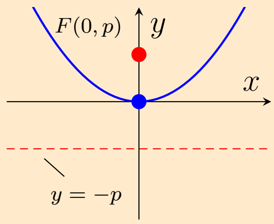
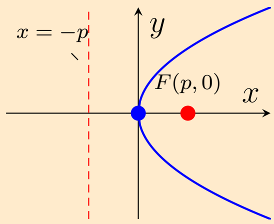
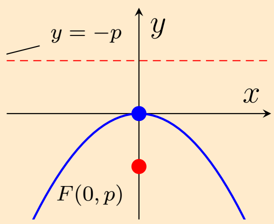
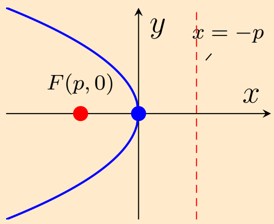
(B) Write the expressions for the distances FP and QP.

(C) From the definition of the parabola, which of the following must be true?

$$FP < QP, \qquad FP = QP, \qquad FP + QP = 0, \qquad FP > QP$$

(D) Use the above information to write an equation for the given parabola solved for y^2.

Standard form of an equation of parabola with vertex at the origin

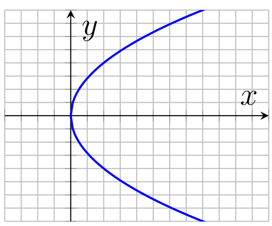
	Axis of symmetry: y-axis	Axis of symmetry: x-axis
Equation:	$x^2 = 4py$	$y^2 = 4px$
Vertex:	$(0, 0)$	$(0, 0)$
Focus:	$(0, p)$	$(p, 0)$
Directrix:	$y = -p$	$x = -p$
Axis of symmetry:	$x = 0$	$y = 0$
Graph: $p > 0$		
Graph: $p < 0$		

We are accustomed to writing an equation of a quadratic function with vertex at the origin in the form $y = ax^2$. The benefit of writing the equation as $x^2 = 4py$ is that we can identify the value of p which gives us the distance between vertex and the focus.

Objective 55: Identify the focus and directrix of a parabola

p is the distance from the vertex to the focus. p is also the distance from the vertex to the directrix. The distance from the focus to the directrix is $2p$.

Example IV.1.2. *Given a parabola defined by $y^2 = 8x$, find the focus and directrix.*



Vocabulary

1. The line perpendicular to the directrix and passing through the focus is called the **axis of symmetry**.

2. The **vertex** of the parabola is the point of intersection of the parabola and the axis of symmetry.

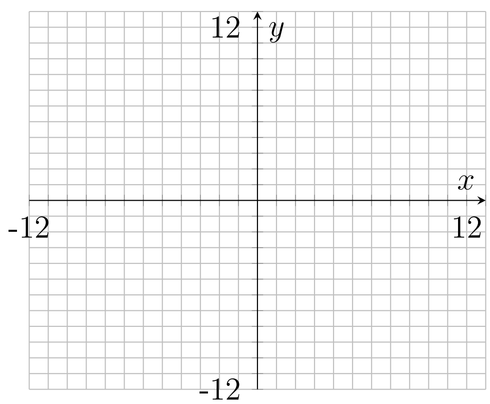
3. The distance between the vertex and the focus of a parabola is called **the focal length** and is often represented by $|p|$.

Objective 56: Graph a parabola with vertex at the origin

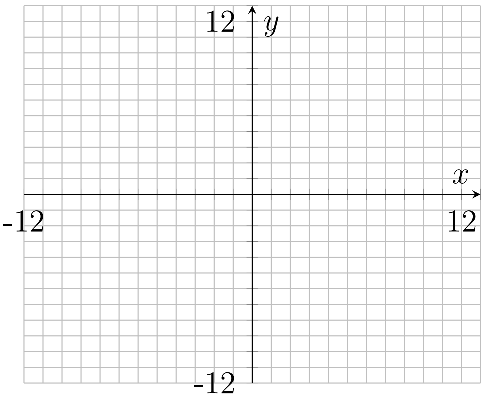
The line segment passing through the focus, perpendicular to the axis of symmetry with endpoints on the parabola is called the **latus rectum** and helps us determine how wide the parabola is at the focus.

The length of the latus rectum is called the **focal diameter** and is equal to $|4p|$.

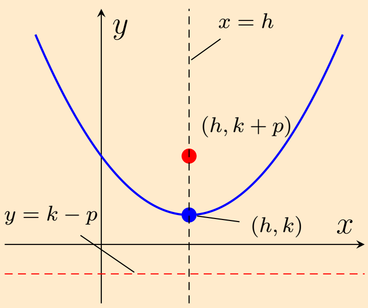
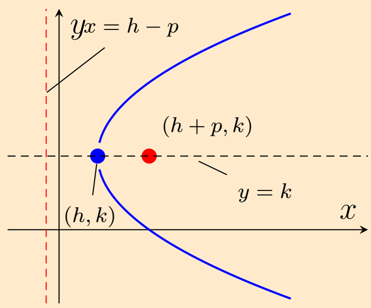
Example IV.1.3. *Graph $x^2 = 16y$. Identify the vertex, focus, and focal diameter. Identify the endpoints of the latus rectum and write equations for the directrix and axis of symmetry.*



EXAMPLE IV.1.4. *Graph $3y^2 = -36x$. Identify the vertex, focus, and focal diameter.*



Standard form of an equation of a parabola with vertex (h, k)

	Vertical axis of symmetry	Horizontal axis of symmetry
Equation:	$(x-h)^2 = 4p\,(y-k)$	$(y-k)^2 = 4p\,(x-h)$
Vertex:	(h, k)	(h, k)
Focus:	$(h, k+p)$	$(h+p, k)$
Directrix:	$y = k - p$	$x = h - p$
Axis of symmetry:	$x = h$	$y = k$
Graph:		

- For $(x-h)^2 = 4p\,(y-k)$, if $p > 0$, the parabola opens upward, as sketched above.
  If $p < 0$, the parabola opens downward.
- For $(y-k)^2 = 4p\,(x-h)$, if $p > 0$, the parabola opens to the right, as sketched above.
  If $p < 0$, the parabola opens to the left.

Objective 57: Graph a parabola with vertex (h, k)

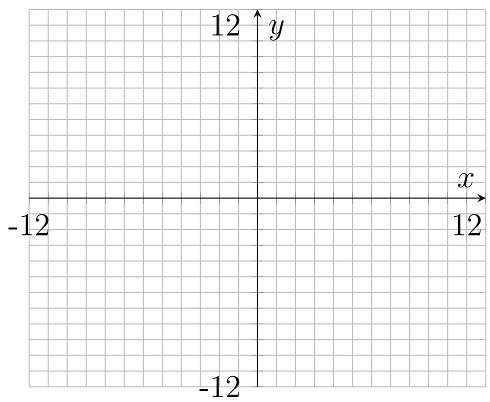
To graph a parabola with vertex (h, k),

(1) Locate the vertex and identify the orientation of the parabola.

(2) Identify the focus, directrix, and endpoints of the latus rectum.

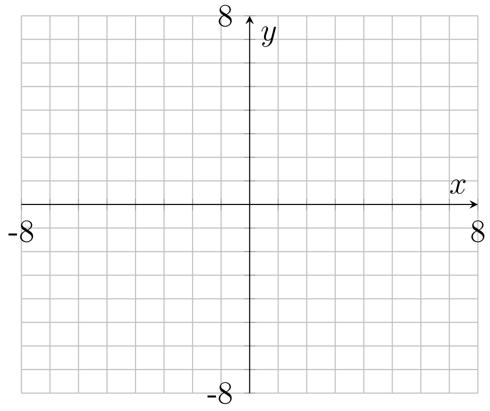
(3) Sketch the parabola through the vertex and endpoints of the latus rectum.

Example IV.1.5. *Graph the parabola* $(y-5)^2 = 8(x+2)$.



Example IV.1.6. *Determine the standard form of an equation of the parabola with focus* $(-5, 8)$ *and directrix* $y = 6$.

Example IV.1.7. *Identify the vertex and focus for* $-3y^2 + 18y - x - 24 = 0$ *and sketch a graph of the parabola.*



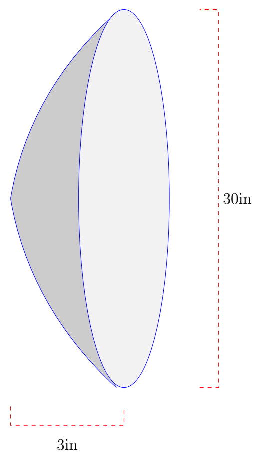
Objective 58: Use parabolas in applications

Searchlights, satellites, and telescopes are paraboloids with parabolic cross sections. For example, a satellite dish uses the reflective property of a parabola to gather radio waves and direct the signal to a common focus to make the signal stronger.

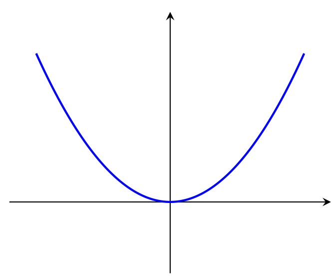
Example IV.1.8. *A satellite dish is in the shape of a paraboloid. Cross sections taken parallel to the direction the dish opens are parabolic. Cross sections taken perpendicular to the direction the dish opens are circular. The diameter of the dish is* 30 *in. and the depth is* 3 *in.*

(A) Find an equation of a parabolic cross section through the vertex of the dish.

(B) Where should the receiver be placed?



Example IV.1.9. *A suspension bridge has a cable shaped like a parabola. The bridge is supported by two towers. The towers are* 108 *meters apart and each tower is* 9 *meters high. What is the height of the cable when it is* 36 *meters from the center of the bridge?*



Students – Check your understanding

CYU 84. *A(n) ____________ is the collection of all points in the plane such that the distance from each point to a fixed point equals its distance to a fixed line.*

A. *Ellipse*

B. *Parabola*

C. *Circle*

D. *Hyperbola*

CYU 85. *Which of the following is the directrix of the parabola given by* $y^2 = 6x$?

A. $x = 3/2$

B. $x = -3/2$

C. $y = 3/2$

D. $y = -3/2$

CYU 86. *Find the equation of the parabola with focus* $(-1, -2)$ *and directrix* $y = 3$.

A. $\left(y - \frac{1}{2}\right)^2 = -10\,(x + 1)$

B. $\left(y - \frac{1}{2}\right)^2 = -20\,(x + 1)$

C. $(x + 1)^2 = -10\left(y - \frac{1}{2}\right)$

D. $(x + 1)^2 = -20\left(y - \frac{1}{2}\right)$

Students - Check your understanding

CYU 87. *A searchlight is shaped like a paraboloid of revolution. If the light source is located* 2 *feet from the base along the axis of symmetry and the opening is* 8 *feet across, how deep should the searchlight be?*

A. 4.0 *feet*

B. 2.0 *feet*

C. 0.3 *feet*

D. 8.0 *feet*

CYU 88. *A very famous logo consists of two golden parabolas. If a coordinate grid is placed over the logo, we discover that each parabola intersects the x-axis at* 4 *units from the center and the height of each parabola is exactly* 8 *units when measured* 2 *units from the origin. Find the equation of each parabola in standard form.*