AUTO-07P :
CONTINUATION AND BIFURCATION SOFTWARE FOR ORDINARY DIFFERENTIAL EQUATIONS

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Preface

This is a guide to the software package AUTO for continuation and bifurcation problems in ordinary differential equations. Earlier versions of AUTO were described in Doedel (1981), Doedel & Kernévez (1986*a*), Doedel & Wang (1995), Wang & Doedel (1995), Doedel, Champneys, Fairgrieve, Kuznetsov, Sandstede & Wang (1997), Doedel, Paffenroth, Champneys, Fairgrieve, Kuznetsov, Oldeman, Sandstede & Wang (2000). For a description of the basic algorithms see Doedel, Keller & Kernévez (1991*a*), Doedel, Keller & Kernévez (1991*b*). AUTO incorporates the HomCont algorithms of Champneys & Kuznetsov (1994), Champneys, Kuznetsov & Sandstede (1996) for the bifurcation analysis of homoclinic orbits, and the BPcont algorithms of Dercole (2008) for the continuation of branch points in both symmetric and non-symmetric problems. The Floquet multiplier algorithms were written by Fairgrieve (1994), Fairgrieve & Jepson (1991). A GUI was written by Wang (1994). The Python CLUI is the work of Randy Paffenroth.

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The development of HomCont has benefitted from help and advice from, among others, W.-J. Beyn, Universität Bielefeld, M. J. Friedman, University of Alabama, A. Rucklidge, University of Cambridge, M. Koper, University of Utrecht, C. J. Budd, University of Bath, and Financial support for this collaboration was also received from the U.K. Engineering and Physical Science Research Council and the Nuffield Foundation.

License

AUTO is available under the terms of the BSD license:

Note that the three-dimensional plotting tool PLAUT04 optionally depends on libraries that are covered by the GNU General Public License (GPL), in particular, Coin, SoQt and Qt. In that case the PLAUT04 binaries are also covered by the GPL.

Chapter 1

Installing AUTO.

1.1 Installation.

The AUTO file auto07p-0.9.1.tar.gz is available via `http://cmvl.cs.concordia.ca/auto`. Here it is assumed that you are using the Unix (e.g. *bash*) shell and that the file auto07p-0.9.1.tar.gz is in your main directory. See below for OS-specific notes.

While in your main directory, enter the commands `gunzip auto07p-0.9.1.tar.gz`, followed by `tar xvfo auto07p-0.9.1.tar`. This will result in a directory auto, with one subdirectory, auto/07p. Type `cd auto/07p` to change directory to auto/07p. Then type `./configure` , to check your system for required compilers and libraries. Once the configure script has finished you may then type `make` to compile AUTO and its ancillary software. The configure script is designed to detect the details of your system which AUTO requires to compile successfully. If either the configure script or the make command should fail, you may assist the configure script by giving it various command line options. Please type `./configure --help` for more details. Upon compilation, you may type `make clean` to remove unnecessary files.

To run AUTO you need to set your environment variables correctly. Assuming AUTO is installed in your home directory, the following commands set your environment variables so that you will be able to run the AUTO commands, and may be placed into your .login, .profile, or .cshrc file, as appropriate. If you are using a `sh` compatible shell, such as `sh`, `bash`, `ksh`, or `ash` enter the command `source $HOME/auto/07p/cmds/auto.env.sh`. On the other hand, if you are using a `csh` compatible shell, such as `csh` or `tcsh`, enter the command `source $HOME/auto/07p/cmds/auto.env`.

The Graphical User Interface (GUI94) requires the X-Window system and Motif or LessTif. Note that the GUI is not strictly necessary, since AUTO can be run very effectively using the Unix Command Language User Interface (CLUI). Moreover, long or complicated sequences of AUTO calculations can be programmed using the alternative Python CLUI. The GUI is not compiled by default. To compile AUTO with the GUI, type `./configure --enable-gui` and then `make` in the directory auto/07p.

To use the Python CLUI and the "@pp" PyPLAUT plotter it is strongly recommended to install NumPy (`http://numpy.scipy.org`), TkInter, and Matplotlib (`http://matplotlib.sourceforge.net/`). Note that Matplotlib 0.99 or higher is recommended because it supports 3D plotting in addition to 2D plotting. Python itself needs to be at least version 2.3 or higher,

but 2.4 or higher is strongly recommended and required for NumPy and Matplotlib. As of this writing Python 3.x is not yet supported by Matplotlib, and therefore not recommended. For enhanced interactive use of the Python CLUI it is also worth installing IPython (`http://ipython.scipy.org`).

The graphic tool for 3D AUTO data visualization, PLAUT04, is compiled by default, but depends on a few libraries that may not be in a standard installation of a typical Unix-like system. These libraries may be available as optional packages, though. In order of preference these are:

1. Coin3D (version 2.2 or higher), SoQt (1.1.0 or higher), and simage (1.6 or higher). With SoQt 1.5.0 or higher, simage is no longer required.

2. Coin3D with the SoXt library, which interfaces with (Open)Motif or LessTif (version 2.0 or higher) instead of Qt. The user interface has a few problems with LessTif though, in particular it is likely to crash on 64-bit machines, so the Qt version or (Open)Motif is recommended.

3. One can download SGI's implementation of the Open Inventor libraries from: `ftp://oss.sgi.com/projects/inventor/download/` Because SGI's implementation for Linux cannot show text correctly, we recommend that Coin is used instead of SGI's implementation.

The configure script checks for these libraries and outputs a warning if any of these libraries cannot be found. It first checks for SoQt, and then for SoXt, unless you pass `--disable-plaut04-qt` as an option to configure. If the libraries are not available you can still compile all other components of AUTO using make.

The Fortran code uses several routines that were not standardardized prior to the Fortran 2003 standard, for timing, flushing output, and accessing command-line arguments. The configure script first looks if the F2003 routines are supported (src/f2003.f90), then checks for a set of routines that are widely implemented across Unix compilers (src/unix.f90), and if that fails too, uses a set of dummy replacement routines (src/compat.f90), which could be edited for some obscure installations.

The PostScript conversion command `@ps` is compiled by default. Alternatively you can type `make` in the directory auto/07p/tek2ps. To generate the on-line manual, type `make` in auto/07p/doc, which depends on the presence of xfig's (transfig) fig2dev utility.

To prepare AUTO for transfer to another machine, type `make superclean` in the directory auto/07p before creating the tar-file. This will remove all executable, object, and other non-essential files, and thereby reduce the size of the package.

Some LAPACK routines used by AUTO for computing eigenvalues and Floquet multipliers are included in the package (Anderson, Bai, Bischof, Blackford, Demmel, Dongarra, Du Croz, Greenbaum, Hammarling, McKenney & Sorensen (1999)). The Python CLUI includes a slightly modified version of the Pointset and Point classes from PyDSTool by R. Clewley, M.D. LaMar, and E. Sherwood (`http://pydstool.sourceforge.net`)

1.1.1 Installation on Linux/Unix

A free Fortran 95 compiler, Gfortran, is shipped with most recent Linux distributions, or can be obtained at `http://gcc.gnu.org/wiki/Gfortran`. The following packages (and their de-

pendencies) are recommended for Fedora:

- Python: python-matplotlib-tk and ipython.
- PLAUT04: SoQt-devel. For SoQt versions older than 1.5.0, to see pictures of stars, the earth and the moon instead of white blobs, compile simage from source (see `www.coin3d.org`; needs libjpeg-devel).
- PLAUT: xterm.
- GUI94: lesstif-devel or openmotif-devel.
- manual: LaTeX(tetex or texlive) and transfig.

and the following for Ubuntu and Debian:

- Python: python-matplotlib and ipython.
- PLAUT04: SoQt (libsoqt-dev or libsoqt4-dev) and libsimage-dev.
- PLAUT: xterm
- GUI94: lesstif2-dev or libmotif-dev.
- manual: LaTeX(tetex or texlive) and transfig.

Other distributions may have packages with similar names.

If you need to compile and install one of the above PLAUT04 libraries from the source code available at the above web site, and you find that, after that, PLAUT04 still does not work then you might need to adjust the environment variable `LD_LIBRARY_PATH` to include the location of these libraries, for instance `/usr/local/lib`.

1.1.2 Installation on Mac OS X

AUTO runs on Mac OS X using the above instructions provided that you have the development tools installed (see the Mac OS X Install DVD, Optional Installs, Xcode). You do not need to start an X server to run AUTO. Furthermore, the following packages are recommended:

- Gfortran: See `http://gcc.gnu.org/wiki/GFortranBinaries`. Alternatively, see `hpc.sourceforge.net` or `r.research.att.com/tools/`. At the time of writing the first of these is recommended to be able to benefit from parallelization.
- Python: you can install **32-bit** Python 2.7, NumPy, and Matplotlib .dmg files from `www.python.org`, `numpy.scipy.org`, and `matplotlib.sf.net`, respectively.
  For example, download from
  `http://www.python.org/ftp/python/2.7.2/python-2.7.2-macosx10.3.dmg`,
  `http://sourceforge.net/projects/numpy/files/NumPy/1.6.1/numpy-1.6.1-py2.7-python.org-macosx10.3.dmg/download`,
  and `http://sourceforge.net/projects/matplotlib/files/matplotlib/matplotlib-1.1.0/matplotlib-1.1.0-py2.7-python.org-macosx10.3.dmg/download`

The default system Python in Mac OS X is usable, but may be too old for Matplotlib and NumPy. There are other alternatives, for instance the Enthought Python Distribution at `www.enthought.com`. Fink should work but native graphics provided by the previous alternatives seem to work better.
To be able to plot in the Python CLUI in some versions of OS X, AUTO uses pythonw instead of python. This should happen automatically.

- PLAUT04: Get a Qt .dmg from `http://qt.nokia.com/downloads/qt-for-open-source-cpp-development-on-mac-os-x`. Similarly, you can get a binary Coin package from `coin3d.org`. After that you can compile SoQt (at least version 1.5.0) from the source code at `coin3d.org`. For example, download `http://ftp.coin3d.org/coin/bin/macosx/all/Coin-3.1.3-gcc4.dmg`, `http://get.qt.nokia.com/qt/source/qt-mac-opensource-4.8.0.dmg`, and `http://ftp.coin3d.org/coin/src/all/SoQt-1.5.0.tar.gz`.

  Try to make sure that the native (Aqua) Qt is used by setting `$QTDIR`, if you also have fink installed.

- PLAUT: In pre-Leopard OS X it appears that you do not see fonts. To solve this issue you need to obtain a different version of xterm; see `http://sourceforge.net/project/showfiles.php?group_id=21781`.

- GUI94: Perhaps possible using Fink but not attempted.

- manual: LaTeX and transfig (comes with xfig).

Notes for 64-bit Snow Leopard to be able to compile PLAUT04:

- To compile and install SoQt, after installing Qt and Coin3D, run

```
./configure CFLAGS="-m32" CXXFLAGS="-m32" LDFLAGS="-m32" FFLAGS="-m32"
make
sudo make install
```

- Then, in the AUTO-07p folder configure and compile AUTO as described above.

1.1.3 Installation on Windows

A native, light-weight solution for running AUTO on Windows is to use GFortran, MSYS 1.0.11 or higher (see `http://www.mingw.org`), combined with a native Win32 version of Python, obtained at `http://www.python.org`. To install this setup:

- Install Python (as of this writing, preferably version 2.7, not 3.2!) from `www.python.org`, NumPy from `numpy.scipy.org`, and Matplotlib from `matplotlib.sf.net`, which all come with installers.

- Install MinGW (make sure to include msys-base, gcc and fortran) using the MinGW Graphical Installer at `http://www.mingw.org`.

- Start MSYS using the Start menu (Start > All Programs → MinGW → MinGW Shell) or by clicking on its desktop icon, which puts you in a home directory, where you can unpack AUTO using gunzip and tar, as described above.
- Make sure that the `gfortran` and `python` binaries are in your `PATH`, and that their directories are at the front of it. You can do this, for instance, using the shell command `export PATH="/c/Python27:/bin:/c/Program Files/gfortran/bin:$PATH"` . You can also inspect, edit and then source the file auto/07p/cmds/auto.env.sh to achieve this.
- Now you should be able to run configure and make to compile AUTO as shown above.

You can use AUTO using shell commands from the default MSYS shell environment. You can also start the CLUI by double clicking on the file auto.py in the python folder of AUTO in Windows Explorer, or by creating a shortcut to it.

Alternatively, AUTO runs on Windows as above using the Unix-like environment Cygwin (see `http://www.cygwin.com`), but the non-Cygwin setup is more responsive and is much easier to setup for Matplotlib. You can however use its X server and lesstif to compile and run the old PLAUT and GUI94, if you so desire.

With some effort it is possible to compile PLAUT04 on Windows (without an X server) using Coin, SoQt, and Qt. You can also find precompiled PLAUT04 binaries at `http://sourceforge.net/projects/auto-07p/files`.

1.2 Restrictions on Problem Size.

There are no size restrictions in the file auto/07p/include/auto.h any more. This file now contains the default effective number of equation parameters `NPAR`, set to 36 upon installation. It can be overridden in constant files. See also Section 11.1. The default can be changed by editing `auto.h`. This must be followed by recompilation by typing `make` in the directory `auto/07p/src`.

Note that in certain cases the *effective dimension* may be greater than the user dimension. For example, for the continuation of folds, the effective dimension is `2NDIM`+1 for algebraic equations, and `2NDIM` for ordinary differential equations, respectively. Similarly, for the continuation of Hopf bifurcations, the effective dimension is `3NDIM`+2.

1.3 Compatibility with Earlier Versions.

Unlike earlier versions, AUTO can no longer be compiled using a pure Fortran 77 compiler, but you need at least a Fortran 90 compiler. A free Fortran 95 compiler, GFortran, is shipped with most recent Linux distributions, or can be obtained at `http://gcc.gnu.org/wiki/GFortran`, which contains binaries for Linux, Mac OS X and Windows. AUTO was also tested with the free compiler `g95`, and there exist various commercial Fortran 9x compilers as well.

The AUTO input files are now called `c.xxx` (the constants file), and `h.xxx` (the HomCont constants file, only used with HomCont); the output files are called `b.xxx` (the bifurcation-diagram-file), `s.xxx` (the solution-file), and `d.xxx` (the diagnostics-file). The command `@rn` can be used to rename all these files from their old names. There are also minor changes in the formatting of these files compared to recent versions of AUTO, such as AUTO97 and AUTO2000.

The main change compared to AUTO97 is that there is now a programmable Python CLUI. The constants file can be written using a completely new, more flexible syntax, but the old syntax is still accepted, and files can be converted using the command `@cnvc` (see Section 5).

Due to the replacement of EISPACK routines by LAPACK routines for the computation of eigenvalues and eigenvectors, the sign of the eigenvectors may have flipped sometimes with respect to earlier versions. This affects the sign of some HomCont test functions and the initial direction when using the homotopy method (you may have to flip the sign of the starting distance in the routine STPNT). Eigenvectors are now normalized to avoid future problems and improve consistency.

Parameter derivatives in `DFDP` are now significant when using HomCont. If only the derivatives with respect to phase space variables are specified in `DFDU`, please use the setting `JAC=-1`.

When upgrading from AUTO2000, you can continue to use equations-files written in C. However, there is now a strict difference between indexing of the array `par[]` in the C file and the references to it using `PAR()` in constants files and output, using `par[i]=PAR(I+1)`. In practise this means that you do not have to change the C file, but need to add 1 to all parameter indices in the constant files, namely `ICP`, `THL`, and `UZR`. For example, the period is referenced by `par[10]` in the C file, but by `PAR(11)` in the constants file. Equation files written in C are used in the homoclinic branch switching demo in Chapter 27.

A detailed list of user-visible changes can be found in the file $AUTO_DIR/CHANGELOG.

1.4 Parallel Version.

AUTO contains code which allows it to run in on parallel computers. Namely, it can use either OpenMP to run most of its code in parallel on shared-memory multi-processors, or the MPI message passing library. When the configure script is run it will try to detect if the Fortran compiler supports OpenMP; examples are Gfortran 4.2 or later and the Intel Fortran Compiler. If it is successful the necessary compiler flags are used to enable OpenMP in AUTO. To force the configure script not to use OpenMP, one may type `./configure --without-openmp`, and then type `make`. On the other hand, unless there is some particular difficulty, we recommend that that the configure script be used without arguments, since the parallel version of AUTO may easily be controlled, and even run in a serial mode, through the use of the environment variable `OMP_NUM_THREADS`.

For example, to run the AUTO executable auto.exe in serial mode you just type `export OMP_NUM_THREADS=1`. To run the same command in parallel on 4 processors you type `export OMP_NUM_THREADS=4`. Without any `OMP_NUM_THREADS` set the number of processors that AUTO will use can be equal to the actual number of processors on the system, or can be equal to one; this is system-dependent.

The MPI message passing library is not used by default. You can enable it by typing `./configure --with-mpi`. If OpenMP and MPI are both used then AUTO uses mixed mode, with MPI parallelisation occurring at the top level.

Running the MPI version is somewhat more complex because of the fact that MPI normally uses some external program for starting the computational processes. The exact name and command line options of this external program depends on your MPI installation. A common name for this MPI external program is mpirun, and a common command line option which

defines the number of computational processes is -np. Accordingly, if you wanted to run the MPI version of AUTO on four processors, with the above external program, you would type mpirun -np 4 file.exe. Please see your local MPI documentation for more detail.

Both the Python CLUI and the commands in the auto/07p/cmds directory described in Chapter 5 may be used with the MPI version as well, by setting the AUTO_COMMAND_PREFIX environment variable. For example, to run AUTO in parallel using the MPI library on 4 processors just type `export AUTO_COMMAND_PREFIX='mpirun -np 4'` before you run the Python CLUI `auto` or the commands in auto/07p/cmds normally. The previous example assumed you are using the `sh` shell or the `bash` shell; for other shells you should modify the commands appropriately, for example `setenv AUTO_COMMAND_PREFIX 'mpirun -np 4'` for the `csh` and `tcsh` shells. Alternatively, inside the Python CLUI and scripts you can use `import os` followed by `os.environ["AUTO_COMMAND_PREFIX"]="mpirun -np 4"`.

Chapter 2

Overview of Capabilities.

2.1 Summary.

AUTO can do a limited bifurcation analysis of algebraic systems

$$f(u,p) = 0, \qquad f(\cdot,\cdot), u \in \mathrm{R}^n. \tag{2.1}$$

The main algorithms in AUTO, however, are aimed at the continuation of solutions of systems of ordinary differential equation (ODEs) of the form

$$u'(t) = f\big(u(t),p\big), \qquad f(\cdot,\cdot), u(\cdot) \in \mathrm{R}^n, \tag{2.2}$$

subject to boundary (including initial) conditions and integral constraints. Above, p denotes one or more free parameters.

These boundary value algorithms also allow AUTO to do certain stationary solution and wave calculations for the partial differential equation (PDE)

$$u_t = Du_{xx} + f(u,p), \qquad f(\cdot,\cdot), u(\cdot) \in \mathrm{R}^n, \tag{2.3}$$

where D denotes a diagonal matrix of diffusion constants.

The basic algorithms used in AUTO, as well as related algorithms, can be found in Keller (1977), Keller (1986), Doedel, Keller & Kernévez (1991*a*), Doedel, Keller & Kernévez (1991*b*).

Below, the basic capabilities of AUTO are specified in more detail. Some representative demos are also indicated.

2.2 Algebraic Systems.

Specifically, for (2.1) AUTO can :

- Compute solution families.
  (Demo `ab`; Run 2.)

- Locate branch points, continue these in two or three parameters, and automatically compute bifurcating families.
  (Demos `pp2`; Run 1, and `apbp`.)

- Locate Hopf bifurcation points, continue these in two parameters, detect whether the Hopf bifurcation is sub- or supercritical, and detect zero-Hopf, Bogdanov-Takens, and generalized Hopf (Bautin) bifurcations.
  (Demos `pp3` and `ppp`.)

- Locate folds (limit points), continue these in two parameters, and detect cusp, zero-Hopf, and Bogdanov-Takens bifurcations.

- Locate branch points, folds, period-doubling, and torus (Neimark-Sacker) bifurcations, continue these in two or three parameters, and switch branches at branch points and period-doubling bifurcations for fixed points of the discrete dynamical system $u^{(k+1)} = f(u^{(k)}, p)$
  (Demo `dd2`.)

- Find extrema of an objective function along solution families and successively continue such extrema in more parameters.
  (Demo `opt`.)

2.3 Ordinary Differential Equations.

For the ODE (2.2) the program can :

- Compute families of stable and unstable periodic solutions and compute the Floquet multipliers, that determine stability, along these families. Starting data for the computation of periodic orbits are generated automatically at Hopf bifurcation points.
  (Demo `ab`; Run 2.)

- Locate folds, branch points, period doubling bifurcations, and bifurcations to tori, along families of periodic solutions. Branch switching is possible at branch points and at period doubling bifurcations.
  (Demos `tor`, `lor`.)

- Continue folds, period-doubling bifurcations, and bifurcations to tori, in two parameters, detecting 1:1, 1:2, 1:3 and 1:4 resonances.
  (Demos `plp`, `pp3`, `tor`.)

  The continuation of orbits of fixed period is also possible. This is the simplest way to compute curves of homoclinic orbits, if the period is sufficiently large.
  (Demo `pp2`.)

  The continuation of branch points in two parameters is only possible in non-generic problems, characterized by problem-specific symmetries.
  (Demo `lcbp`, Run 2.)

  Generically, in non-symmetric problems, branch points are continued in three parameters.
  (Demos `lcbp`, Run 3, and `abcb`.)

- Do each of the above for *rotations*, i.e., when some of the solution components are periodic modulo a phase gain of a multiple of 2π.
(Demo `pen`.)

- Follow curves of homoclinic orbits and detect and continue various codimension-2 bifurcations, using the HomCont algorithms of Champneys & Kuznetsov (1994), Champneys, Kuznetsov & Sandstede (1996).
(Demos `san`, `mnt`, `kpr`, `cir`, `she`, `rev`.)

- Locate extrema of an integral objective functional along a family of periodic solutions and successively continue such extrema in more parameters.
(Demo `ops`.)

- Compute curves of solutions to (2.2) on $[0, 1]$, subject to general nonlinear boundary and integral conditions. The boundary conditions need not be separated, i.e., they may involve both $u(0)$ and $u(1)$ simultaneously. The side conditions may also depend on parameters. The number of boundary conditions plus the number of integral conditions need not equal the dimension of the ODE, provided there is a corresponding number of additional parameter variables.
(Demos `exp`, `int`.)

- Determine folds and branch points along solution families to the above boundary value problem. Branch switching is possible at branch points. Curves of folds and branch points can be computed.
(Demos `bvp`, `int`, `sspg`.)

2.4 Parabolic PDEs.

For (2.3) the program can :

- Trace out families of spatially homogeneous solutions. This amounts to a bifurcation analysis of the algebraic system (2.1). However, AUTO uses a related system instead, in order to enable the detection of bifurcations to wave train solutions of given wave speed. More precisely, bifurcations to wave trains are detected as Hopf bifurcations along fixed point families of the related ODE

$$\begin{aligned} u'(z) &= v(z), \\ v'(z) &= -D^{-1}\big[c\ v(z) + f\big(u(z), p\big)\big], \end{aligned} \tag{2.4}$$

  where $z = x - ct$, with the wave speed c specified by the user.
  (Demo `wav`; Run 2.)

- Trace out families of periodic wave solutions to (2.3) that emanate from a Hopf bifurcation point of Equation 2.4. The wave speed c is fixed along such a family, but the wave length L, i.e., the period of periodic solutions to (2.4), will normally vary. If the wave length L becomes large, i.e., if a homoclinic orbit of Equation 2.4 is approached, then the wave tends to a solitary wave solution of (2.3).
  (Demo `wav`; Run 3.)

- Trace out families of waves of fixed wave length L in two parameters. The wave speed c may be chosen as one of these parameters. If L is large then such a continuation gives a family of approximate solitary wave solutions to (2.3).
  (Demo `wav`; Run 4.)

- Do time evolution calculations for (2.3), given periodic initial data on the interval $[0, L]$. The initial data must be specified on $[0, 1]$ and L must be set separately because of internal scaling. The initial data may be given analytically or obtained from a previous computation of wave trains, solitary waves, or from a previous evolution calculation. Conversely, if an evolution calculation results in a stationary wave then this wave can be used as starting data for a wave continuation calculation.
  (Demo `wav`; Run 5.)

- Do time evolution calculations for (2.3) subject to user-specified boundary conditions. As above, the initial data must be specified on $[0, 1]$ and the space interval length L must be specified separately. Time evolution computations of (2.3) are adaptive in space and in time. Discretization in time is not very accurate : only implicit Euler. Indeed, time integration of (2.3) has only been included as a convenience and it is not very efficient.
  (Demos `pd1`, `pd2`.)

- Compute curves of stationary solutions to (2.3) subject to user-specified boundary conditions. The initial data may be given analytically, obtained from a previous stationary solution computation, or from a time evolution calculation.
  (Demos `pd1`, `pd2`.)

In connection with periodic waves, note that (2.4) is just a special case of (2.2) and that its fixed point analysis is a special case of (2.1). One advantage of the built-in capacity of AUTO to deal with problem (2.3) is that the user need only specify f, D, and c. Another advantage is the compatibility of output data for restart purposes. This allows switching back and forth between evolution calculations and wave computations.

2.5 Discretization.

AUTO discretizes ODE boundary value problems (which includes periodic solutions) by the method of orthogonal collocation using piecewise polynomials with 2-7 collocation points per mesh interval (de Boor & Swartz (1973)). The mesh automatically adapts to the solution to equidistribute the local discretization error (Russell & Christiansen (1978)). The number of mesh intervals and the number of collocation points remain constant during any given run, although they may be changed at restart points. The implementation is AUTO-specific. In particular, the choice of local polynomial basis and the algorithm for solving the linearized collocation systems were specifically designed for use in numerical bifurcation analysis.

Chapter 3

User-Supplied Files.

The user must prepare the two files described below. This can be done with the GUI described in Chapter 9, or independently.

3.1 The Equations-File `xxx.f90`, or `xxx.f`, or `xxx.c`

A source file `xxx.f90` containing the Fortran routines `FUNC`, `STPNT`, `BCND`, `ICND`, `FOPT`, and `PVLS`. Here `xxx` stands for a user-selected name. If any of these routines is irrelevant to the problem then its body need not be completed. Examples are in `auto/07p/demos`, where, e.g., the file `ab/ab.f90` defines a two-dimensional dynamical system, and the file `exp/exp.f90` defines a boundary value problem. The simplest way to create a new equations-file is to copy an appropriate demo file. For a fully documented equations-file see `auto/07p/demos/cusp/cusp.f90` or `auto/07p/gui/aut.f90`. In GUI mode, this file can be directly loaded with the GUI-button *Equations/New*; see Section 9.2.

The equations-file can either be written in fixed-form (old-style) Fortran (.f), free-form Fortran (.f90) or in C (.c).

3.2 The Constants-File `c.xxx`

AUTO-constants for `xxx.{f,f90,c}` are normally expected in a corresponding file `c.xxx`. Specific examples include `ab/c.ab` and `exp/c.exp` in `auto/07p/demos`. See Chapter 10 for the significance of each constant.

3.3 User-Supplied Routines.

The purpose of each of the user-supplied routines in the file `xxx.{f90,f}` is described below.

- `FUNC` : defines the function $f(u,p)$ in (2.1), (2.2), or (2.3).

- `STPNT` : This routine is called only if `IRS`=0 (see Section 10.8.10 for `IRS`), which typically is the case for the first run, or when a system is manually extended. A system is extended if `NDIM` (see Section 10.2.1) increases between runs. It defines a starting solution (u,p) of (2.1) or (2.2). The starting solution should not be a branch point.
  (Demos `ab`, `exp`, `frc`, `lor`.)
  It extends an existing solution into higher dimensions in the demos `p2c` and `c2c`.

- `BCND` : A routine `BCND` that defines the boundary conditions.
  (Demo `exp`, `kar`.)

- `ICND` : A routine `ICND` that defines the integral conditions.
  (Demos `int`, `lin`.)

- `FOPT` : A routine `FOPT` that defines the objective functional.
  (Demos `opt`, `ops`.)

- `PVLS` : A routine `PVLS` for defining "solution measures". This routine, using a "`LOGICAL, SAVE :: first = .TRUE.`" variable can also be used for initialization, as it is called first.
  (Demo `pvl`.)

In a C language equation file, these routines are written using lowercase letters; with Fortran you can use any case.

3.4 User-Supplied Derivatives.

If AUTO-constant `JAC` equals 0 then derivatives need not be specified in `FUNC`, `BCND`, `ICND`, and `FOPT`; see Section 10.2.5. If `JAC=1` then derivatives must be given. If `JAC=-1` then the parameter derivatives may be omitted in `FUNC`. This may be necessary for sensitive problems, and is recommended for computations in which AUTO generates an extended system. Derivatives are specified as follows, where zero entries may be omitted:

`FUNC` – Derivatives with respect to phase space variables are specified in `DFDU(1:NDIM, 1:NDIM)`.

– Parameter derivatives go into `DFDP(1:NDIM, 1:NPAR)`.

`BCND` – Derivatives with respect to the two boundary conditions are specified in `DBC(1:NBC, 1:NBC)` and `DBC(1:NBC, NBC+1:2*NBC)`, respectively.

– Parameter derivatives go into `DBC(1:NBC, 2*NBC+1:2*NBC+NPAR)`.

`ICND` – Derivatives with respect to the integral conditions are specified in `DINT(1:NINT, 1:NINT)`.

– Parameter derivatives go into `DINT(1:NINT, NINT+1:NINT+NPAR)`.

Examples of user-supplied derivatives can be found in demos `dd2`, `int`, `plp`, `opt`, and `ops`.

Chapter 4

Running AUTO using Python Commands.

4.1 Typographical Conventions

This chapter uses the following conventions. All code examples will be in in the following font.

```
AUTO> copydemo("ab")
Copying demo ab ... done
```

To distinguish commands which are typed to the Unix shell from those which are typed to the AUTO command line user interface (CLUI) we will use the following two prompts.

`>`	Commands which follow this prompt are for the Unix shell.
`AUTO>`	Commands which follow this prompt are for the AUTO CLUI.

4.2 General Overview.

The AUTO command line user interface (CLUI) is similar to the command language described in Section 5 in that it facilitates the interactive creating and editing of equations-files and constants-files. It differs from the other command language in that it is based on the object-oriented scripting language Python (see Lutz (1996)) and provides extensive programming capabilities. This chapter will provide documentation for the AUTO CLUI commands, but is not intended as a tutorial for the Python language. We will attempt to make this chapter self contained by describing all Python constructs that we use in the examples, but for more extensive documentation on the Python language, including tutorials and pointers to further documentation, please see Lutz (1996) or the web page `http://www.python.org` which contains an excellent tutorial at `http://www.python.org/doc/current/tut/tut.html`.

To use the CLUI for a new equation, change to an empty directory. For an existing equations-file, change to its directory. (*Do not activate the CLUI in the directory auto/07p or in any of its subdirectories.*) Then type

```
auto.
```

```
> auto
Python 2.5.2 (r252:60911, Nov 14 2008, 19:46:32)
[GCC 4.3.2] on linux2
Type "help", "copyright", "credits" or "license" for more information.
(AUTOInteractiveConsole)
AUTO>
```

Figure 4.1: Typing `auto` at the Unix shell prompt starts the AUTO CLUI.

If your command search path has been correctly set (see Section 1.1), this command will start the AUTO CLUI interactive interpretor and provide you with the AUTO CLUI prompt.

If you have IPython installed (`http://ipython.scipy.org`), then you can get a friendlier interface using the command `auto -i`, enabling TAB completion, persistent command-line history and other features.

In addition to the examples in the following sections there are several example scripts which can be found in auto/07p/demos/python and are listed in Table 4.1. These scripts are fully annotated and provide good examples of how AUTO CLUI scripts are written. The scripts in auto/07p/demos/python/n-body are especially lucid examples and perform various related parts of a calculation involving the gravitational N-body problem. Scripts which end in the suffix .auto are called "basic" scripts and can be run by typing `auto scriptname.auto`. The scripts shown in Section 4.3 and Section 4.5 are examples of basic scripts. Scripts which end in the suffix .xauto are called "expert" scripts and can be run by typing `autox scriptname.xauto`. More information on expert scripts can be found in Section 4.6. See the README file in that directory for more information.

4.3 First Example

We begin with a simple example of the AUTO CLUI. In this example we copy the ab demo from the AUTO installation directory and run it. For more information on the ab demo see Section 12.8. The commands listed in Table 4.2 will copy the demo files to your work directory and run the first part of the demo. The results of running these commands are shown in Figure 4.2.

Let us examine more closely what action each of the commands performs. First, `demo('ab')` (Section 4.14.6 in the reference) copies the files in $AUTO_DIR/demo/ab into the work directory.

Next, `ab = load(equation='ab')` (Section 4.14.1 in the reference) informs the AUTO CLUI that the name of the user defined function file is ab.f90. The commands `load`, and the closely related `run`, are two of the most commonly used commands in the AUTO CLUI, since they read and parse the user files which are manipulated by other commands. The AUTO CLUI stores this setting in the variable `ab` until it is changed by a command, such as another `load` command. The idea of storing information is one of the ideas that sets the CLUI apart from the command language described in Section 5.

Next, `ab = load(ab, constants='ab.1')` parses the AUTO constants file c.ab.1 and reads it into memory. Note that *changes to the file c.ab.1 after it has been loaded in will not be used*

Script	Description
demo1.auto	The demo script from Section 4.3.
demo2.auto, demo3.auto, and demo4.auto	The demo scripts from Section 4.5.
userScript.xauto	The expert demo script from Figure 4.14.
userScript.py	The loadable expert demo script from Figure 4.15.
branches.auto	The branch manipulating script from Figure 4.18.
fullTest.auto	A script which uses the entire AUTO command set, except for the plotting commands.
plotter.auto	A demonstration of some of the plotting capabilities of AUTO.
tutorial.auto	A script which implements the tutorial from Section 12.8.
n-body/compute_lagrange_points_family.auto	A basic script which computes and plots all of the "Lagrange points" as a function of the ratio of the masses of the two planets.
n-body/compute_lagrange_points_0.5.auto	A basic script which computes all of the "Lagrange points" for the case where the masses of the two planets are equal, and saves the data.
n-body/compute_periodic_family.xauto	An expert script which starts at a "Lagrange point" computed by compute_lagrange_points_0.5.auto and continues in the ratio of the masses until a specified mass ratio is reached. It then computes a family of periodic orbits for each pair of purely complex eigenvalues.
n-body/to_matlab.xauto	A script which takes a set of AUTO data files and creates a set of files formatted for importing into Matlab for either plotting or further calculations.

Table 4.1: The various demonstration scripts for the AUTO CLUI.

Unix-COMMAND	ACTION
`auto`	start the AUTO CLUI
AUTO CLUI COMMAND	ACTION
`demo('ab')`	copy the demo files to the work directory
`ab = load(equation='ab')`	load the filename ab.f90 into the variable ab
`ab = load(ab, constants='ab.1')`	load the contents of the file c.ab.1 into the variable ab
`run(ab)`	run AUTO with the current set of files

Table 4.2: Running the demo ab files.

```
> auto
Python 2.5.2 (r252:60911, Nov 14 2008, 19:46:32)
[GCC 4.3.2] on linux2
Type "help", "copyright", "credits" or "license" for more information.
(AUTOInteractiveConsole)
AUTO> demo('ab')
Copying demo ab ... done
Runner configured
AUTO> ab = load(equation='ab')
Runner configured
AUTO> ab = load(ab,constants='ab.1')
Runner configured
AUTO> run(ab)
gfortran -fopenmp -O -c ab.f90 -o ab.o
gfortran -fopenmp -O ab.o -o ab.exe /home/bart/auto/07p/lib/*.o
Starting ab ...

  BR    PT  TY  LAB    PAR(2)         L2-NORM         U(1)          U(2)
   1     1  EP    1   8.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1    31  UZ    2   1.40000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    36  UZ    3   1.50000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    41  UZ    4   1.60000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    46  UZ    5   1.70000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    51  UZ    6   1.80000E+01   0.00000E+00   0.00000E+00   0.00000E+00

 Total Time    0.181E-01
ab ... done
<_=bifDiag instance at 0x0972198c>
AUTO>
```

Figure 4.2: Typing `auto` at the Unix shell prompt starts the AUTO CLUI. The rest of the commands are interpreted by the AUTO CLUI.

by AUTO unless it is loaded in again after the changes are made.

Finally, `run(ab)` (Section 4.14.1 in the reference) uses the user defined functions loaded by the `load(equation='ab')` command, and the AUTO constants loaded by the `load(ab, constants = 'ab.1')` to run AUTO.

The `run` command returns a bifurcation diagram structure. It can be referenced using the special _ variable in interactive sessions, or assigned as `result=run(ab)`. The result can then be referred to in further calculations, plotted, and saved.

Figure 4.2 showed two of the file types that the `load` command can read into memory, namely the user defined function file and the AUTO constants file (Section 3). There are two other files types that can be read in using the `load` command, and they are the restart solution file (Section 6) and the HomCont parameter file (Section 20.2). The `load` command can also directly load AUTO constants.

Note that the name given to the load command is not the same as the filename which is read in, for example `load(constants='ab.1')` reads in the file `c.ab.1`. This difference is a result of the automatic transformation of the filenames by the AUTO CLUI into the standard names used by AUTO. The standard filename transformations are shown in Table 4.3.

Long name	Short name	Name entered	Transformed file name
equation	e	foo	foo.f90/foo.f/foo.c
constants	c	foo	c.foo
solution	s	foo	s.foo
bifurcationDiagram	b	foo	b.foo
diagnostics	d	foo	d.foo
homcont	h	foo	h.foo

Table 4.3: This table shows the standard AUTO CLUI filename translations. In load and run commands either the long name or the short name may be used for loading the appropriate files.

Since the `load` command is so common, there are various shorthand versions of it. First, there are short versions of the various arguments as shown in Table 4.3. For example, the command `load(constants='ab.1')` can be shortened to `load(c='ab.1')`. Next, several different files may be loaded at once using the same `load` command. For example, the two commands in Figure 4.3 have the same effect as the single command in Figure 4.4. Last, you can bypass the `load` command, unless the intermediate result is needed, and use the `run` command directly on the `load` arguments, as in Figure 4.5.

```
AUTO> ab = load(e='ab')
Runner configured
AUTO> ab = load(ab,c='ab.1)
Runner configured
```

Figure 4.3: Loading two files individually.

```
AUTO> ab = load(e='ab',c='ab.1')
Runner configured
```

Figure 4.4: Loading two files at the same time.

```
AUTO> runab1 = run(e='ab',c='ab.1')
Runner configured
```

Figure 4.5: Loading two files at the same time and run using them.

Also, since it is common that several files will be loaded that have the same base name `load('ab')` performs the same action as `load(e='ab', c='ab', s='ab', h='ab')`. Note, for the command `load('ab')` it is not required that all of the files exist. Information from all existing files is used only if they exist, and no error message will be given for non-existing files. However, later `run` commands may cause AUTO to err with incomplete information.

4.4 Scripting

Section 4.3 showed commands being interactively entered at the AUTO CLUI prompt, but since the AUTO CLUI is based on Python one has the ability to write scripts for performing sequences of commands automatically. A Python script is very similar to the interactive mode shown in Section 4.3 except that the commands are placed in a file and read all at once. For example, if the commands from Figure 4.2 where placed into the file demo1.auto, in the format shown in Figure 4.6, then the commands could be run all at once by typing `auto demo1.auto`. See Figure 4.7 for the full output.

```
demo('ab')
ab = load(equation='ab')
ab = load(ab, constants='ab.1')
run(ab)
```

Figure 4.6: The commands from Figure 4.2 and they would appear in a AUTO CLUI script file. The source for this script can be found in $AUTO_DIR/demos/python/demo1.auto.

4.5 Second Example

In Section 4.3 we showed a very simple AUTO CLUI script, in this Section we will describe a more complex example, which introduces several new AUTO CLUI commands as well as a

```
> cat demo1.auto
demo('ab')
ab = load(equation='ab')
ab = load(ab, constants='ab.1')
run(ab)

> auto demo1.auto
Copying demo ab ... done
Runner configured
Runner configured
Runner configured
gfortran -fopenmp -O -c ab.f90 -o ab.o
gfortran -fopenmp -O ab.o -o ab.exe /home/bart/auto/07p/lib/*.o
Starting ab ...

  BR    PT  TY  LAB    PAR(2)         L2-NORM         U(1)          U(2)
   1     1  EP    1   8.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1    31  UZ    2   1.40000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    36  UZ    3   1.50000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    41  UZ    4   1.60000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    46  UZ    5   1.70000E+01   0.00000E+00   0.00000E+00   0.00000E+00
   1    51  EP    6   1.80000E+01   0.00000E+00   0.00000E+00   0.00000E+00

 Total Time    0.193E-01
ab ... done
>
```

Figure 4.7: This Figure starts by listing the contents of the demo1.auto file using the Unix `cat` command. The file is then run through the AUTO CLUI by typing auto demo1.auto and the output is shown.

basic Python construct for looping. We will not provide an exhaustive reference for the Python language, but only the very basics. For more extensive documentation we refer the reader to Lutz (1996) or the web page `http://www.python.org`. In this section we will describe each line of the script in detail, and the full text of the script is in Figure 4.8.

The script begins with a section, extracted into Figure 4.9, which performs a task identical to that shown in Figure 4.2 except that the shorthand discussed in Section 4.3 is used for the `run` command.

Up to this point all of the commands presented have had analogs in the command language discussed in Section 5, and the AUTO CLUI has been designed in this way to make it easy for users to migrate from the old command language to the AUTO CLUI. The next section of the script extracted into Figure 4.9, introduces a new command, namely `branchpoints = bvp("BP")`, which is the first command which has no analog in the old command language. The command `bvp("BP")`, given the output variable `bvp` from the first run, returns a Python object

```
demo('bvp')

bvp = run('bvp')
branchpoints = bvp("BP")
for solution in branchpoints:
    bp = load(solution, ISW=-1, NTST=50)
    # Compute forwards
    print "Solution label", bp["LAB"], "forwards"
    fw = run(bp)
    # Compute backwards
    print "Solution label", bp["LAB"], "backwards"
    bw = run(bp,DS='-')
    both = fw + bw
    merged = merge(both)
    bvp = bvp + merged

bvp=relabel(bvp)
save(bvp, 'bvp')
plot(bvp)
wait()
```

Figure 4.8: This Figure shows a more complex AUTO CLUI script. The source for this script can be found in $AUTO_DIR/demos/python/demo2.auto.

```
demo('bvp')

bvp = run('bvp')
```

Figure 4.9: The first part of the complex AUTO CLUI script.

which represents a list of all branchpoint solutions. Accordingly, this list is stored in the Python variable `branchpoints`. Note, variables in Python are different from those in languages such as C in that their type does not have to be declared before they are created.

```
branchpoints = bvp("BP")
for solution in branchpoints:
    bp = load(solution, ISW=-1, NTST=50)
    # Compute forwards
    print "Solution label", bp["LAB"], "forwards"
    fw = run(bp)
    # Compute backwards
    print "Solution label", bp["LAB"], "backwards"
    bw = run(bp,DS='-')
    both = fw + bw
    merged = merge(both)
    bvp = bvp + merged
```

Figure 4.10: The second part of the complex AUTO CLUI script.

The next command, `for solution in branchpoints:` is the Python syntax for loops. The `branchpoints` object is a list of the branch point solutions from the first run. The command `for solution in branchpoints:` is used to loop over all solutions in the branchpoints variable by setting the variable `solution` to be one of the solutions in `branchpoints` and then calling the rest of the code in the block.

Python differs from most other computer languages in that blocks of code are not defined by some delimiter, such as `END DO` in Fortran, but by indentation. In Figure 4.8 the commands starting with `bvp=relabel(bvp)` are not part of the loop, because they are indented differently. This can be confusing first time users of Python, but it has the advantage that the code is forced to have a consistent indentation style.

The next command in the script, `bp = load(solution, ISW=-1, NTST=50)` loads a solution with modified AUTO constants. All other constants are the same as they were in the first run. The `ISW` value is changed to `-1` (see Section 10.8.4), so that a branch switch is performed, and the `NTST` value is changed to `50` (see Section 10.3.1). Only "in memory" versions of the AUTO constants are modified; the original file `c.bvp` is *not* modified.

Some diagnostics are then printed to the screen using a standard Python `print` command: the label number of the branch point that we switch at can be found by using `bp["LAB"]`. In addition, as can be seen in Figure 4.10, the `#` character is the Python comment character. When the Python interpretor encounters a `#` character it ignores everything from that character to the end of the line.

We then use a `fw=run(bp)` command to perform the calculation of the bifurcating branch from solution `bp`. We print additional information and use the command `bw = run(bp,DS='-')` to change the AUTO initial step size from positive to negative, which causes AUTO to compute the bifurcating branch in the other direction (see Section 10.5.1). This output is appended to the existing output in the Python variable `bvp` after some more processing. First the command

`both = fw + bw` concatenates, using standard Python list syntax, the forwards and backwards branches. Subsequently `merged = merge(both)` merges the two branches into one continuous branch where the backwards branch is flipped. Finally, the command `bvp = bvp + merged` appends the merged branch to the existing results.

```
bvp=relabel(bvp)
save(bvp, 'bvp')
plot(bvp)
wait()
```

Figure 4.11: The third part of the complex AUTO CLUI script.

The last section of the script, extracted into Figure 4.11, does some postprocessing and plotting. First, the command `bvp=relabel(bvp)` relabels so all solutions in the `bvp` object have unique labels starting at 1. The command `save(bvp,'bvp')` (Section 4.14.1 in the reference) saves the results of the AUTO runs into files using the base name `bvp` and the filename extensions in Table 4.3. For example, in this case the bifurcation diagram will be saved as `b.bvp`, the solution will be saved as `s.bvp`, and the diagnostics will be saved as `d.bvp`.

Now that the section of script shown in Figure 4.11 has finished computing the bifurcation diagram, the command `plot(bvp)` brings up a plotting window (Section 4.14.2 in the reference), and the command `wait()` causes the AUTO CLUI to wait for input. You may now exit the AUTO CLUI by pressing any key in the window in which you started the AUTO CLUI.

For convenience, some of these commands have shorter forms. For instance, the `load`, `run`, `merge`, `relabel`, `save`, and `plot` commands have the shorter forms `ld`, `r`, `mb`, `rl`, `sv`, and `pl`, respectively. All algebraic and functional expressions can be combined in the usual way. Combining these techniques, a shorter version of the complex AUTO CLUI script is given in Figure 4.12.

Even shorter forms are possible to save you typing using features borrowed from ipython, both in `auto -i` and in plain `auto`. Those forms use auto-parentheses and auto-quotes: for a command that does not assign to a variable you can skip the parentheses, and by using a "," on the first character of a line you force all parameters to be quoted. For example, you can just type `pl bvp` to plot the bifurcation diagram and solutions in the object `bvp`. Beware that these extra short forms are only possible in normal auto scripts and at the AUTO CLUI prompt, and only if they start in the first column, but not in the "expert" scripts described in the next section. The even shorter version of the complex AUTO CLUI script is given in Figure 4.13. It should be clear that these super-short forms save you typing at the command prompt but do not help readability in scripts.

4.6 Extending the AUTO CLUI

The code in Figure 4.8 performed a very useful and common procedure, it started an AUTO calculation and performed additional continuations at every point which AUTO detected as a bifurcation. Unfortunately, the script as written can only be used for the `bvp` demo. In this

```
demo('bvp')

bvp=r('bvp')
for solution in bvp('BP'):
    bp = ld(solution,NTST=50,ISW=-1)
    # Compute forwards and backwards
    bvp = bvp + mb(r(bp)+r(bp,DS='-'))

bvp=rl(bvp)
pl(bvp)
sv(bvp,'bvp')
wait()
```

Figure 4.12: This Figure shows a shorter version of the more complex AUTO CLUI script given above. The source for this script can be found in $AUTO_DIR/demos/python/demo3.auto.

```
,demo bvp

bvp=r('bvp')
for solution in bvp('BP'):
    bp = ld(solution,NTST=50,ISW=-1)
    # Compute forwards and backwards
    bvp = bvp + mb(r(bp)+r(bp,DS='-'))

bvp=rl(bvp)
pl bvp
sv bvp, 'bvp'
wait
```

Figure 4.13: This Figure shows an even shorter version of the more complex AUTO CLUI script given above. The source for this script can be found in $AUTO_DIR/demos/python/demo4.auto.

section we will generalize the script in Figure 4.8 for use with any demo, and demonstrate how it can be imported back into the interactive mode to create a new command for the AUTO CLUI. Several examples of such "expert" scripts can be found in auto/07p/demos/python/n-body.

Just as loops and conditionals can be used in Python, one can also define functions. For example, Figure 4.14 is a functional version of script from Figure 4.8. The changes are actually quite minor. The first line, `from auto import *`, includes the definitions of the AUTO CLUI commands, and must be included in all AUTO CLUI scripts which define functions. The next line, `def myRun(demoname):`, begins the function definition, and creates a function named `myRun` which takes one argument `demoname`. The rest of the script is the same except that it has been indented to indicate that it is part of the function definition, all occurrences of string `'bvp'` have been replaced with the variable `demoname`, and the variable `bvp` was replaced by the variable `r`. Finally we have added a line `myRun('bvp')` which actually calls the function we have created and runs the same computation as the original script.

```
from auto import *

def myRun(demoname):

    demo(demoname)

    r = run(demoname)
    branchpoints = r("BP")
    for solution in branchpoints:
        bp = load(solution, ISW=-1, NTST=50)
        # Compute forwards
        print "Solution label", bp["LAB"], "forwards"
        fw = run(bp)
        # Compute backwards
        print "Solution label", bp["LAB"], "backwards"
        bw = run(bp,DS='-')
        both = fw + bw
        merged = merge(both)
        r = r + merged

    r=relabel(r)
    save(r, demoname)
    plot(r)
    wait()

myRun('bvp')
```

Figure 4.14: This Figure shows a complex AUTO CLUI script written as a function. The source for this script can be found in $AUTO_DIR/demos/python/userScript.xauto.

While the script in Figure 4.14 is only slightly different then the one showed in Figure 4.8 it is much more powerful. Not only can it be used as a script for running any demo by modifying the last line, it can be read back into the interactive mode of the AUTO CLUI and used to create a new command, as in Figure 4.15. First, we create a file called userScript.py which contains the script from Figure 4.14, with one minor modification. We want the function only to run when we use it interactively, not when the file userScript.py is read in, so we remove the last line where the function is called. We start the AUTO CLUI with the Unix command `auto`, and once the AUTO CLUI is running we use the command `from userScript import *`, to import the file userScript.py into the AUTO CLUI. The `import` command makes all functions in that file available for our use (in this case `myRun` is the only one). It is important to note that `from userScript import *` does *not* use the .py extension on the file name. After importing our new function, we may use it just like any other function in the AUTO CLUI, for example by typing `myRun('bvp')`.

4.7 Bifurcation Diagram Objects

The `run` and `loadbd` commands (see Section 4.14.1 in the reference for details) return a Python structure which we refer to as a *bifurcation diagram object*. It represents the information that is also stored in AUTO's output files (fort.7, fort.8, and fort.9, or b.*, s.*, and d.*), and in AUTO's constant files. For example, the command `loadbd('ab')` returns an object corresponding to the files b.ab, s.ab, and d.ab. (if you are using the standard filename translations from Table 4.3). The command `run('ab')` returns an object corresponding to the output of the run.

The bifurcation diagram object encapsulates all this information in an easy to use form. This object is a list of all of the branches in the appropriate bifurcation diagram file, and each branch behaves like an array (a PyDSTool Pointset subclass, to be precise). This array can be viewed as a list of all of the points in the appropriate bifurcation diagram file, and each point is a Python dictionary with entries for each piece of data for the point. For example, the sequence of commands in Figure 4.16, prints out the label of the first point of a branch in a bifurcation diagram. The query-able parts of the object are listed in Table 4.4.

The individual elements of the array may be accessed in a number of ways: by index of the point using the `[]` syntax, a column using `['columnname']` syntax, or by label number or type name plus one-based index using the `()` syntax. For example, assume that the parsed object is contained in a variable `data`, and the first branch is in a variable `br=data[0]`. The first point may then be accessed using the command `br[0]`, while the column with label `PAR(1)` may be accessed using the command `br['PAR(1)']`. The point with label `57` may be accessed using the command `br(57)`, and the second Hopf bifurcation point using the command `br('HB2')` Using the `()` syntax you can also obtain new lists of points: `br('HB')` gives a list of all Hopf bifurcation points, `br([1,4])` gives the points with labels 1 and 4, and `br(['UZ',4])` gives all user defined points and label 4.

The `['columnname']` syntax is especially useful for plotting, as is illustrated in Figure 4.17. Here the Python package matplotlib is directly used to plot a branch. Of course, the command `plot` may also be used, but sometimes more control may be needed.

Additionally three special keys can be used to query a branch: use `br['BR']` to obtain the branch number, `br['TY']` for the branch type, and `br['TY number']` for the corresponding

```
> cp \$AUTO\_DIR/python/demo/userScript.py .
> ls
userScript.py
> cat userScript.py
# This is an example script for the AUTO07p command line user
# interface.  See the "Command Line User Interface" chapter in the
# manual for more details.
from auto import *

def myRun(demoname):

    demo(demoname)

    r = run(demoname)
    branchpoints = r("BP")
    for solution in branchpoints:
        bp = load(solution, ISW=-1, NTST=50)
        # Compute forwards
        print "Solution label", bp["LAB"], "forwards"
        fw = run(bp)
        # Compute backwards
        print "Solution label", bp["LAB"], "backwards"
        bw = run(bp,DS='-')
        both = fw + bw
        merged = merge(both)
        r = r + merged

    r=relabel(r)
    save(r, demoname)
    plot(r)
    wait()

> auto
Python 2.5.2 (r252:60911, Nov 14 2008, 19:46:32)
[GCC 4.3.2] on linux2
Type "help", "copyright", "credits" or "license" for more information.
(AUTOInteractiveConsole)
AUTO> from userScript import *
AUTO> myRun('bvp')
...
```

Figure 4.15: This Figure shows the functional version of the AUTO CLUI from Figure 4.14 being used as an extension to the AUTO CLUI. The source code for this script can be found in $AUTO_DIR/python/demo/userScript.py

```
 AUTO> demo('lrz')
 Copying demo lrz ... done
 Runner configured
 AUTO> data=run('lrz')
 gfortran -fopenmp -O -c lrz.f -o lrz.o
 gfortran -fopenmp -O lrz.o -o lrz.exe /home/bart/auto/07p/lib/*.o
 Starting lrz ...
(...)
lrz ... done
 AUTO> print data

  BR    PT  TY  LAB    PAR(1)        L2-NORM         U(1)          U(2)          U(3)
   1     1  EP    1   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1     5  BP    2   1.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1    13  EP    3   3.16000E+01   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00

  BR    PT  TY  LAB    PAR(1)        L2-NORM         U(1)          U(2)          U(3)
   2    42  HB    4   2.47368E+01   2.62685E+01   7.95602E+00   7.95602E+00   2.37368E+01
   2    45  EP    5   3.26008E+01   3.41635E+01   9.17980E+00   9.17980E+00   3.16008E+01

  BR    PT  TY  LAB    PAR(1)        L2-NORM         U(1)          U(2)          U(3)
   2    42  HB    6   2.47368E+01   2.62685E+01  -7.95602E+00  -7.95602E+00   2.37368E+01
   2    45  EP    7   3.26008E+01   3.41635E+01  -9.17980E+00  -9.17980E+00   3.16008E+01

 AUTO> br=data[1]
 AUTO> print br
  BR    PT  TY  LAB    PAR(1)        L2-NORM         U(1)          U(2)          U(3)
   2    42  HB    4   2.47368E+01   2.62685E+01   7.95602E+00   7.95602E+00   2.37368E+01
   2    45  EP    5   3.26008E+01   3.41635E+01   9.17980E+00   9.17980E+00   3.16008E+01
 AUTO> print br[0]
 {'TY number': 0, 'PT': -1, 'index': 0, 'section': 0, 'LAB': 0, 'BR': 2,
  'data': [0.99999994000000003, 0.0, 0.0, 0.0, 0.0], 'TY name': 'No Label'}
 AUTO> print br['PAR(1)']
 [  0.99999994   1.001875     1.00747645   1.01786791   1.03426011
    1.05801491   1.08738983   1.12195713   1.16544975   1.21923418
(...)
   13.69994168  15.15715248  16.76398367  18.53533839  20.48761501
   22.63885621  24.73684367  27.10974373  29.72303281  32.60076345]
 AUTO> print br(4)
 {'TY number': 3, 'PT': 42, 'index': 41, 'section': 0, 'LAB': 4, 'BR': 2,
  'data': [24.736843667999999, 26.268502943000001, 7.9560197197999996,
   7.9560197197999996, 23.736843667999999], 'TY name': 'HB'}
```

Figure 4.16: This figure shows an example of parsing a bifurcation diagram. First the demo involving the Lorenz equations named 'lrz' is copied and we perform its first run. We then print the result, its second branch, the first point on this branch, the column corresponding to 'PAR(1)', and the point with label 4 on this branch.

```
AUTO> import pylab
AUTO> pylab.plot(br['PAR(1)'],br['U(1)'])
[<matplotlib.lines.Line2D object at 0x9b1356c>]
AUTO> pylab.show()
```

Figure 4.17: Following the example in Figure 4.16 we can plot a subset of the bifurcation diagram, that is, the second branch, directly using matplotlib.

Query string	Meaning
TY name	The short name for the solution type (see Table 4.5).
TY number	The number of the solution type (see Table 4.5).
BR	The branch number.
PT	The point number.
LAB	The solution label, if any.
section	A unique identifier for each branch in a file with multiple branches.
data	An array which contains the AUTO output.

Table 4.4: This table shows the strings that can be used to query a bifurcation diagram object and their meanings.

Type	Short Name	Number
No Label	No Label	
Branch point (algebraic problem)	BP	1
Fold (algebraic problem)	LP	2
Hopf bifurcation (algebraic problem)	HB	3
Regular point (every NPR steps)	RG	4
User requested point	UZ	-4
Fold (ODE)	LP	5
Bifurcation point (ODE)	BP	6
Period doubling bifurcation (ODE)	PD	7
Bifurcation to invariant torus (ODE)	TR	8
Normal begin or end	EP	9
Abnormal termination	MX	-9

Table 4.5: This table shows the various types of points that can be in solution and bifurcation diagram files, with their short names and numbers.

Type	Short Name	Number
Bogdanov-Takens bifurcation (algebraic problem)	BT	-21, -31
Cusp (algebraic problem)	CP	-22
Generalized Hopf bifurcation (algebraic problem)	GH	-32
Zero-Hopf bifurcation (algebraic problem)	ZH	-13, -23, -33
1:1 Resonance bifurcation (ODE, maps)	R1	-25, -55, -85
1:2 Resonance bifurcation (ODE, maps)	R2	-76, -86
1:3 Resonance bifurcation (ODE, maps)	R3	-87
1:4 Resonance bifurcation (ODE, maps)	R4	-88
Fold-flip bifurcation (maps)	LPD	28, 78
Fold-torus bifurcation (maps)	LTR	23, 83
Flip-torus bifurcation (maps)	PTR	77, 87
Torus-torus bifurcation (maps)	TTR	88

Table 4.6: This table shows the various types of codimension-two points as in Table 4.5. The absolute value of the number divided by 10 gives the type of the branch on which the codimension-two bifurcation occurs.

number. These keys are also useful to change bifurcation diagram files, as is illustrated in Figure 4.18. Note that you can interactively change branch numbers, solution numbers and delete branches and solutions using the `@lb` command (see Chapter 5).

4.7.1 Solutions

You can also obtain solutions from a bifurcation diagram structure, by using the `()` syntax directly on the diagram instead of on individual branches. For example, in the above example the command `s=data()` returns an object which encapsulates all solutions in a easy to use form.

The object returned in this way is a list of all of the solutions in the appropriate bifurcation solution file, and each solution is a Python dictionary with entries for each piece of data for the solution. For example, the sequence of commands in Figure 4.19, prints out the label of the first solution in a bifurcation solution. The query-able parts of the object are listed in Table 4.7.

The individual elements of the list may, again, be accessed in two ways, either by the index of the solution using the `[]` syntax or by label number or type name using the `()` syntax. For example, assume that the parsed object is contained in a variable `s=data()`. The first solution may be accessed using the command `s[0]`, while the solution with label `57` may be accessed using the command `s(57)`.

Individual solutions can be also be obtained directly from the bifurcation diagram object `data` in the same way as for individual points above, by using `s=data(label)`. For example, the solution with label `57` may be accessed using the command `data(57)`, and the second Hopf bifurcation solution using the command `data('HB2')`. All individual solutions can then be used as a starting point for a new run.

Similarly, using the `()` syntax you can also obtain new lists of solutions: `data('HB')` gives a list of all Hopf bifurcation solutions, `data([1,4])` gives the solutions with labels 1 and 4, and `data(['UZ',4])` gives all user defined solutions and label 4. A for loop can then iterate through

```
# get and run the ab demo
demo('ab')
auto('ab.auto')
# load data from b.ab, s.ab, and d.ab
ab = loadbd('ab')
# change the branch number to either 1 or 2 depending on IPS
for branch in ab:
    branch['BR'] = branch.c['IPS']
# delete the last branch
del ab[-1]
# subtract the first branch from all other branches with respect to PAR(1)
ab = subtract(ab, ab, 'PAR(1)')
# plot the branches, coloring by branch number
plot(ab, coloring_method='branch', color_list='black red')
wait()
# save data to b.abnew, s.abnew, and d.abnew
save(ab, 'abnew')
```

Figure 4.18: This example shows how to change branch numbers and delete a branch in the 'ab' demo output file. Using the `coloring_method='branch'` setting you can then give one color to all branches of fixed points and one other color to all branches of periodic orbits. The source for this script can be found in $AUTO_DIR/demos/python/branches.auto.

```
AUTO> s=data()
AUTO> sol=s[3] #or s(4), data(4), s('HB1'), or data('HB1')
AUTO> print sol
  BR    PT  TY  LAB ISW NTST NCOL NDIM IPS IPRIV
   2    42  HB    4   1    1    0    3   1     0
Pointset lrz (parameterized)
Independent variable:
t:  [ 0.]
Coordinates:
U(1):  [ 7.95601972]
U(2):  [ 7.95601972]
U(3):  [ 23.73684367]
Labels by index: Empty
Active ICP: [1]
rldot: [0.69738311435]
udotps: Pointset lrz (non-parameterized)
Coordinates:
UDOT(1):  [ 0.11686228]
UDOT(2):  [ 0.11686228]
UDOT(3):  [ 0.69738311]
Labels by index: Empty
PAR(1:5):      2.4736843668E+01   2.6666666667E+00   1.0000000000E+01   0   0
PAR(6:10):     0.0000000000E+00   0.0000000000E+00   0.0000000000E+00   0   0
PAR(11:11):    6.5283032822E-01
AUTO> print sol['LAB']
4
AUTO> print sol['L2-NORM'] # or sol.b['L2-NORM']
26.268502943
AUTO> sol[0]
{'u': [7.9560197197999996, 7.9560197197999996, 23.736843667999999],
  't': 0.0, 'u dot': [0.11686227712, 0.11686227712, 0.69738311435]}
AUTO> sol['t']
array([ 0.])
AUTO> sol['U(1)']
array([ 7.95601972])
AUTO> sol.PAR(1) # or sol['PAR(1)']
24.736843667999999
AUTO> pylab.plot(sol['U(1)'],sol['U(2)'],'+')
[<matplotlib.lines.Line2D object at 0x9c3348c>]
AUTO> pylab.show()
```

Figure 4.19: This figure shows an example of parsing solutions. The first command, `s=data()`, extracts a list of all solutions stored in the bifurcation diagram object `data` from the 'lrz' demo in Figure 4.16 and puts it into the variable `s`. The command `sol=s[3]` obtains the fourth solution. Next, `print sol` displays it. The last commands illustrate how to extract components from this fourth solution: its label, its first point, its time array, its first coordinate array, its first parameter, and a plot of the first two coordinates.

Query string	Meaning
data	An array which contains the AUTO output. Each array entry is a Python dictionary with a scalar entry 't' denoting time, and sub-arrays for the solution vector 'u' and the solution direction vector 'u dot'. Other syntax: `s[0]`, `s['t']`, `s['U(1)']`.
BR	The number of the branch to which the solution belongs.
IPRIV	A private field for use by toolboxes.
IPS	The user-specified problem type. See Section 10.8.12.
ISW	The ISW value used to start the calculation. See Section 10.8.4.
LAB	The label of the solution.
NCOL	The number of collocation points used to compute the solution. See Section 10.3.2.
NDIM	The user-specified number of dimensions. See Section 10.2.1.
NTST	The number of mesh intervals used to compute the solution. See Section 10.3.1.
Parameters parameters p	The value of all of the parameters for the solution. 'p' and 'parameters' are aliases. Other syntax: `s.PAR(1)`, `s['PAR(1)']`
PT	The number of the point in the given branch.
TY	A short string which describes the type of the solution (see Table 4.5).
TY number	A number which describes the type of the solution (see Table 4.5).
Active ICP	The values of the (one-based) indices of the free parameters.
rldot	The values of the parameter direction vector.

Table 4.7: This table shows the strings that can be used to query a solution object and their meanings.

the list and provide new starting solutions.

Solutions can be deleted using the commands `dlb`, `dsp`, `klb`, and `ksp`, described in the reference, Section 4.14.5. The command `relabel`, as described before, gives each solution a unique label starting at 1. Finally, you can change individual solution labels using the `relabel` *method*: `data.relabel(9, 57)` relabels label 9 to label 57, making sure that the label changes both in the bifurcation diagram and in the solution.

4.7.2 Summary and reference

We have defined the following objects:

Bifurcation diagram object : `bd=run(...)`, `bd=loadbd(...)`.

Branch : `bd[0], bd[1], bd[2], ....`

Branch AUTO constants : `bd[0].c, bd[1].c, bd[2].c, ....`
`bd.c` refers to `bd[0].c`.

Branch column : `bd['PAR(1')], bd[0]['PAR(1)'], bd[1]['L2-NORM'], ....`
Here `bd['PAR(1)']` is a shortcut for `bd[0]['PAR(1)']`.

Point : `bd[0][0], bd[1](5), bd[1]('UZ1'), bd(1).b, bd('UZ1').b, ....`

Point list : `bd[0]('UZ'), bd[0]([1,2]), bd[0](['UZ','HB1',7]), ....`

Solution : `bd(5), bd('UZ1'), bd()[0], ....`

Solution list : `s=bd(), s('UZ'), bd('UZ'), bd([1,2]), bd(['UZ','HB1',7]), ....`

Solution column : `bd('UZ1')['t'], bd('UZ1')['U(1)'], ....`

Solution measures : `bd('UZ1')['L2-NORM'], ....`
This example is a shortcut for `bd('UZ1').b['L2-NORM']`: here you can use any column name from the bifurcation diagram.

Solution point : `bd('UZ1')(0), bd('UZ1')[0], ....`
Here the `bd('UZ1')(t)` notation gives the point at time `t`.

Solution AUTO constants : `bd(5).c, bd('UZ1').c, bd()[0].c, ....`
These constants are copied from the corresponding branch constants, removing the constants `IRS`, `PAR`, `U`, `sv`, `s`, and `dat`, because those constants need to change between runs. The AUTO constant `IRS` is automatically set to the solution label.

4.8 Importing data from Python or external tools.

A solution can be created from a Python list or Numerical Python array using the `load` command. The syntax is slightly different depending on whether you create a point or an orbit.

Use `s = load(u,PAR=p)` where `u` is a Python list or a numpy array representing a solution. For a point such an array is just the point itself, for example, `u = [x, y, z]` or `u = [0, 0, 0]`. For an orbit such an array must be given column-wise, as `u = [[t0, ..., tn], [x0, ..., xn], [y0, ..., yn], ...]`. The `PAR=p` keyword argument takes a dictionary `p` as described in Section 10.8.9, for instance, `PAR={1:5.0, 2:0.0}` to set `PAR(1)=5.0` and `PAR(2)=0.0`.

You can check your new solution using the command `print s`.

An external orbit from an ASCII file can be directly imported by AUTO via the `dat` AUTO constant, see Section 10.8.7.

4.9 Exporting output data for use by Python or external visualization tools.

The bifurcation and solution file classes have three methods that are particularily useful for creating data which can be used in other programs. First, there is a method called `toArray` which takes a bifurcation diagram or solution and returns a Python array (a list of lists). Second, the method `toarray` returns a Numerical Python (numpy) array, which works for branches, points and solutions (but not for lists of branches). Third, there is a method called `writeRawFilename` which will create a standard ASCII file which contains the bifurcation diagram or the solution. In the solution ASCII file, the first element of each row will be the 't' value and the following elements will be the values of the components at that 't' value. Such ASCII files can be readily parsed and plotted by external tools such as Gnuplot and MATLAB.

For example, we assume that a bifurcation diagram object is contained in a variable `bd`, for instance, using `bd=loadbd('ab')`. If one wanted to have the bifurcation diagram returned as a Python list one would type `bd.toArray()`. Similarily, if one wanted to write out the bifurcation diagram to the file `outputfile` one would type `bd.writeRawFilename('outputfile')`.

To get the solution with label `57` returned as a numpy array one would type `bd(57).toarray()`. Similarily, if one wanted to write out the solution to the file `outputfile` one would type `bd(57).writeRawFilename('outputfile')`.

4.10 The .autorc or autorc File

Much of the default behavior of the AUTO CLUI can be controlled by the .autorc file. The .autorc file can exist in either the main AUTO directory, the users home directory, or the current directory. In the current directory it can also have the name autorc, that is, without the dot. For any option which is defined in more then one file, the .autorc file in the current directory (if it exists) takes precedence, followed by the .autorc file in the users home directory (if it exists), and then the .autorc file in the main AUTO directory. Hence, options may be defined on either a per directory, per user, or global basis.

The first section of the .autorc file begins with the line [AUTO_command_aliases] and this section defines short names, or aliases, for the AUTO CLUI commands. Each line thereafter is a definition of a command, similiar to branchPoint =commandQueryBranchPoint. The right hand side of the assignment is the internal AUTO CLUI name for the command, while the left hand side is the desired alias. Aliases and internal names may be used interchangably, but the intention is that the aliases will be more commonly used. A default set of aliases is provided, and these aliases will be used in the examples in the rest of this Chapter. The default aliases are listed in the reference in Section 4.14.

NOTE: Defaults for the plotting tool may be included in the .autorc file as well.

4.11 Plotting Tool

The plotting tool can be run by using the command plot(bd) to plot a bifurcation diagram object bd after a calculation has been run, or using the command plot() to plot the files fort.7 and fort.8, or using the command plot('foo') to plote the data in the files s.foo and b.foo.

The menu bar provides two buttons. The File button brings up a menu which allows the user to save the current plot as a Postscript file or to quit the plotting tool. The Options button allows the plotter configuration options to be modified. The available options are decribed in Table 4.8. In addition, the options can be set and figures can be saved from within the CLUI. For example, the set of commands in Figure 4.20 shows how to create a plot, change its background color to black, and save it. The demo script auto/07p/demo/python/plotter.py contains several examples of changing options in plotters. The special argument hide=True to plot does not produce an on-screen plot, which is useful for quick automatic generation of saved figure files.

If you are using matplotlib, then you will find seven icons that allow you to use various zoom functions: a home button to go back to the original plot, back and forward buttons to go back and forwards between zooms, a button to select pan/zoom mode, a button to select rectangular zoom mode, a button to which brings up sliders that adjust margins, and a floppy disk button that you can use to save the plot to a file. In "zoom to rect" mode, the left mouse button may be held down to create a box in the plot. When the left button is released the plot will zoom to the selected portion of the diagram. Similarly, the right mouse button can be used to zoom out. In "pan/zoom" mode, dragging with the left mouse button pressed pans (shifts) the graph, whereas dragging with the right mouse button zooms in and out.

If you are *not* using matplotlib, then pressing the right mouse button in the plotting window brings up a menu of buttons which control several aspects of the plotting window. The top two toggle buttons control what function the left button performs. The print value button causes the left button to print out the numerical value underneath the pointer when it is clicked. When zoom button is checked the left mouse button may be held down to create a box in the plot. When the left button is released the plot will zoom to the selected portion of the diagram. The unzoom button returns the diagram to the default zoom. The Postscript button allows the user to save the plot as a Postscript file. The Configure... button brings up the dialog for setting configuration options.

Query string	Meaning
1_1_resonance_symbol	The symbol to use for 1:1 resonance points.

1_2_resonance_symbol	The symbol to use for 1:2 resonance points.
1_3_resonance_symbol	The symbol to use for 1:3 resonance points.
1_4_resonance_symbol	The symbol to use for 1:4 resonance points.
azimuth	Azimuth of the axes in 3D plots.
background	The background color of the plot.
bifurcation_column_defaults	A set of bifurcation columns the user is likely to use.
bifurcation_coordnames	Names to use instead of PAR(1),... for bifurcation diagrams.
bifurcation_diagram	A parsed bifurcation diagram file to plot.
bifurcation_diagram_filename	The filename of the bifurcation diagram to plot.
bifurcation_symbol	The symbol to use for bifurcation points.
bifurcation_x	The column to plot along the X-axis for bifurcation diagrams.
bifurcation_y	The column to plot along the Y-axis for bifurcation diagrams.
bifurcation_z	The column to plot along the Z-axis for bifurcation diagrams.
bogdanov_takens_symbol	The symbol to use for Bogdanov-Takens points.
bottom_margin	The margin between the graph and the bottom edge.
color_list	A list of colors to use for multiple plots.
coloring_method	color_list index: 'branch' (BR), 'type' (TY), or 'curve' (seq.).
cusp_symbol	The symbol to use for Cusp points.
d0, d1, d2, d3, d4	Redefine d0, d1, d2, etc. setting to use with PyPLAUT (`@pp`).
dashes	List of dash, no-dash lengths for dashed lines.
decorations	Turn on or off the axis, tick marks, etc.
default_option	Default d0, d1, d2, etc. setting to use with PyPLAUT (`@pp`).
elevation	Elevation of the axes in 3D plots.
error_symbol	The symbol to use for error points.
even_tick_length	The length of the even tick marks.
flip_torus_symbol	The symbol to use for flip-torus points.
fold_flip_symbol	The symbol to use for fold-flip points.
fold_torus_symbol	The symbol to use for fold-torus points.
foreground	The background color of the plot.
generalized_hopf_symbol	The symbol to use for Generalized Hopf points.
grid	Turn on or off the grid.
height	Height of the graph.
hopf_symbol	The symbol to use for Hopf bifurcation points.
index	An array of indices to plot.
label	An array of labels to plot, or 'all' for all labels.
labelnames	A dictionary mapping names in fort.7 to axis labels.
label_defaults	A set of labels that the user is likely to use.
left_margin	The margin between the graph and the left edge.
letter_symbols	Whether to use letter (True) or symbols (False) for special points.
limit_point_symbol	The symbol to use for limit points.
line_width	Width to use for lines and curves.
mark_t	The t value to marker with a small ball.

maxx	The upper bound for the x-axis of the plot.
maxy	The upper bound for the y-axis of the plot.
maxz	The upper bound for the z-axis of the plot.
minx	The lower bound for the x-axis of the plot.
miny	The lower bound for the y-axis of the plot.
minz	The lower bound for the z-axis of the plot.
odd_tick_length	The length of the odd tick marks.
period_doubling_symbol	The symbol to use for period doubling bifurcation points.
ps_colormode	The PostScript output mode: 'color', 'gray' or 'monochrome'.
right_margin	The margin between the graph and the left edge.
runner	The runner object from which to get data.
smart_label	Whether to use a smart but slower label placement algorithm.
special_point_colors	An array of colors used to mark special points.
special_point_radius	The radius of the spheres used to mark special points.
solution	A parsed solution file to plot.
solution_column_defaults	A set of solution columns the user is likely to use.
solution_coordnames	Variable names to use instead of U(1),... for solutions.
solution_filename	The filename of the solution to plot.
solution_indepvarname	Variable name to use instead of 't' for solutions.
solution_x	The column to plot along the X-axis for solutions.
solution_y	The column to plot along the Y-axis for solutions.
solution_z	The column to plot along the Z-axis for solutions.
stability	Turn on or off stability information using dashed curves.
symbol_font	The font to use for marker symbols.
symbol_color	The color to use for the marker symbols.
tick_label_template	A string which defines the format of the tick labels.
tick_length	The length of the tick marks.
top_margin	The margin between the graph and the top edge.
top_title	The label for the top title.
top_title_fontsize	The font size for the top title.
torus_symbol	The symbol to use for torus bifurcation points.
torus_torus_symbol	The symbol to use for torus-torus points.
type	The type of the plot, either "solution" or "bifurcation".
use_labels	Whether or not to display label numbers in the graph.
use_symbols	Whether or not to display bifurcation symbols in the graph.
user_point_symbol	The symbol to use for user defined output points.
width	Width of the graph.
xlabel	The label for the x-axis.
xlabel_fontsize	The font size for the x-axis label.
xticks	The number of ticks on the x-axis.
ylabel	The label for the y-axis.
ylabel_fontsize	The font size for the y-axis label.

yticks	The number of ticks on the y-axis.
zero_hopf_symbol	The symbol to use for zero-Hopf points.
zlabel	The label for the z-axis.
zlabel_fontsize	The font size for the z-axis label.
zticks	The number of ticks on the z-axis.

Table 4.8: This table shows the options that can be set for the PyPLAUT plotting window and their meanings.

4.12 The Plotting Tool PLAUT04

The AUTO plotting tool PLAUT04 as described in Chapter 8, can be run from the Python CLUI using the command `plot3` or `commandPlotter3D`, in a similar fashion to `plot`. It does not use the options that are used for the plotting window produced by `plot`.

4.13 Quick Reference

In this section we have created a table of all of the AUTO CLUI commands, their abbreviations, and a one line description of what function they perform. Each command may be entered using its full name or any of its aliases.

Command, Aliases	Long name	Description
append ap	commandAppend	Append data files.
cat	commandCat	Print the contents of a file
cd	commandCd	Change directories.
clean cl	commandClean	Clean the current directory.
demo dm	commandCopyAndLoadDemo	Copy a demo into the current directory and load it.
copy cp	commandCopyDataFiles	Copy data files.
copydemo	commandCopyDemo	Copy a demo into the current directory.
save sv	commandCopyFortFiles	Save data files.
gui	commandCreateGUI	Show AUTOs graphical user interface.
delete, dl	commandDeleteDataFiles	Delete data files.
df deletefort	commandDeleteFortFiles	Clear the current directory of fort files.
dlb	commandDeleteLabels	Delete special labels.
dsp	commandDeleteSpecialPoints	Delete special points.
double	commandDouble	Double a solution.
man	commandInteractiveHelp	Get help on the AUTO commands.

klb	commandKeepLabels	Keep special labels.
ksp	commandKeepSpecialPoints	Keep special points.
ls	commandLs	List the current directory.
merge mb	commandMergeBranches	Merge branches in data files.
move mv	commandMoveFiles	Move data-files to a new name.
cn constantsget	commandParseConstantsFile	Get the current continuation constants.
bt diagramandsolutionget	commandParseDiagramAndSolutionFile	Parse both bifurcation diagram and solution.
dg diagramget	commandParseDiagramFile	Parse a bifurcation diagram.
sl solutionget	commandParseSolutionFile	Parse solution file:
plot p2 pl	commandPlotter	plotting of data.
plot3 p3	commandPlotter3D	PLAUT04 plotting of data.
branchpoint br bp	commandQueryBranchPoint	Print the "branch-point function".
eigenvalue ev eg	commandQueryEigenvalue	Print eigenvalues of Jacobian (algebraic case).
floquet fl	commandQueryFloquet	Print the Floquet multipliers.
hopf hb hp	commandQueryHopf	Print the value of the "Hopf function".
iterations it	commandQueryIterations	Print the number of Newton interations.
limitpoint lm lp	commandQueryLimitpoint	Print the value of the "limit point function".
note nt	commandQueryNote	Print notes in info file.
secondaryperiod sc sp	commandQuerySecondaryPeriod	Print value of "secondary-periodic bif. fcn".
stepsize ss st	commandQueryStepsize	Print continuation step sizes.
quit q	commandQuit	Quit the AUTO CLUI.
relabel rl	commandRelabel	Relabel data files.
run r rn	commandRun	Run AUTO.
ch changeconstant cc	commandRunnerConfigFort2	Modify continuation constants.
hch	commandRunnerConfigFort12	Modify HomCont continuation constants.
load ld	commandRunnerLoadName	Load files into the AUTO runner.
printconstant pc pr	commandRunnerPrintFort2	Print continuation parameters.
hpr	commandRunnerPrintFort12	Print HomCont continuation parameters.
shell	commandShell	Run a shell command.
splabs	commandSpecialPointLabels	Return special labels.
subtract sb	commandSubtractBranches	Subtract branches in data files.

triple tr	commandTriple	Triple a solution.
us userdata	commandUserData	Covert user-supplied data files.
wait	commandWait	Wait for the user to enter a key.
auto execfile ex		Execute an AUTO CLUI script.
demofile dmf		Execute an AUTO CLUI script, line by line (demo mode).

For convenience, you can use "!" to run a shell command. Moreover the common shell commands `clear`, `less`, `mkdir`, `rmdir`, `cp`, `mv`, `rm`, `ls`, `cd`, and `cat`, and all AUTO Unix commands that are described in Chapter 5 and start with an "@"-sign can be entered directly, without the "!".

```
AUTO> p=plot()
Created plot
AUTO> p.config(bg="black")
AUTO> p.savefig("black.eps")
AUTO> p=plot(hide=True)
Created plot
AUTO> p.savefig("white.eps")
```

Figure 4.20: This example shows how a plotter is created, how the background color may be changed to black, how a figure is saved, and how an invisible plot is saved to a file. All other configuration options are set similarily. Note, the above commands assume that the files fort.7 and fort.8 exist in the current directory.

4.14 Reference

4.14.1 Basic commands.

run Run AUTO.

Type `r=run([data],[options])` to run AUTO from solution data with the given AUTO constants or file keyword options.

The results are stored in the bifurcation diagram r which you can later print with `print r`, obtain branches from via r[0], r[1], ..., and obtain solutions from via r(3), r(5), r('LP2'), where 3 and 5 are label numbers, and 'LP2' refers to the second LP label.

`run(data)` runs AUTO in the following way for different types of `data`:

- A solution: AUTO starts from solution `data`, with AUTO constants `data.c`.
- A bifurcation diagram: AUTO start from the solution specified by the AUTO constant `IRS`, or if `IRS` is not specified, the last solution in `data`, `data()[-1]`, with AUTO constants `data()[-1].c`.
- A string: AUTO uses the solution in the file s.data together with the constants in the files c.data, and h.data. Not all of these files need to be present.

If no solution `data` is specified, then the global values from the `load` command (below) are used instead, where options which are not explicitly set retain their previous value.

Keyword argument options can be AUTO constants, such as `DS=0.05`, or `ISW=-1`, or specify a constant or solution file. These override the constants in s.c, where applicable. See `load`:
`run(s,options)` is equivalent to `run(load(s,options))`

Example: given a bifurcation diagram bd, with a branch point solution, switch branches and stop at the first Hopf bifurcation:
`hb = run(bd('BP1'),ISW=-1,STOP='HB1')`

Special keyword arguments are `sv` and `ap`; `sv` is also an AUTO constant:
`run(bd('BP1'),ISW=-1,STOP='HB1',sv='hb',ap='all')`
saves to the files b.hb, s.hb and d.hb, and appends to b.all, s.all, and d.all.

Aliases: r rn commandRun

load Load files into the AUTO runner or return modified solution data.

Type `result=load([options])` to modify the AUTO runner.
Type `result=load(data,[options])` to return possibly modified solution data.

The type of the result is a solution object.

`load(data,[options])` returns a solution in the following way for different types of data:

- A solution: load returns the solution data, with AUTO constants modified by options.
- A bifurcation diagram or a solution list: returns the solution specified by the AUTO constant `IRS`, or if `IRS` is not specified, the last solution in `s`.
- A string: AUTO uses the solution in the file 's.s' together with the constants in the files 'c.s', and 'h.s'. Not all of these files need to be present.
- A Python list array or a numpy array representing a solution, returns a solution with the given contents. Such an array must be given column-wise, as [[t0, ..., tn], [x0, ..., xn], [y0, ..., yn], ...], or for a point solution as [x, y, z, ...].

There are many possible options:

```
Long name   Short name    Description
-----------------------------------------
equation    e             The equations file
constants   c             The AUTO constants file
homcont     h             The Homcont parameter file
solution    s             The restart solution file
            NDIM,IPS,etc  AUTO constants.
            BR,PT,TY,LAB  Solution constants.
```

If data is not specified or data is a string then options which are not explicitly set retain their previous value. For example one may type: `s=load(e='ab',c='ab.1')` to use ab.f90, ab.f, or ab.c as the equations file and c.ab.1 as the constants file.

Type `s=load('name')` to load all files with base 'name'. This does the same thing as running `s=load(e='name',c='name,h='name',s='name')`.

You can also specify AUTO Constants, e.g., `DS=0.05`, or `IRS=2`. Special values for `DS` are `'+'` (forwards) and `'-'` (backwards).
Example: `s = load(s,DS='-')` changes `s.c['DS']` to `-s.c['DS']`.

Aliases: ld commandRunnerLoadName

loadbd Load bifurcation diagram files.

Type `b=loadbd([options])` to load output files or output data. There are three possible options:

```
Long name   Short name    Description
-----------------------------------------
bifurcationdiagram   b    The bifurcation diagram file
solution    s            The solution file or list of solutions
diagnostics d            The diagnostics file
```

Type `loadbd('name')` to load all files with base 'name'. This does the same thing as running
`loadbd(b='name',s='name,d='name')`.
`plot(b)` will then plot the 'b' and 's' components.

Returns a bifurcation diagram object representing the files in b.

Aliases: bd commandParseOutputFiles

save Save data files.

Type `save(x,'xxx')` to save bifurcation diagram `x` to the files b.xxx, s.xxx, d.xxx. Existing files with these names will be overwritten. If `x` is a solution, a list of solutions, or does not contain any bifurcation diagram or diagnostics data, then only the file s.xxx is saved to.

Type `save('xxx')` to save the output-files fort.7, fort.8, fort.9, to b.xxx, s.xxx, d.xxx. Existing files with these names will be overwritten.

Aliases: commandCopyFortFiles

append Append data files.

Type `append(x,'xxx')` to append bifurcation diagram x to the data-files b.xxx, s.xxx, and d.xxx. This is equivalent to the command
`save(x+load('xxx'),'xxx')`

Type `append('xxx',xxx)` to append existing data-files s.xxx, b.xxx, and d.xxx to bifurcation diagram x. This is equivalent to the command
`x=load('xxx')+x`

Type `append('xxx')` to append the output-files fort.7, fort.8, fort.9, to existing data-files s.xxx, b.xxx, and d.xxx.

Type `append('xxx','yyy')` to append existing data-files s.xxx, b.xxx, and d.xxx to data-files s.yyy, b.yyy, and d.yyy.

Aliases: ap commandAppend

4.14.2 Plotting commands.

plot Plotting of data.

Type `plot(x)` to run the graphics program PyPLAUT for the graphical inspection of bifurcation diagram or solution data in x.

Type `plot('xxx')` to run the graphics program PyPLAUT for the graphical inspection of the data-files b.xxx and s.xxx.

Type `plot()` to run the graphics program for the graphical inspection of the output-files fort.7 and fort.8.

Values also present in the file autorc, such as `color_list = "black green red blue orange"` can be provided as keyword arguments, as well as `hide=True` which hides the on-screen plot.

The return value, for instance, p for `p=plot(x)` will be the handle for the graphics window. It has p.config() and p.savefig() methods that allow you to configure and save the plot. When plotting, see `help(p.config)` and `help(p.savefig)` for details.

Aliases: p2 pl commandPlotter

plot3 Plotting of data using PLAUT04.

Type `plot3(x)` to run the graphics program PLAUT04 for the graphical inspection of bifurcation diagram or solution data in x.

Type `plot3('xxx')` to run the graphics program PLAUT04 for the graphical inspection of the data-files b.xxx and s.xxx.

Type `plot3()` to run the graphics program PLAUT04 for the graphical inspection of the output-files fort.7 and fort.8.

Type `plot3(...,r3b=True)` to run PLAUT04 in restricted three body problem mode.

Aliases: p3 commandPlotter3D

4.14.3 File-manipulation.

copy Copy data files.

Type `copy(name1, name2)`, or `copy(name1, name2, name3)`, or `copy(name1, name2, name3, name4)` to copy the data-files dir1/c.xxx, dir1/b.xxx, dir1/s.xxx, and dir1/d.xxx, to dir2/c.yyy, dir2/b.yyy, dir2/s.yyy, and dir2/d.yyy.

Type `copy('xxx','yyy')` to copy the data-files c.xxx, d.xxx, b.xxx, and h.xxx to c.yyy, d.yyy, b.yyy, and h.yyy.

The values of dir1/?.xxx and dir2/?.yyy are as follows, depending on whether name1 is a directory or name2 is a directory:

`copy(name1, name2)`
no directory names: ./?.name1 and ./?.name2
name1 is a directory: name1/?.name2 and ./?.name2
name2 is a directory: ./?.name1 and name2/?.name1

`copy(name1, name2, name3)`
name1 is a directory: name1/?.name2 and ./?.name3
name2 is a directory: ./?.name1 and name2/?.name3

`copy(name1, name2, name3, name4)`
name1/?.name2 and name3/?.name4

Aliases: cp commandCopyDataFiles

move Move data-files to a new name.

Type `move(name1, name2)`, or `move(name1, name2, name3)`, or `move(name1, name2, name3, name4)` to move the data-files dir1/b.xxx, dir1/s.xxx, and dir1/d.xxx, to dir2/b.yyy, dir2/s.yyy, and dir2/d.yyy, and copy the constants file dir1/c.xxx to dir2/c.yyy.

The values of dir1/?.xxx and dir2/?.yyy are determined in the same way as for `copy` above.

Aliases: mv commandMoveFiles

df Clear the current directory of fort files.

Type `df()` to clean the current directory. This command will delete all files of the form fort.*.

Aliases: deletefort commandDeleteFortFiles

clean Clean the current directory.

Type `clean()` to clean the current directory. This command will delete all files of the form fort.*, *.*~, *.o, and *.exe.

Aliases: cl commandClean

delete Delete data files.

Type `delete('xxx')` to delete the data-files d.xxx, b.xxx, and s.xxx (if you are using the default filename templates).

Aliases: dl commandDeleteDataFiles

4.14.4 Diagnostics.

limitpoint Print the value of the "limit point function".

Type `limitpoint(x)` to list the value of the "limit point function" in the diagnostics of the bifurcation diagram object `x`. This function vanishes at a limit point (fold).

Type `limitpoint()` to list the value of the "limit point function" in the output-file fort.9.

Type `limitpoint('xxx')` to list the value of the "limit point function" in the info file d.xxx.

Aliases: lm lp commandQueryLimitpoint

branchpoint Print the "branch-point function".

Type `branchpoint(x)` to list the value of the "branch-point function" in the diagnostics of the bifurcation diagram object `x`. This function vanishes at a branch point.

Type `branchpoint()` to list the value of the "branch-point function" in the output-file fort.9.

Type `branchpoint('xxx')` to list the value of the "branch-point function" in the info file d.xxx.

Aliases: br bp commandQueryBranchPoint

hopf Print the value of the "Hopf function".

Type `hopf(x)` to list the value of the "Hopf function" in the diagnostics of the bifurcation diagram object `x`. This function vanishes at a Hopf bifurcation point.

Type `hopf()` to list the value of the "Hopf function" in the output-file fort.9.

Type `hopf('xxx')` to list the value of the "Hopf function" in the info file d.xxx.

Aliases: hb hp commandQueryHopf

secondaryperiod Print value of "secondary-periodic bif. fcn".

Type `secondaryperiod(x)` to list the value of the "secondary-periodic bifurcation function" in the diagnostics of the bifurcation diagram object `x`. This function vanishes at period-doubling and torus bifurcations.

Type `secondaryperiod()` to list the value of the "secondary-periodic bifurcation function" in the output-file fort.9.

Type `secondaryperiod('xxx')` to list the value of the "secondary-periodic bifurcation function" in the info file d.xxx.

Aliases: sc sp commandQuerySecondaryPeriod

iterations Print the number of Newton interations.

Type `iterations(x)` to list the number of Newton iterations per continuation step in the diagnostics of the bifurcation diagram object `x`.

Type `iterations()` to list the number of Newton iterations per continuation step in fort.9.

Type `iterations('xxx')` to list the number of Newton iterations per continuation step in the info file d.xxx.

Aliases: it commandQueryIterations

note Print notes in info file.

Type `note(x)` to show any notes in the diagnostics of the bifurcation diagram object `x`.

Type `note()` to show any notes in the output-file fort.9.

Type `note('xxx')` to show any notes in the info file d.xxx.

Aliases: nt commandQueryNote

stepsize Print continuation step sizes.

Type `stepsize(x)` to list the continuation step size for each continuation step in the diagnostics of the bifurcation diagram object `x`.

Type `stepsize()` to list the continuation step size for each continuation step in fort.9.

Type `stepsize('xxx')` to list the continuation step size for each continuation step in the info file d.xxx.

Aliases: ss st commandQueryStepsize

eigenvalue Print eigenvalues of Jacobian (algebraic case).

Type `eigenvalue(x)` to list the eigenvalues of the Jacobian in the diagnostics of the bifurcation diagram object `x`. (Algebraic problems.)

Type `eigenvalue()` to list the eigenvalues of the Jacobian in fort.9.

Type `eigenvalue('xxx')` to list the eigenvalues of the Jacobian in the info file d.xxx.

Aliases: ev eg commandQueryEigenvalue

floquet Print the Floquet multipliers.

Type `floquet(x)` to list the Floquet multipliers in the diagnostics of the bifurcation diagram object `x`. (Differential equations.)

Type `floquet()` to list the in the output-file fort.9.

Type `floquet('xxx')` to list the Floquet multipliers in the info file d.xxx.

Aliases: fl commandQueryFloquet

4.14.5 File-maintenance.

relabel Relabel data files.

Type `y=relabel(x)` to return the python object x, with the solution labels sequentially relabelled starting at 1, as a new object y.

Type `relabel('xxx')` to relabel s.xxx and b.xxx. Backups of the original files are saved.

Type `relabel('xxx','yyy')` to relabel the existing data-files s.xxx and b.xxx and save then to s.yyy and b.yyy; d.xxx is copied to d.yyy.

Aliases: rl commandRelabel

double Double a solution.

Type `double()` to double the solution in fort.8.

Type `double('xxx')` to double the solution in s.xxx.

Aliases: db commandDouble

triple Triple a solution.

Type `triple()` to triple the solution in fort.8.

Type `triple('xxx')` to triple the solution in s.xxx.

Aliases: tr commandTriple

us Convert user-supplied data files.

Type `us('xxx')` to convert a user-supplied data file 'xxx.dat' to AUTO format. The converted file is called 's.dat'. The original file is left unchanged. AUTO automatically sets the period in `PAR(11)`. Other parameter values must be set in 'STPNT'. (When necessary, `PAR(11)` may also be redefined there.) The constants-file file 'c.xxx' must be present, as the AUTO-constants 'NTST' and 'NCOL' are used to define the new mesh.

Note: this technique has been obsoleted by the 'dat' AUTO constant in Section 10.8.7.

Aliases: userdata commandUserData

dlb Delete special labels.

Type `dlb(x,list)` to delete the special points in list from the Python object x, which must be a solution list or a bifurcation diagram.

Type `dlb(list,'xxx')` to delete from the data-files b.xxx and s.xxx.
Type `dlb(list,'xxx','yyy')` to save to b.yyy and s.yyy instead of ?.xxx.
Type `dlb(list)` to delete from fort.7 and fort.8.

`list` is a label number or type name code, or a list of those, such as 1, or [2, 3], or 'UZ' or ['BP', 'LP'], or it can be None or omitted to mean the special points ['BP', 'LP', 'HB', 'PD', 'TR', 'EP', 'MX']

Alternatively a boolean user-defined function `f` that takes a solution can be specified for `list`, such as

```
def f(s):
    return s["PAR(9)"]<0
```

where all solutions are deleted that satisfy the given condition, or

```
def f(s1,s2):
    return abs(s1["L2-NORM"] - s2["L2-NORM"]) < 1e-4
```

where all solutions are compared with each other and `s2` is deleted if the given condition is satisfied, which causes pruning of solutions that are close to each other.

Type information is kept in the bifurcation diagram for plotting.

Alias: commandDeleteLabels

klb Keep special labels.

Type `klb(x,list)` to only keep the special points in list in the Python object x, which must be a solution list or a bifurcation diagram.

Type `klb(list,'xxx')` to keep them in the data-files b.xxx and s.xxx.
Type `klb(list,'xxx','yyy')` to save to b.yyy and s.yyy instead of ?.xxx.
Type `klb(list)` to keep them in fort.7 and fort.8.

`list` is a label number or type name code, or a list of those, such as 1, or [2, 3], or 'UZ' or ['BP', 'LP'], or it can be None or omitted to mean ['BP', 'LP', 'HB', 'PD', 'TR', 'EP', 'MX'], deleting 'UZ' and regular points.

Alternatively a boolean user-defined function `f` that takes a solution can be specified for `list`, such as

```
def f(s):
    return s["PAR(9)"]<0
```

where only solutions are kept that satisfy the given condition.

Type information is kept in the bifurcation diagram for plotting.

Alias: commandKeepLabels

dsp Delete special points.

Type `dsp(x,list)` to delete the special points in list from the Python object x, which must be a solution list or a bifurcation diagram.

Type `dsp(list,'xxx')` to delete from the data-files b.xxx, and s.xxx.
Type `dsp(list,'xxx','yyy')` to save to b.yyy and s.yyy instead of ?.xxx.
Type `dsp(list)` to delete from fort.7 and fort.8.

`list` is a label number or type name code, or a list of those, such as 1, or [2, 3], or 'UZ' or ['BP', 'LP'], or it can be None or omitted to mean the special points ['BP', 'LP', 'HB', 'PD', 'TR', 'EP', 'MX']
Alternatively a boolean user-defined function `f` that takes a solution can be specified for `list`, such as

```
def f(s):
    return s["PAR(9)"]<0
```

where all solutions are deleted that satisfy the given condition, or

```
def f(s1,s2):
    return abs(s1["L2-NORM"] - s2["L2-NORM"]) < 1e-4
```

where all solutions are compared with each other and `s2` is deleted if the given condition is satisfied, which causes pruning of solutions that are close to each other.

Type information is *not* kept in the bifurcation diagram.

Alias: commandDeleteSpecialPoints

ksp Keep special points.

Type `ksp(x,list)` to only keep the special points in list in the Python object x, which must be a solution list or a bifurcation diagram.

Type `ksp(list,'xxx')` to keep them in the data-files b.xxx, and s.xxx.
Type `ksp(list,'xxx','yyy')` to save to b.yyy and s.yyy instead of ?.xxx.
Type `ksp(list)` to keep them in fort.7 and fort.8.

`list` is a label number or type name code, or a list of those, such as 1, or [2, 3], or 'UZ' or ['BP', 'LP'], or it can be None or omitted to mean ['BP', 'LP', 'HB', 'PD', 'TR', 'EP', 'MX'], deleting 'UZ' and regular points.
Alternatively a boolean user-defined function `f` that takes a solution can be specified for `list`, such as

```
def f(s):
    return s["PAR(9)"]<0
```

where only solutions are kept that satisfy the given condition.

Type information is *not* kept in the bifurcation diagram.

Alias: commandKeepSpecialPoints

merge Merge branches in data files.

Type `y=merge(x)` to return the python object x, with its branches merged into continuous curves, as a new object y.

Type `merge('xxx')` to merge branches in s.xxx, b.xxx, and d.xxx. Backups of the original files are saved.

Type `merge('xxx','yyy')` to merge branches in the existing data-files s.xxx, b.xxx, and d.xxx and save them to s.yyy, b.yyy, and d.yyy.

Aliases: mb commandMergeBranches

subtract Subtract branches in data files.

Type `z=subtract(x,y,ref)` to return the python object x, where, using interpolation, the first branch in y is subtracted from all branches in x, as a new object z. Use 'ref' (e.g., 'PAR(1)') as the reference column in y (only the first monotonically increasing or decreasing part is used).

Type `subtract('xxx','yyy','ref')` to subtract, using interpolation, the first branch in b.yyy from all branches in b.xxx, and save the result in b.xxx. A Backup of the original file is saved.

Use optional arguments branch=m, and point=n, to denote the branch and first point on that branch within y or b.yyy, where m,n are in 1,2,3,....

Aliases: sb commandSubtractBranches

4.14.6 Copying a demo.

demo Copy a demo into the current directory and load it.

Type `demo('xxx')` to copy all files from auto/07p/demos/xxx to the current user directory. Here 'xxx' denotes a demo name; e.g., 'abc'. To avoid the overwriting of existing files, always run demos in a clean work directory. NOTE: This command automatically performs the `load` command as well.

Aliases: dm commandCopyAndLoadDemo

copydemo Copy a demo into the current directory.

Type `copydemo('xxx')` to copy all files from auto/07p/demos/xxx to the current user directory. Here 'xxx' denotes a demo name; e.g., 'abc'. To avoid the overwriting of existing files, always run demos in a clean work directory.

Aliases: copydemo commandCopyDemo

4.14.7 Python data structure manipulation functions.

All commands here, except for 'man', 'gui', and 'wait' are only provided for backwards compatibility. Alternatives are given.

man Get help on the AUTO commands.

Type `man` to list all commands with a online help. Type `man xxx` or `help xxx` to get help for command `xxx`.

Aliases: commandInteractiveHelp

cn Get the current continuation constants.

Type `cn('xxx')` to get a parsed version of the constants file `c.xxx`. This is equivalent to the command
`loadbd('xxx').c`

Aliases: constantsget commandParseConstantsFile

dg Parse a bifurcation diagram.

Type `dg('xxx')` to get a parsed version of the diagram file `b.xxx`. This is equivalent to the command `loadbd('xxx')` but without the solutions in `s.xxx` and without the diagnostics in `d.xxx`.

Aliases: diagramget commandParseDiagramFile

sl Parse solution file:

Type `sl('xxx')` to get a parsed version of the solution file `s.xxx`. This is equivalent to the command
`loadbd('xxx')()`

Aliases: solutionget commandParseSolutionFile

bt Parse both bifurcation diagram and solution.

Type `bt('xxx')` to get a parsed version of the diagram file `b.xxx` and solution file `s.xxx`. This is equivalent to the command `loadbd('xxx')` but without the diagnostics in `d.xxx`.

Aliases: diagramandsolutionget commandParseDiagramAndSolutionFile

ch Modify continuation constants.

Type `ch('xxx',yyy)` to change the constant `'xxx'` to have value `yyy`. This is equivalent to the command
`s=load(s,xxx=yyy)`
where `s` is a solution.

Aliases: changeconstant cc commandRunnerConfigFort2

hch Modify HomCont continuation constants.

Type `hch('xxx',yyy)` to change the HomCont constant `'xxx'` to have value `yyy`. This is equivalent to the command
`s=load(s,xxx=yyy)`
where `s` is a solution.

Aliases: commandRunnerConfigFort12

pr Print continuation parameters.

Type `pr()` to print all the parameters. Type `pr('xxx')` to return the parameter 'xxx'. These commands are equivalent to the commands
`print s.c`
`print s.c['xxx']`
where `s` is a solution.

Aliases: pc pr printconstant commandRunnerPrintFort2

hpr Print HomCont continuation parameters.

Type `hpr()` to print all the HomCont parameters. Type `hpr('xxx')` to return the HomCont parameter 'xxx'. These commands are equivalent to the commands
`print s.c`
`print s.c['xxx']`
where `s` is a solution.

Aliases: commandRunnerPrintFort12

splabs Return special labels.

Type `splabs('xxx',typename)` to get a list of labels with the specified typename, where typename can be one of 'EP', 'MX', 'BP', 'LP', 'UZ', 'HB', 'PD', 'TR', or 'RG'. This is equivalent to the command
`load('xxx')(typename)`
which gives a list of the solutions themselves; `load('xxx')(typename).getLabels()` returns the list of labels.

Or use `splabs(s,typename)` where `s` is a parsed solution from `sl()`. This is equivalent to the command
`s(typename).getLabels()`

Aliases: commandSpecialPointLabels

wait Wait for the user to enter a key.

Type `wait()` to have the AUTO interface wait until the user hits any key (mainly used in scripts).

Aliases: commandWait

quit Quits the AUTO CLUI

Aliases: q commandQuit

gui Show AUTO's graphical user interface.

Type `gui()` to start AUTO's graphical user interface.

NOTE: This command is not implemented yet.

Aliases: commandCreateGUI

4.14.8 Shell Commands.

cat Print the contents of a file

Type `cat xxx` to list the contents of the file `xxx`.

Aliases: commandCat

cd Change directories.

Type `cd xxx` to change to the directory `xxx`. This command understands both shell variables and home directory expansion.

Aliases: commandCd

ls List the current directory.

Type `ls` to run the system `ls` command in the current directory. This command will accept whatever arguments are accepted by the Unix command `ls`.

Aliases: commandLs

shell Run a shell command.

Type `shell('xxx')` to run the command `xxx` in the Unix shell and display the results in the AUTO command line user interface.

Aliases: commandShell

Chapter 5

Running AUTO using Unix Commands.

Apart from the Python commands described in the previous chapter, AUTO can also be run with the GUI described in Chapter 9, or using the Unix commands described below. These Unix commands run both directly in the shell and at the `AUTO` Python prompt. The AUTO aliases must have been activated; see Section 1.1; and an equations-file `xxx.f90` and a corresponding constants-file `c.xxx` (see Section 3) must be in the current user directory.
Do not run AUTO in the directory `auto/07p` *or in any of its subdirectories.*

Most commands only need a plain Unix shell but some use Python: the commands that depend on Python are `@cnvc`, `@dlb`, `@dsp`, `@klb`, `@ksp`, `@lbf`, `@ll`, `@ls`, `@mb`, `@pp`, and `@sb`.

5.1 Basic commands.

`@r` : Type `@r xxx` to run AUTO. Restart data, if needed, are expected in `s.xxx`, and AUTO-constants in `c.xxx`. This is the simplest way to run AUTO.

- Type `@r xxx yyy` to run AUTO with equations-file `xxx.f90` and restart data-file `s.yyy`. AUTO-constants must be in `c.xxx`.

- Type `@r xxx yyy zzz` to run AUTO with equations-file `xxx.f90`, restart data-file `s.yyy` and constants-file `c.zzz`.

`@R` : The command `@R xxx` is equivalent to the command `@r xxx` above.

- Type `@R xxx i` to run AUTO with equations-file `xxx.f90`, constants-file `c.xxx.i` and, if needed, restart data-file `s.xxx`.

- Type `@R xxx i yyy` to run AUTO with equations-file `xxx.f90`, constants-file `c.xxx.i` and restart data-file `s.yyy`.

- Use `@@R` on case-insensitive file systems.

`@sv` : Type `@sv xxx` to save the output-files `fort.7`, `fort.8`, `fort.9`, as `b.xxx`, `s.xxx`, `d.xxx`, respectively. Existing files by these names will be deleted.

`@ap` : Type `@ap xxx` to append the output-files `fort.7`, `fort.8`, `fort.9`, to existing data-files `b.xxx`, `s.xxx`, `d.xxx`, resp.

- Type `@ap xxx yyy` to append `b.xxx`, `s.xxx`, `d.xxx`, to `b.yyy`, `s.yyy`, `d.yyy`, resp.

`@ll` : Type `@ll` to list all solutions in fort.8.
Type `@ll xxx` to list all solutions in s.xxx.

`@ls` : Type `@ls` to list the abbreviated contents of fort.7.
Type `@ls xxx` to list the abbreviated contents of b.xxx.
The contents are shown in a similar format as the screen output of AUTO runs.

`@lbf` : Type `@lbf` to list the contents of fort.7.
Type `@lbf xxx` to list the contents of b.xxx.
The contents are shown with less accuracy (6 instead of 11 significant figures) than in the actual file for easier viewing.

5.2 Plotting commands.

`@pp` : Type `@pp xxx` to run the graphics program PyPLAUT (See Chapter 7) for the graphical inspection of the data-files `b.xxx` and `s.xxx`.

- Type `@pp` to run the graphics program PyPLAUT for the graphical inspection of the output-files `fort.7` and `fort.8`.

`@pl` : The command `@pl` is equivalent to `@pp` but runs the graphics program PLAUT04 instead. (See Chapter 8)

`@r3b` : The command `@r3b` is equivalent to `@pp` but runs the graphics program PLAUT04 instead in R3B mode. (See Chapter 8)

`@p` : The command `@p` is equivalent to `@pp` but runs the graphics program PLAUT instead. (See Chapter 7)

`@ps` : Type `@ps fig.x` to convert a saved PLAUT figure `fig.x` from compact PLOT10 format to PostScript format. The converted file is called `fig.x.ps`. The original file is left unchanged.

5.3 File-manipulation.

`@cp` : Type `@cp name1 name2`, or `@cp name1 name2 name3`, or `@cp name1 name2 name3 name4` to copy the data-files dir1/c.xxx, dir1/b.xxx, dir1/s.xxx, and dir1/d.xxx, to dir2/c.yyy, dir2/b.yyy, dir2/s.yyy, and dir2/d.yyy.

The values of dir1/?.xxx and dir2/?.yyy are as follows, depending on whether name1 is a directory or name2 is a directory:

`@cp name1 name2`
no directory names: ./?.name1 and ./?.name2
name1 is a directory: name1/?.name2 and ./?.name2
name2 is a directory: ./?.name1 and name2/?.name1

`@cp name1 name2 name3`
name1 is a directory: name1/?.name2 and ./?.name3
name2 is a directory: ./?.name1 and name2/?.name3

`@cp name1 name2 name3 name4`
name1/?.name2 and name3/?.name4

`@mv` : Type `@mv name1 name2`, or `@mv name1 name2 name3)`, or `@mv name1 name2 name3 name4` to move the data-files dir1/b.xxx, dir1/s.xxx, and dir1/d.xxx, to dir2/b.yyy, dir2/s.yyy, and dir2/d.yyy, and copy the constants file dir1/c.xxx to dir2/c.yyy.

The values of dir1/?.xxx and dir2/?.yyy are determined in the same way as for `@cp` above.

`@df` : Type `@df` to delete the output-files `fort.7`, `fort.8`, `fort.9`.

`@cl` : Type `@cl` to clean the current directory. This command will delete all files of the form `fort.*`, `*.o`, `*.*~`, and `*.exe`.

`@dl` : Type `@dl xxx` to delete the data-files `b.xxx`, `s.xxx`, `d.xxx`.

`@cnvc` : Type `@cnvc xxx yyy` to convert old-format constants files c.xxx and h.xxx to a new-format constant file c.yyy.
The command `@cnvc xxx` overwrites the file c.xxx with the new style file and deletes h.xxx if it exists.

`@rn` : Type `@rn` to rename, within the current directory, all old-named constants, HomCont, bifurcation diagram, and solution files starting with r., s., p., and q. to files with the new prefixes c., h., b., and s..

`@rc` : Type `@rc` to do a recovery by swapping the backup files fort.7~ and fort.8~ with the files fort.7 and fort.8.
Type `@rc xxx` to do a recovery by swapping the backup files b.xxx and s.xxx with the files b.xxx~ and s.xxx~.

`@gz` : Type `@gz` to compress, using `gzip` all output files in the current directory.

`@uz` : Type `@uz` to decompress, using `unzip` all output files in the current directory.

`@sr` : Type `@sr xxx y` to copy c.xxx to c.xxx.y.

5.4 Diagnostics.

`@lp` : Type `@lp` to list the value of the "limit point function" in the output-file `fort.9`. This function vanishes at a limit point (fold).

- Type `@lp xxx` to list the value of the "limit point function" in the data-file `d.xxx`. This function vanishes at a limit point (fold).

`@bp` : Type `@bp` to list the value of the "branch-point function" in the output-file `fort.9`. This function vanishes at a branch point.

- Type `@bp xxx` to list the value of the "branch-point function" in the data-file `d.xxx`. This function vanishes at a branch point.

`@hb` : Type `@hb` to list the value of the "Hopf function" in the output-file `fort.9`. This function vanishes at a Hopf bifurcation point.

- Type `@hb xxx` to list the value of the "Hopf function" in the data-file `d.xxx`. This function vanishes at a Hopf bifurcation point.

`@ho` : The command `@ho` is an alias to `@hb` above.

`@sp` : Type `@sp` to list the value of the "secondary-periodic bifurcation function" in the output-file `fort.9`. This function vanishes at period-doubling and torus bifurcations.

- Type `@sp xxx` to list the value of the "secondary-periodic bifurcation function" in the data-file `d.xxx`. This function vanishes at period-doubling and torus bifurcations.

`@it` : Type `@it` to list the number of Newton iterations per continuation step in `fort.9`.

- Type `@it xxx` to list the number of Newton iterations per continuation step in `d.xxx`.

`@st` : Type `@st` to list the number of stable eigenvalues or stable Floquet multipliers per continuation step in `fort.9`.

`@ss` : Type `@st` to list the continuation step size for each continuation step in `fort.9`.

- Type `@st xxx` to list the continuation step size for each continuation step in `d.xxx`.

`@ev` : Type `@ev` to list the eigenvalues of the Jacobian in `fort.9`. (Algebraic problems.)

- Type `@ev xxx` to list the eigenvalues of the Jacobian in `d.xxx`. (Algebraic problems.)

`@fl` : Type `@fl` to list the Floquet multipliers in the output-file `fort.9`. (Differential equations.)

- Type `@fl xxx` to list the Floquet multipliers in the data-file `d.xxx`. (Differential equations.)

`@no` : Type `@no` to show any notes in `fort.9`.

- Type `@no xxx` to show any notes in `d.xxx`.

5.5 File-editing.

`@e7` : To use the vi editor to edit the output-file `fort.7`.

`@e8` : To use the vi editor to edit the output-file `fort.8`.

`@e9` : To use the vi editor to edit the output-file `fort.9`.

`@j7` : To use the SGI jot editor to edit the output-file `fort.7`.

`@j8` : To use the SGI jot editor to edit the output-file `fort.8`.

`@j9` : To use the SGI jot editor to edit the output-file `fort.9`.

5.6 File-maintenance.

`@rl` : Type `@rl` to sequentially relabel solutions with the numbers $1, 2, 3, \ldots$ in the output-files `fort.7` and `fort.8`. The original files are backed up as `fort.7`$\sim$ and `fort.8`$\sim$.

- Type `@rl xxx` to relabel solutions in the data-files `b.xxx` and `s.xxx`. The original files are backed up as `b.xxx`$\sim$ and `s.xxx`$\sim$.

- Type `@rl xxx yyy` to relabel solutions in the data-files `b.xxx` and `s.xxx`. The modified files are written as `b.yyy` and `s.yyy`.

`@lb` : Type `@lb` to run an interactive utility program for listing, deleting and relabeling solutions and branches in the output-files `fort.7` and `fort.8`. The original files are backed up as `fort.7`$\sim$ and `fort.8`$\sim$.

- Type `@lb xxx` to list, delete and relabel solutions and branches in the data-files `b.xxx` and `s.xxx`. The original files are backed up as `b.xxx`$\sim$ and `s.xxx`$\sim$.

- Type `@lb xxx yyy` to list, delete and relabel solutions in the data-files `b.xxx` and `s.xxx`. The modified files are written as `b.yyy` and `s.yyy`.

`@LB` : Type `@LB` or `@@LB` on case-insensitive file systems is equivalent to `@lb` above but moves instead of copies files so that it is much quicker but interrupts may be harmful.

`@fc` : Type `@fc xxx` to convert a user-supplied data file `xxx.dat` to AUTO format. The converted file is called `s.dat`. The original file is left unchanged. AUTO automatically sets the period in `PAR(11)`. Other parameter values must be set in `STPNT`. (When necessary, `PAR(11)` may also be redefined there.) The constants-file file `c.xxx` must be present, as the AUTO-constants `NTST` and `NCOL` (Sections 10.3.1 and 10.3.2) are used to define the new mesh.

Note: this technique has been obsoleted by the 'dat' AUTO constant in Section 10.8.7.

`@db` : Type `@db` to double the solution in fort.8. Type `@db xxx` to double the solution in s.xxx.

`@tr` : Type `@tr` to triple the solution in fort.8. Type `@tr xxx` to triple the solution in s.xxx.

`@dlb` : Type `@dlb` to delete all special labels in fort.7 and fort.8. Backups are made.
Type `@dlb xxx` to delete all special labels in b.xxx and s.xxx. Backups are made.
Type `@dlb xxx yyy` to delete all special labels in b.xxx and s.xxx. The output is written to b.yyy and s.yyy.
Optionally, give an argument of the form `-UZ`, `-HB`, `-LP`, `-EP`, `-PD`, `-TR`, `-BP`, `-MX`, or `-RG` to remove all labels with the given type, as in `@dlb -UZ xxx`; otherwise only labels with type UZ and regular labels are kept.
Type information is kept in the bifurcation diagram for plotting.

`@klb` : Type `@klb` to keep all special labels in fort.7 and fort.8. Backups are made.
Type `@klb xxx` to keep all special labels in b.xxx and s.xxx. Backups are made.
Type `@klb xxx yyy` to keep all special labels in b.xxx and s.xxx. The output is written to b.yyy and s.yyy.
Optionally, give an argument of the form `-UZ`, `-HB`, `-LP`, `-EP`, `-PD`, `-TR`, `-BP`, `-MX`, or `-RG` to keep all labels with the given type, as in `@klb -UZ xxx` and remove all others; otherwise all labels are kept except for labels with type UZ and regular labels.
Type information is kept in the bifurcation diagram for plotting.

`@dsp` : The command `@dsp` is equivalent to the command `@dlb` above, except that type information is *not* kept in the bifurcation diagram for plotting.

`@ksp` : The command `@ksp` is equivalent to the command `@klb` above, except that type information is *not* kept in the bifurcation diagram for plotting.

`@dlp` : The command `@dlp` is equivalent to the command `@dsp -LP` above.

`@kbp` : The command `@kbp` is equivalent to the command `@ksp -BP` above.

`@klp` : The command `@klp` is equivalent to the command `@ksp -LP` above.

`@kuz` : The command `@kuz` is equivalent to the command `@ksp -UZ` above.

`@rd` : Type `@rd` to reduce the solution in fort.7 and fort.8.
Type `@rd xxx` to reduce the solution in b.xxx and s.xxx.
Reducing means that all even regular point solutions (from `NPR`: the 2nd, 4th, 6th, etc.) are deleted from the files.

`@RD` : The commands `@RD` and `@@RD` (for case-insensitive file systems) are equivalent to `@rd` above but are faster, though not reliable when interrupting, by using moves instead of copies.

`@mb` : Type `@mb` to merge branches into continuous curves in fort.7, fort.8, and fort.9. Backups of the original files are saved.
Type `@mb xxx` to merge branches in s.xxx, b.xxx, and d.xxx. Backups of the original files are saved.
Type `@mb xxx yyy` to merge branches in s.xxx, b.xxx, and d.xxx, and save them to s.yyy, b.yyy, and d.yyy.

`@sb` : Type `@sb xxx yyy ref` to subtract, using interpolation, the first branch in `b.yyy` from all branches in `b.xxx`, and save the result in `b.xxx`. Use `ref` (e.g., `PAR(1)`) as the reference column in `b.yyy`. (only the first monotonically increasing or decreasing part is used). A Backup of the original file is saved.

Use optional fourth and fifth arguments `m`, and `n`, to denote the branch m and first point n on that branch within `b.yyy`, where m, n are in $\{1, 2, 3, \ldots\}$.

`@zr` : Type `@zr xxx` to zero all small numbers, with absolute value less than 10^{-16} in `s.xxx`. A backup file is made.

5.7 HomCont commands.

Note that the `@h` and `@H` are obsolete with new-style constants files, where HomCont constants can be included in the main constant file with a `c.` prefix.

`@h` : Use `@h` instead of `@r` when using HomCont, i.e., when `IPS`=9 (see Chapter 20). Type `@h xxx` to run AUTO/HomCont. Restart data, if needed, are expected in `s.xxx`, AUTO-constants in `c.xxx` and HomCont-constants in `h.xxx`.

- Type `@h xxx yyy` to run AUTO/HomCont with equations-file `xxx.f90` and restart data-file `s.yyy`. AUTO-constants must be in `c.xxx` and HomCont-constants in `h.xxx`.

- Type `@h xxx yyy zzz` to run AUTO/HomCont with equations-file `xxx.f90`, restart data-file `s.yyy` and constants-files `c.zzz` and `h.zzz`.

`@H` : The command `@H xxx` is equivalent to the command `@h xxx` above.

- Type `@H xxx i` to run AUTO/HomCont with equations-file `xxx.f90` and constants-files `c.xxx.i` and `h.xxx.i` and, if needed, restart data-file `s.xxx`.

- Type `@H xxx i yyy` to run AUTO/HomCont with equations-file `xxx.f90`, constants-files `c.xxx.i` and `h.xxx.i`, and restart data-file `s.yyy`.

- Use `@@H` on case-insensitive file systems.

5.8 Copying a demo.

`@dm` : Type `@dm xxx` to copy all files from `auto/07p/demos/xxx` to the current user directory. Here `xxx` denotes a demo name; e.g., `abc`. Note that the `@dm` command also copies `.auto` files to the current user directory. To avoid the overwriting of existing files, always run demos in a clean work directory.

5.9 Viewing the manual.

`@mn` : Use gv or evince to view the PDF version of this manual.

Chapter 6

Output Files.

AUTO writes to standard output and three output-files :

- standard output: A summary of the computation is written to standard output, which usually corresponds to the window in which AUTO is run. Only special, labeled solution points are noted, namely those listed in Tables 6.1 and 6.2 The letter codes in the Table are used in the screen output. The numerical codes are used internally and in the `fort.7` and `fort.8` output-files described below.

BP	(1)	Branch point (algebraic systems)
LP	(2)	Fold (algebraic systems)
HB	(3)	Hopf bifurcation
	(4)	User-specified regular output point
UZ	(-4)	Output at user-specified parameter value
LP	(5)	Fold (differential equations)
BP	(6)	Branch point (differential equations)
PD	(7)	Period doubling bifurcation
TR	(8)	Torus bifurcation
EP	(9)	End point of family; normal termination
MX	(-9)	Abnormal termination; no convergence

Table 6.1: Solution Types.

- `fort.7` : The `fort.7` output-file contains the bifurcation diagram. Its format is the same as the `fort.6` (screen) output, but the `fort.7` output is more extensive, as every solution point has an output line printed.

- `fort.8` : The `fort.8` output-file contains complete graphics and restart data for selected, labeled solutions. The information per solution is generally much more extensive than that in `fort.7`. The `fort.8` output should normally be kept to a minimum.

- `fort.9` : Diagnostic messages, convergence history, eigenvalues, and Floquet multipliers are written in `fort.9`. It is strongly recommended that this output be habitually in-

BT	(-21)	Bogdanov-Takens bifurcation on fold curve (algebraic systems)
BT	(-31)	Bogdanov-Takens bifurcation on Hopf curve
CP	(-22)	Cusp bifurcation on fold curve (algebraic systems)
GH	(-32)	Generalized Hopf bifurcation on Hopf curve
ZH	(-13)	Zero-Hopf on BP curve (algebraic systems)
ZH	(-23)	Zero-Hopf on Fold curve (algebraic systems)
ZH	(-33)	Zero-Hopf on Hopf curve
R1	(-25)	1:1 resonance on Fold (maps)
R1	(-55)	1:1 resonance on Fold (periodic solutions)
R1	(-85)	1:1 resonance on Torus (periodic solutions, maps)
R2	(-76)	1:2 resonance on PD (periodic solutions, maps)
R2	(-86)	1:2 resonance on Torus (periodic solutions, maps)
R3	(-87)	1:3 resonance on Torus (periodic solutions, maps)
R4	(-88)	1:4 resonance on Torus (periodic solutions, maps)
LPD	(28)	Fold-flip bifurcation on Fold (maps)
LPD	(78)	Fold-flip bifurcation on PD (maps)
LTR	(23)	Fold-torus bifurcation on Fold (maps)
LTR	(83)	Fold-torus bifurcation on Torus (maps)
PTR	(77)	Flip-torus bifurcation on PD (maps)
PTR	(87)	Flip-torus bifurcation on Torus (maps)
TTR	(88)	Torus-torus bifurcation on Torus (maps)

Table 6.2: Codimension-two solution types. Note that the absolute value of the numerical code divided by 10 gives the type of the curve on which the special point occurs.

spected. The amount of diagnostic data can be controlled via the AUTO-constant `IID`; see Section 10.9.9.

The user has some control over the `fort.6` (screen) and `fort.7` output via the AUTO-constant `IPLT` (Section 10.9.10). Furthermore, the routine `PVLS` can be used to define "solution measures" which can then be printed by "parameter overspecification"; see Section 10.7.10. For an example see demo `pvl`.

The AUTO-commands `@sv`(`sv`), `@ap`(`ap`), and `@df`(`df`) can be used to manipulate the output-files `fort.7`, `fort.8`, and `fort.9`. Furthermore, the AUTO-command `@lb`(`rl`) can be used to delete and relabel solutions and branches simultaneously in `fort.7` and `fort.8`. For details see Section 5.

The graphics programs PLAUT, PLAUT04, and the Python CLUI command `plot` can be used to graphically inspect the data in `fort.7` and `fort.8`; see Chapters 7, 8, and 4.

Chapter 7

The Graphics Programs PLAUT and PyPLAUT.

PLAUT and PyPLAUT can be used to extract graphical information from the AUTO output-files `fort.7` and `fort.8`, or from the corresponding data-files `b.xxx` and `s.xxx`. To invoke PLAUT, use the `@p` command defined in Section 5. The PLAUT window (a Tektronix window) will appear, in which PLAUT commands can be entered. To invoke PyPLAUT, use the `@pp` command. The same plotting window as you get by using `plot` in the Python interface appears (see Section 4.11), but you can also manipulate it by typing PLAUT commands in the terminal in which you typed `@pp`. For examples of using PLAUT and PyPLAUT see the tutorial demos `pp2` and `pp3` in sections 14.3 and 14.9, respectively.

The files `.autorc` and `autorc`, as explained in Section 4.11, can be used to customize PyPLAUT's behaviour and appearance.

7.1 Basic PLAUT-Commands.

The principal PLAUT-commands are

`bd0` : This command is useful for an initial overview of the bifurcation diagram as stored in `fort.7`. If you have not previously selected one of the default options *d0, d1, d2, d3*, or *d4* described below then you will be asked whether you want solution labels, grid lines, titles, or labeled axes.

`bd` : This command is the same as the *bd0* command, except that you will be asked to enter the minimum and the maximum of the horizontal and vertical axes. This is useful for blowing up portions of a previously displayed bifurcation diagram.

`ax` : With the *ax* command you can select any pair of columns of real numbers from `fort.7` as horizontal and vertical axis in the bifurcation diagram. (The default is columns 1 and 2). To determine what these columns represent, one can look at the screen ouput of the corresponding AUTO run, or one can inspect the column headings in `fort.7`.

`2d` : Upon entering the *2d* command, the labels of all solutions stored in `fort.8` will be listed and you can select one or more of these for display. The number of solution components is

also listed and you will be prompted to select two of these as horizontal and vertical axis in the display. Note that the first component is typically the independent time or space variable scaled to the interval [0,1].

`sav` : To save the displayed plot in a file. You will be asked to enter a file name. Each plot must be stored in a separate new file. The plot is stored in compact PLOT10 format, which can be converted to PostScript format with the AUTO-commands `@ps`; see Section 7.4.

`cl` : To clear the graphics window.

`lab` : To list the labels of all solutions stored in `fort.8`. Note that PLAUT requires all labels to be distinct. In case of multiple labels you can use the AUTO command `@lb` to relabel solutions in `fort.7` and `fort.8`.

`end` : To end execution of PLAUT.

7.2 Default Options.

After entering the commands *bd0, bd*, or *2d*, you will be asked whether you want solution labels, grid lines, titles, or axes labels. For quick plotting it is convenient to bypass these selections. This can be done by the default commands *d0, d1, d2, d3*, or *d4* below. These can be entered as a single command or they can be entered as prefixes in the *bd0* and *bd* commands. Thus, for example, one can enter the command *d1bd0*.

`d0` : Use solid curves, showing symbols, but no solution labels.

`d1` : Use solid curves, except use dashed curves for unstable solutions and for solutions of unknown stability. Show solution labels and symbols.

`d2` : As *d1*, but without solution labels.

`d3` : As *d1*, but with grid lines.

`d4` : As *d2*, but with grid lines.

If no default option *d0, d1, d2, d3*, or *d4* has been selected or if you want to override a default feature, then the following commands can be used. These can be entered as individual commands or as prefixes. For example, one can enter the command *sydpbd0*.

`sy` : Use symbols for special solution points, for example, open square = branch point, solid square = Hopf bifurcation.

`dp` : "Differential Plot", i.e., show stability of the solutions. Solid curves represent stable solutions. Dashed curves are used for unstable solutions and for solutions of unknown stability. For periodic solutions use solid/open circles to indicate stability/instability (or unknown stability).

`st` : Set up titles and axes labels.

`nu` : Normal usage (reset special options).

7.3 Other PLAUT-Commands.

The full PLAUT program has several other capabilities, for example,

`scr` : To change the diagram size.

`rss` : To change the size of special solution point symbols.

These commands are not available in PyPLAUT.

7.4 Printing PLAUT Files.

`@ps` : Type `@ps fig.1` to convert a saved PLAUT file `fig.1` to PostScript format in `fig.1.ps`.

`@eps` : Type `@eps fig.1` to convert a saved PLAUT file `fig.1` to encapsulated PostScript format in `fig.1.eps`.

In PyPLAUT you can directly save to a variety of file formats, including .eps and .png.

Chapter 8

The Graphics Program PLAUT04.

"PLAUT04" is a graphic tool for AUTO data visualization. Here we explain how to view AUTO data sets with PLAUT04. An AUTO data set contains a solution file, "s.foo", a bifurcation file, "b.foo", and a diagnostic file, "d.foo". Here "foo" denotes a user-chosen data set name. This user's guide includes the following information:

1. A description of the PLAUT04 window system.
2. A list of PLAUT04 configuration options.
3. An example of using PLAUT04.

8.1 Quick start

8.1.1 Starting and stopping Plaut04

Starting

The starting command for PLAUT04 is: "plaut04". A short Unix command is also provided as "@pl". In the Python CLUI, one can start PLAUT04 by typing "plot3()", "p3()", or "commandPlotter3D()".

This command can have no argument, one argument, or two arguments.

If no argument is provided, then the system uses the AUTO default data files, fort.7, fort.8, and fort.9, as inputs.

If one argument is given, it must be the name of the data set which we want to view. This data set should be in the current directory.

When two arguments are given, the first is always the path to the data set, and the second is the data set name.

Note that the AUTO data set name does not mean the full name of an AUTO file. It refers to the postfix of AUTO data files. For example, if we have the AUTO data files: "s.H1", "b.H1", and "d.H1", the AUTO data file name is "H1".

Stopping

One can exit the system by clicking the cross at the top-right corner of the window or from the "File" menu of the system.

8.1.2 Changing the "Type"

Often one will frequently change between the solution diagram and the bifurcation diagram. The "Type" menu helps to complete this change. This menu includes two items, "Solution", and "Bifurcation". There is a marker beside the current diagram. For example, if the current diagram is the solution diagram, but we want to change to the bifurcation diagram, we can do so by clicking "Type → Bifurcation" to switch to the bifurcation diagram.

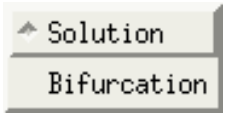


Figure 8.1: The Type Menu

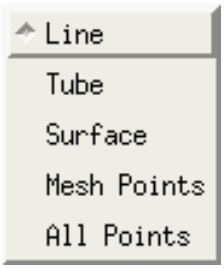


Figure 8.2: The Style Menu

8.1.3 Changing the "Style"

PLAUT04 provides four ways to draw the graphics, *i.e.*, using curves, tubes, points, or as a surface. One can select the style from the "Style" menu. The "Style" menu is shown in Figure 8.2.

8.1.4 Coordinate axes

Figure 8.3 shows the selections of the "Coord" menu. One may use this menu to select to show or not to show coordinate axes, and the type of coordinate axes, in the graphics.

8.1.5 Options

The "Options" Menu provides functions to add or remove widgets from the graphics. It also allows to start/stop solution or orbit animation. The "normalize data" normalizes the raw data to [0,1]. "Preference" lets us set preferences for the GUI (see Figure 8.4).

8.1.6 CR3BP animation

The "Center" Menu allows to animate the motion of the three bodies in different coordinate systems. We can animate the motion in a large-primary-centered inertial coordinate system, or in a small-primary-centered inertial system, or in the bary-centered inertial system. Figure 8.5 displays the layout of the "Center" menu.

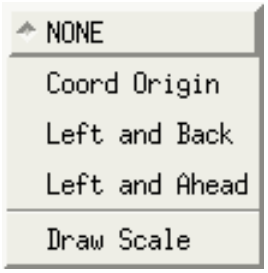


Figure 8.3: The Draw-Coordinate-Axes Menu

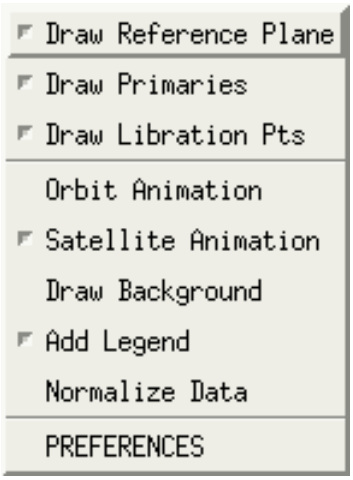


Figure 8.4: The Options Menu

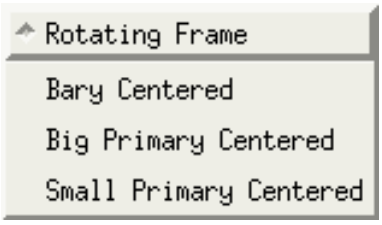


Figure 8.5: The Center Menu

Figure 8.6: The Help Menu

8.1.7 Help

The "Help" menu provides an on-line help on how to use PLAUT04.

8.1.8 Picking a point in the diagram

The picking operation is useful when we want to know data corresponding to a certain point in the diagram. In order to execute a picking operation, we should follow these steps.

- Click the arrow icon to change the mouse to picking state.
- Move the mouse to the point of interest.
- Click the left button of the mouse to pick the point.

Once a point has been picked, a new window is popped up. In this new window, the Floquet multipliers of the point are shown in an x-y plane. Black crosses in the diagram indicate the Floquet Multipliers. The solution, and the values of the corresponding Floquet Multipliers, are given in the lower part of the window. A unit circle is drawn in the diagram. Figure 8.7 is an example of the picking operation. From this diagram, we can see that two Floquet Multipliers are outside the unit circle, two are on the unit circle, and the other two are inside the unit circle.

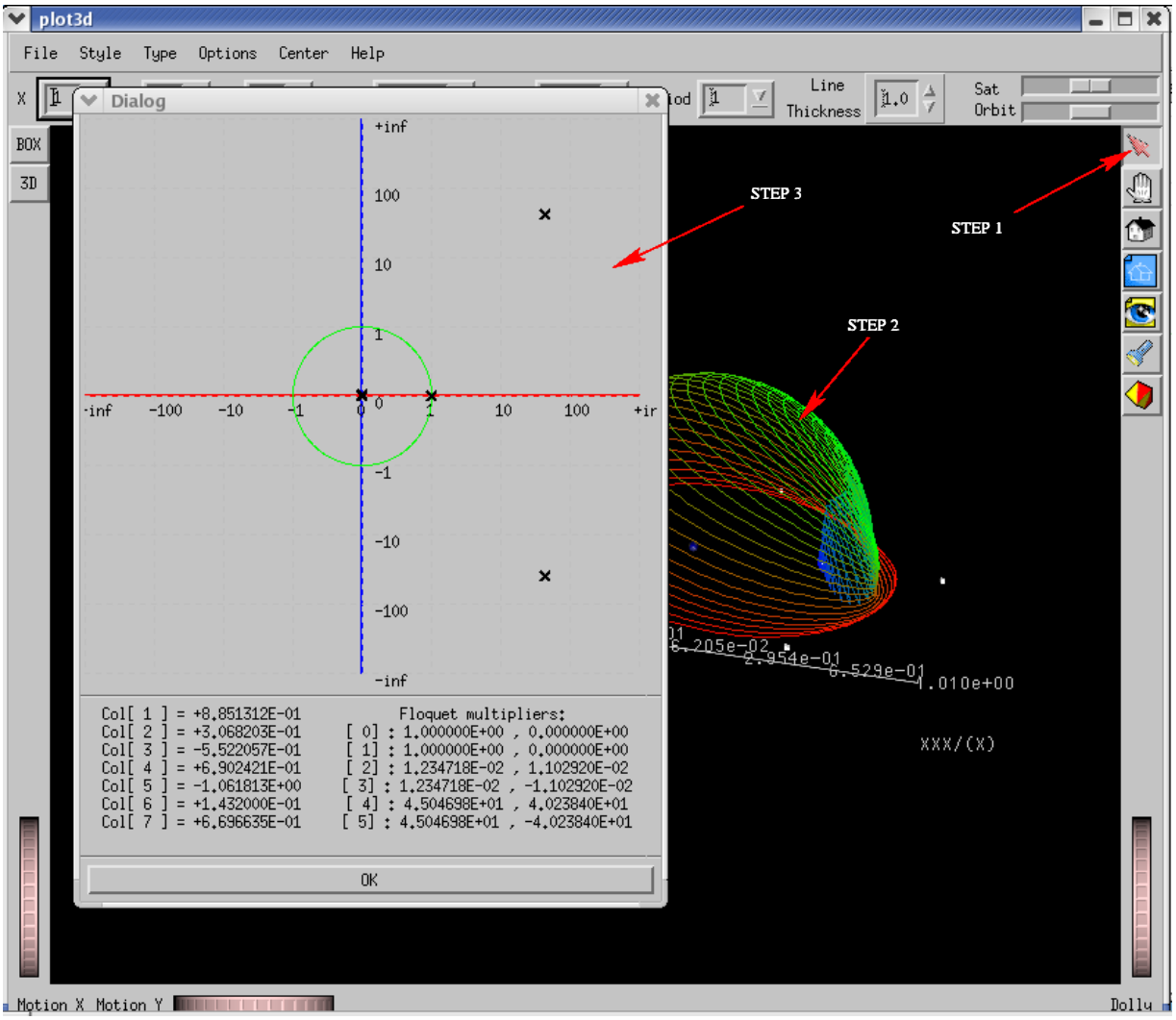


Figure 8.7: Picking a point

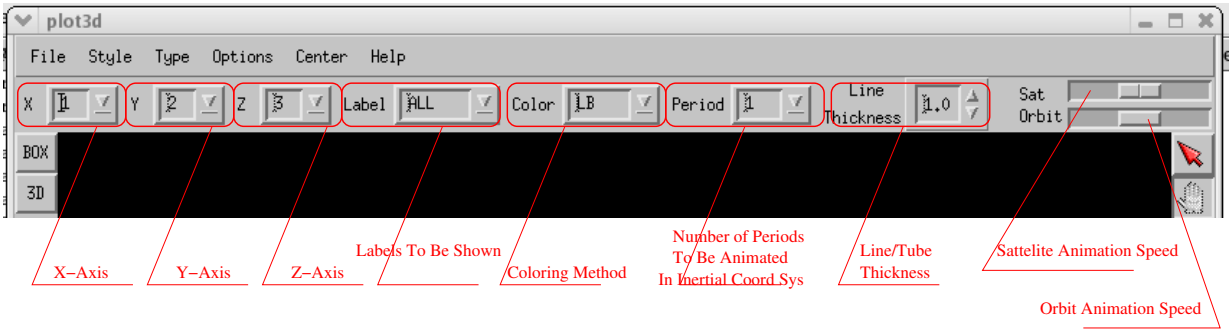


Figure 8.8: Menu-bar layout

8.1.9 Choosing the variables

AUTO can generate large amounts of data. The CR3BP, for example, has 6 variables, *i.e.*, x, y, z, x', y', z', and time. One can choose to draw any combination of these variables in 2 or 3 dimensions using PLAUT04. On the list bar, we can see three dropdown lists with label "X", "Y", and "Z" (See Figure 8.8). Each of these three lists has the exact number of choices, namely, the number of variables of the system plus one. In our case, these lists have 7 choices, which are represented by the integers 0 to 6. 0 represents time. 1 to 6 stand for x, y, z, x', y', and z', respectively. "1" is selected for "X", which indicates that x is drawn on the X-axis. "2" is selected for "Y", which indicates that y is represented on the Y-axis. "3" is selected for "Z", which indicates that z is represented on the Z-axis.

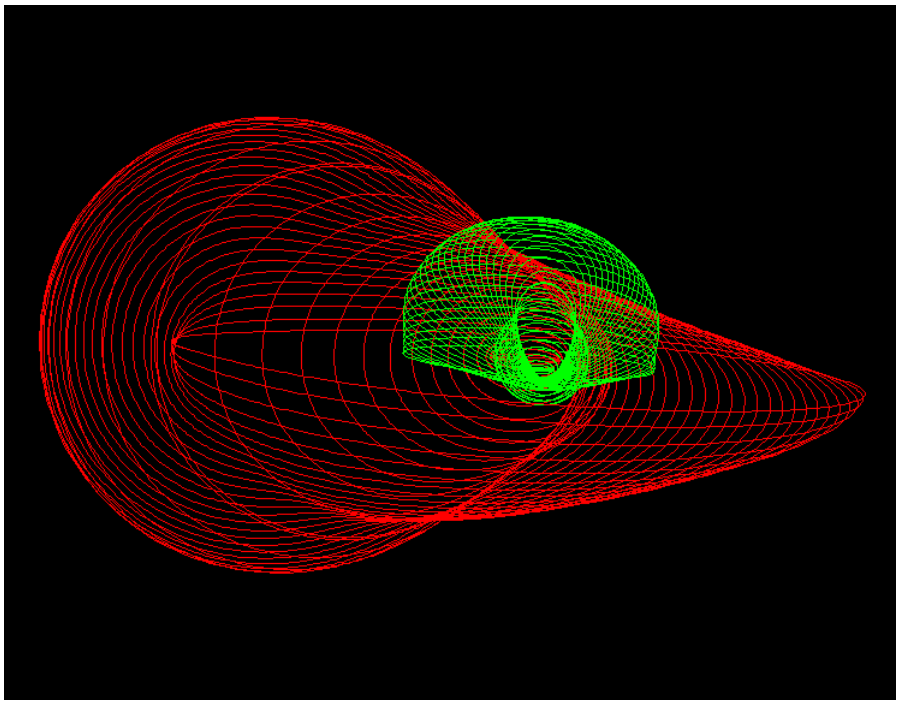


Figure 8.9: Displaying multiple components

We can also show multiple combinations at the same time. For example, if we want to show x-y-z and x'-y'-z' in the same diagram, we can input $1, 4$ in the "X" dropdown list to select x and x' being drawn on the X-axis, input $2, 5$ in the "Y" list to show y and y' on the Y-axis, and input $3, 6$ in the "Z" dropdown list to draw z and z' on the Z-Axis. Note that after finishing the input in the dropdown list box, we must type "ENTER" for the input to be accepted by the system. Figure 8.9 shows the results of the above choices. The combination is flexible. For example, if X is 1, Y is $3, 5$, and Z is $4, 5, 6$, the system will automatically reorganize them to $1-3-4$, $1-5-5$, $1-3-6$ and show the results. If X is $1, 5$, Y is 2, and Z is $3, 4$, the system reorganizes them to $1-2-3$, $5-2-4$.

Different components are drawn with different colors from blue to red.

The default values can be set in the resource file. If no resource file exists, then the system will use "1" for X-axis, "2" for Y-axis, and "3" for Z-axis for both the solution and the bifurcation diagrams.

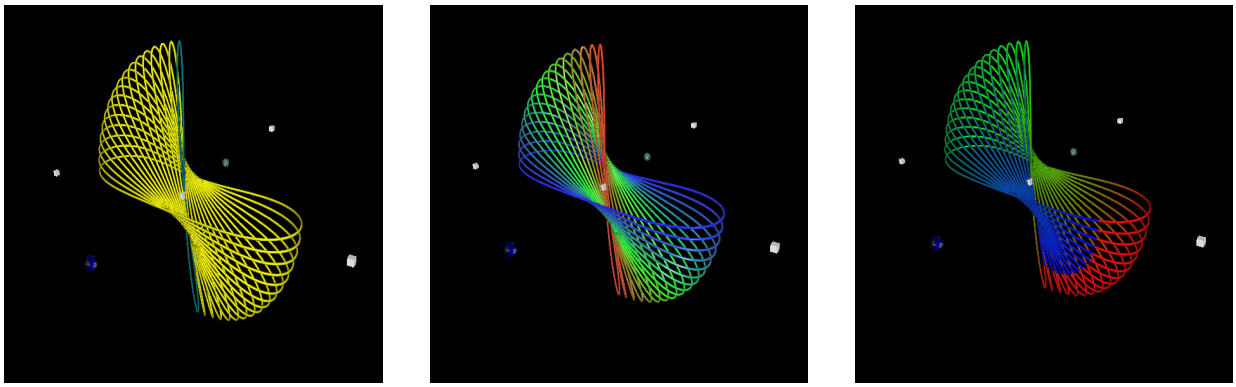
8.1.10 Choosing labels

From the Label list, we can choose the label of the solution to be drawn. If "ALL" is chosen, all solutions are shown in the diagram. If "NONE" is chosen, none of the solutions is shown. "HALF" shows the solutions with odd labels and special solutions only. "SPEC" lets the system show the special solutions only. We can also show selected solutions by inputting their labels in the list box separated by commas. For example, typing 1, 10, 15, 20 will lead the system to show only the solutions with label 1, 10, 15 and 20.

We can set the default value for this list in the PLAUT04 resource file.

8.1.11 Coloring

Many coloring methods are provided. They can be classified into three groups. The first group is coloring by variables. This group provides as many choices as the number of variables of a problem plus 1 for the time. The second group is coloring by parameters. These parameters are defined by the AUTO user. in the AUTO constants file. There are as many choices as the number of parameters defined in the AUTO constants file. The third group includes "TYPE", type of solution, "PONT", point number, "BRAN", the branch to which the solution belongs, and "LABL", label of the solution. Different coloring methods cannot be used at the same time. Figure 8.10 shows the difference between coloring by type and coloring by label. From Figure 8.10(a), we can see that there is only one branching orbit in this family, which is shown in cyan. In Figure 8.10(b), the start solution is colored in blue, and the last solution is colored in red. When using time to color the diagram, 0 is set to blue, while 1 is set to red.



(a) Coloring by "Type" (b) Coloring by "Label" (c) Coloring by "Time"

Figure 8.10: Coloring

We can set the default value in the PLAUT04 resource file.

8.1.12 Number of periods to be animated

Generally only one period is animated when we animate the solution in the inertial frame. However, the SpinBox allows us to change the default value. This is a specially designed function for the CR3BP. It is useful when we animate the motion in the three bodies in the inertial frame.

8.1.13 Changing the line/tube thickness

The "Line Thickness" spinbox allows us to increase or decrease the line/tube thickness in the diagram. The PLAUT04 resource file also provides a way to change the default values of the line/tube thickness.

8.1.14 Changing the animation speed

The "Sat" and "Orbit" scale bar allow us to change the animation speed. Their Maximum and Minimum value can be set in the resource file.

8.1.15 Changing the background picture

A user can set the background with his favorite picture. To do this, a user should copy the picture to the directory "$AUTO_DIR/plaut04/widgets", and then change the name of the file to "background.rgb".

8.2 Setting up the resource file

The PLAUT04 resource file sets default values for almost all controls of PLAUT04. PLAUT04 allows us to write our own resources files and put them in the same directory as the AUTO data files. PLAUT04 first looks for the resource file in the current directory. If it cannot find a resource file there, then it will try to use the one installed in the AUTO root directory. If both these searches fail, then the internal default values will be used.

In order to write a usable resource file, one should follow the following rules:

1. Comment lines start with "#". Comments may take as many lines as desired.

2. Between the "variable name" and the default value, we must use "=" to tell the system that the left side is the "variable name", and the right side is its corresponding default value.

3. If a "variable" has aggregate values, a comma "," must be used between two values.

4. The line type is set using 4-digit hexadecimals, starting with "0x". Its values can range from 0 (invisible) to "0xffff" (solid). The system default is "0xffff" for stable solutions, and "0x3333" for unstable ones. The line pattern is determined by the number of 1s and 0s when the hexadecimal is converted to a 16-bit binary. A "1" indicates that the drawing occurs, and "0" that it does not, on a pixel by pixel basis. For example, the pattern "0xAAAA", in binary is 0000100010001000, and PLAUT04 interprets this as drawing 3 bits off, 1 bit on, 3 bits off, 1 bit on, 3 bits off, 1 bit on and finally 4 bits off. The pattern is read backward because the low order bits are used first.

5. Some variables can only be set to "Yes" or "No". They cannot be assigned other values.

6. No "variable name" should be modified.

It is strongly recommended that the default resource file is used as a template when writing a custom resource file.

Below is a copy of the default resource file.

```
#version 0.0

# Line colors are represented by RGB values from 0 to 1.0.
# DEFAULT color is also used when animationLabel == 0, i.e.,
# when showing all solutions and animating the solution change.
# Point Type    RED  GREEN  BLUE PATTERN
DEFAULT       = 1.0,  1.0,  1.0, 0xffff
BP ALG        = 1.0,  0.0,  0.0, 0xffff
LP ALG        = 0.0,  1.0,  0.0, 0xffff
HB            = 0.0,  0.0,  1.0, 0xffff
UZ4           = 1.0,  1.0,  0.0, 0xffff
UZ-4          = 0.5,  0.5,  0.0, 0xffff
LP DIF        = 0.0,  0.0,  0.5, 0xffff
BP DIF        = 0.0,  0.5,  0.5, 0xffff
PD            = 1.0,  0.0,  1.0, 0xffff
TR            = 0.0,  1.0,  1.0, 0xffff
EP            = 0.3,  0.0,  0.3, 0xffff
MX            = 0.6,  0.0,  0.6, 0xffff
OTHERS        = 1.0,  1.0,  1.0, 0xffff

# Initialize the line pattern for showing stability:
UNSTABLE LINE PATTERN = 0xffff
STABLE LINE PATTERN   = 0xffff

# Initialize the default options:
Draw Reference Plane  = No
Draw Reference Sphere = No
Orbit Animation       = No
Satellite Animation   = No
Draw Primaries        = No
Draw Libration Points = No
Normalize Data        = No
Draw Labels           = No
Show Label Numbers    = No
Draw Background       = No
Draw Legend           = No

# Initialize the default coordinate axes:
#  0 --- None,
#  1 --- at origin
#  2 --- at left and behind
#  3 --- at left and ahead
#  4 --- always at origin
Coordinate Type = 3

# Draw Scale on the Aexs
Draw Scale = Yes

#  Initialize the default graph type:
#  0 --- Solution (fort.8)
```

```
#  1 --- Bifurcation (fort.7)
Graph Type    = 0

# Initialize the default graph style:
#  0 --- LINES,
#  1 --- TUBES,
#  2 --- SURFACE
Graph Style  =  0

# Set the window width and height:
Window Width        = 1000
Window Height       = 1000

# Set X, Y, Z axes for the solution diagram:
# 0 is Time for X,Y,Z.
X Axis Solution             = 1
Y Axis Solution             = 2
Z Axis Solution             = 3

# Set X, Y, Z axes for the bifurcation diagram:
X Axis Bifurcation          = 4
Y Axis Bifurcation          = 5
Z Axis Bifurcation          = 6

#Labeled solutions:
Labels              = 0

# Set coloring method:
# -6 --- STABILITY
# -5 --- POINT
# -4 --- BRANCH
# -3 --- TYPE
# -2 --- LABEL
# -1 --- COMPONENT
# Otherwise, according to the data in the ith column of the solution file.
# It can only be set to an integer value.
#Coloring Method           = -3
# For the solution diagram:
Coloring Method Solution  = -3
# For the bifurcation diagram:
Coloring Method Bifurcation = -3
Number of Period Animated = 1

# Line Width Scaler adjusts the thickness of curves:
Line Width Scaler         = 1.0

# The AniLine Thickness Scaler sets the thickness of animated solution curves:
AniLine Thickness Scaler  = 3.0

# Background color:
Background Color  = 0.0,  0.0,   0.0

# Background transparency:
Background Transparency = 0.0
```

```
# Set the radius of the spheres used for labels:
# The normal size is 1.0.
# For smaller radius, use 0.xxx
# For bigger radius, use  X.XXX
Label Sphere Radius       =  1.0

# Disk Rotation
Disk Rotation = 1.0, 0.0, 0.0, 1.570796

# Disk Position
Disk Position = 0.0, 0.0, 0.0

# Disk Radius
Disk Radius = 1.0

# Disk Height
Disk Height = 0.001

# Disk Transparency [0, 1]
Disk Transparency = 0.7

# Read Disk From File
Disk From File = No

# Sphere Position
Sphere Position = 0.0, 0.0, 0.0

# Sphere Radius
Sphere Radius = 1.0

# Sphere Transparency [0, 1]
Sphere Transparency = 0.7

# Read Sphere From File
Sphere From File = No

# Axes color:
X Axis Color    = 1.0,  0.0,   0.0
Y Axis Color    = 0.0,  1.0,   0.0
Z Axis Color    = 0.0,  0.0,   1.0

# Color of the satellite, large primary, and small primary in animation:
satellite Color             = 1.0,  0.0,   0.0
large primary Color         = 0.0,  1.0,   0.0
large primary tail Color    = 0.0,  1.0,   1.0
small primary Color         = 0.0,  0.0,   1.0
small primary tail Color    = 0.5,  0.5,   0.0

# Surface color:
Surface Color         =  0.0,  1.0,  0.0

# Stable solution color:
Stable Solution Color = 0.0, 0.0, 1.0
```

```
# Stable solution color:
Unstable Solution Color = 1.0, 0.0, 0.0

# Set the radius of the satellite, large primary, and small primary:
# The normal size is 1.0.
# For smaller radius, use 0.xxx
# For bigger radius, use  X.XXX
Satellite Radius        = 1.0
Large Primary Radius    = 1.0
Small Primary Radius    = 1.0
Libration Point Size    = 1.0

# Set the initial, maximum and minimum satellite animation speed:
Sat Animation Speed = 100
Sat Max Animation Speed = 100
Sat Min Animation Speed = 0

# Set the initial, maximum and minimum orbit-change animation speed:
Orbit Animation Speed = 50
Orbit Max Animation Speed = 100
Orbit Min Animation Speed = 0

# Set the active AUTO parameter indices:
parameter ID = 10

# Choose 3D or 2D graph:
#3D  = Yes

# Choose 3D or 2D graph for the bifurcation diagram:
3DBif = Yes

# Choose 3D or 2D graph for the solution diagram:
3DSol = Yes
```

8.3 Example

In this example, we want to view a CR3BP data set. We want the diagram to show the "x" component on the X-axis, "y" component on the Y-axis, and "z" component on the Z-axis for the solution diagram. In the CR3BP, we use the parameters "1 2 3 10 21 22 23" in the AUTO calculations, and we also want to be able to use these to color the diagram, so we set the "parameter indices".

Other preferences include

- The diagram is drawn using Tubes.
- Coordinate axes are not drawn.
- No animation.
- Reference plane, libration points, and primaries are drawn.

- All labels are shown.
- Data is not normalized.

The settings are the settings in the resource file are then as follows:

```
#  Initialize the default options
Draw Reference Plane  = Yes
Orbit Animation       = No
Satellite Animation   = No
Draw Primaries        = Yes
Draw Libration Points = Yes
Normalize Data        = No
Draw Background       = No

#  Initialize the default graph type
#  0 --- Solution(fort.8)  1 --- Bifurcation(fort.7)
Graph Type    = 0

# initialize the default graph style
#  0 --- LINES,  1 --- TUBES, 2 ---- SURFACE  3--- nurbs curve
graph Style  =  1

# set X, Y, Z, and Label
# 0 is Time for X,Y,Z. 0 is "All" for Label

Solution X Axis              = 1
Solution Y Axis              = 2
Solution Z Axis              = 3

Labels                       = 0

#set the parameter indices
parameter ID = 1,  2,  3, 10,  15, 21, 22,  23
```

Based on the above settings, the solution diagram for the CR3BP family Ł1 for $\mu = 0.01215$ appears in Figure 8.11.

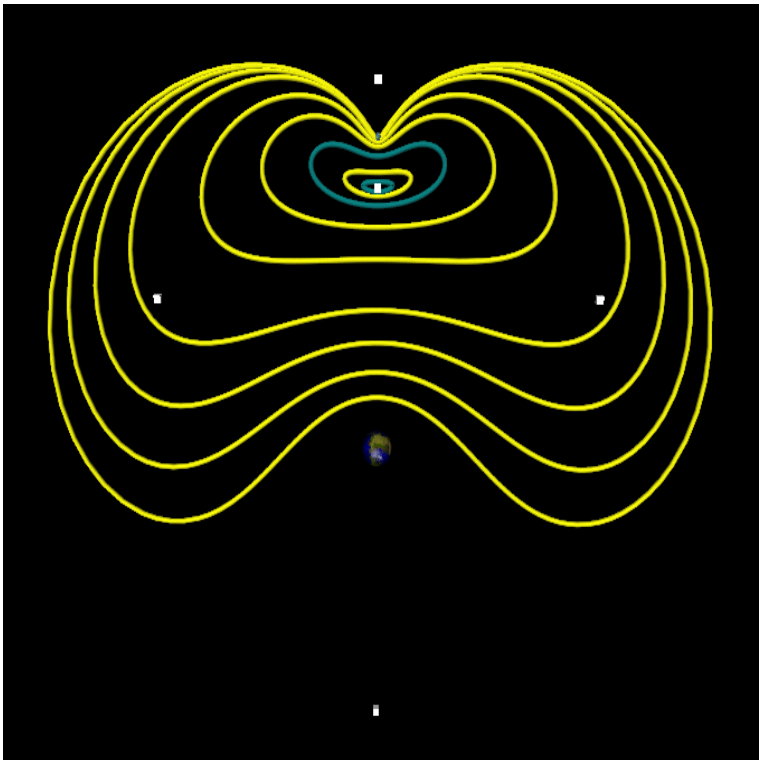


Figure 8.11: Example

Chapter 9

The Graphical User Interface GUI94.

9.1 General Overview.

The AUTO graphical user interface (GUI) is a tool for creating and editing equations-files and constants-files; see Section 3 for a description of these files. The GUI can also be used to run AUTO and to manipulate and plot output-files and data-files; see Section 5 for corresponding commands. To use the GUI for a new equation, change to an empty work directory. For an existing equations-file, change to its directory. (*Do not activate the GUI in the directory* `auto/07p` *or in any of its subdirectories.*) Then type

@*auto*,

or its abbreviation @*a*. Here we assume that the AUTO aliases have been activated; see Section 1.1. The GUI includes a window for editing the equations-file, and four groups of buttons, namely, the *Menu Bar* at the top of the GUI, the *Define Constants*-buttons at the center-left, the *Load Constants*-buttons at the lower left, and the *Stop- and Exit*-buttons.

Note : Most GUI buttons are activated by point-and-click action with the *left* mouse button. If a beep sound results then the *right* mouse button must be used.

9.1.1 The Menu bar.

It contains the main buttons for running AUTO and for manipulating the equations-file, the constants-file, the output-files, and the data-files. In a typical application, these buttons are used from left to right. First the *Equations* are defined and, if necessary, *Edited*, before being *Written*. Then the AUTO-constants are *Defined*. This is followed by the actual *Run* of AUTO. The resulting output-files can be *Saved* as data-files, or they can be *Appended* to existing data-files. Data-files can be *Plotted* with the graphics program PLAUT, and various file operations can be done with the *Files*-button. Auxiliary functions are provided by the *Demos*-, *Misc*-, and *Help*-buttons. The Menu Bar buttons are described in more detail in Section 9.2.

9.1.2 The Define-Constants-buttons.

These have the same function as the *Define*-button on the Menu Bar, namely to set and change AUTO-constants. However, for the *Define*-button all constants appear in one panel, while for

the Define Constants-buttons they are grouped by function, as in Chapter 10, namely *Problem* definition constants, *Discretization* constants, convergence *Tolerances*, continuation *Step Size*, diagram *Limits*, designation of free *Parameters*, constants defining the *Computation*, and constants that specify *Output* options.

9.1.3 The Load-Constants-buttons.

The *Previous*-button can be used to load an existing AUTO-constants file. Such a file is also loaded, if it exists, by the *Equations*-button on the *Menu Bar*. The *Default*-button can be used to load default values of all AUTO-constants. Custom editing is normally necessary.

9.1.4 The Stop- and Exit-buttons.

The *Stop*-button can be used to abort execution of an AUTO-run. This should be done only in exceptional circumstances. Output-files, if any, will normally be incomplete and should be deleted. Use the *Exit*-button to end a session.

9.2 The Menu Bar.

9.2.1 Equations-button.

This pull-down menu contains the items *Old*, to load an existing equations-file, *New*, to load a model equations-file, and *Demo*, to load a selected demo equations-file. Equations-file names are of the form `xxx.f90`. The corresponding constants-file `c.xxx` is also loaded if it exists. The equation name `xxx` remains active until redefined.

9.2.2 Edit-button.

This pull-down menu contains the items *Cut* and *Copy*, to be performed on text in the GUI window highlighted by click-and-drag action of the mouse, and the item *Paste*, which places editor buffer text at the location of the cursor.

9.2.3 Write-button.

This pull-down menu contains the item *Write*, to write the loaded files `xxx.f90` and `c.xxx`, by the active equation name, and the item *Write As* to write these files by a selected new name, which then becomes the active name.

9.2.4 Define-button.

Clicking this button will display the full AUTO-constants panel. Most of its text fields can be edited, but some have restricted input values that can be selected with the right mouse button. Some text fields will display a subpanel for entering data. To actually apply changes made in the panel, click the *OK*- or *Apply*-button at the bottom of the panel.

9.2.5 Run-button.

Clicking this button will write the constants-file `c.xxx` and run AUTO. If the equations-file has been edited then it should first be rewritten with the *Write*-button.

9.2.6 Save-button.

This pull-down menu contains the item *Save*, to save the output-files `fort.7`, `fort.8`, `fort.9`, as `b.xxx`, `s.xxx`, `d.xxx`, respectively. Here `xxx` is the active equation name. It also contains the item *Save As*, to save the output-files under another name. Existing data-files with the selected name, if any, will be overwritten.

9.2.7 Append-button.

This pull-down menu contains the item *Append*, to append the output-files `fort.7`, `fort.8`, `fort.9`, to existing data-files `b.xxx`, `s.xxx`, `d.xxx`, respectively. Here `xxx` is the active equation name. It also contains the item *Append To*, to append the output-files to other existing data-files.

9.2.8 Plot-button.

This pull-down menu contains the items *Plot*, to run the plotting program PLAUT for the data-files `b.xxx` and `s.xxx`, where `xxx` is the active equation name, and the item *Name*, to run PLAUT with other data-files.

9.2.9 Files-button.

This pull-down menu contains the item *Restart*, to redefine the restart file. Normally, when restarting from a previously computed solution, the restart data is expected in the file `s.xxx`, where `xxx` is the active equation name. Use the *Restart*-button to read the restart data from another data-file in the immediately following run. The pull-down menu also contains the following items :

- *Copy*, to copy `b.xxx`, `s.xxx`, `d.xxx`, `c.xxx`, to `b.yyy`, `s.yyy`, `d.yyy`, `c.yyy`, resp.;
- *Append*, to append data-files `b.xxx`, `s.xxx`, `d.xxx`, to `b.yyy`, `s.yyy`, `d.yyy`, resp.;
- *Move*, to move `b.xxx`, `s.xxx`, `d.xxx`, `c.xxx`, to `b.yyy`, `s.yyy`, `d.yyy`, `c.yyy`, resp.;
- *Delete*, to delete data-files `b.xxx`, `s.xxx`, `d.xxx`;
- *Clean*, to delete all files of the form `fort.*`, `*.o`, and `*.exe`.

9.2.10 Demos-button.

This pulldown menu contains the items *Select*, to view and run a selected AUTO demo in the demo directory, and *Reset*, to restore the demo directory to its original state. Note that demo files can be copied to the user work directory with the *Equations/Demo*-button.

9.2.11 Misc.-button.

This pulldown menu contains the items *Tek Window* and *VT102 Window*, for opening windows; *Emacs* and *Xedit*, for editing files, and *Print*, for printing the active equations-file `xxx.f90`.

9.2.12 Help-button.

This pulldown menu contains the items *AUTO-constants*, for help on AUTO-constants, and *User Manual*, for viewing the user manual; i.e., this document.

9.3 Using the GUI.

AUTO-commands are described in Section 5 and illustrated in the demos. In Table 9.1 we list the main AUTO-commands together with the corresponding GUI button.

`@r`	*Run*
`@sv`	*Save*
`@ap`	*Append*
`@p`	*Plot*
`@cp`	*Files/Copy*
`@mv`	*Files/Move*
`@cl`	*Files/Clean*
`@dl`	*Files/Delete*
`@dm`	*Equations/Demo*

Table 9.1: Command Mode - GUI correspondences.

The AUTO-command `@r xxx yyy` is given in the GUI as follows : click *Files/Restart* and enter *yyy* as data. Then click *Run*. As noted in Section 5, this will run AUTO with the current equations-file `xxx.f90` and the current constants-file `c.xxx`, while expecting restart data in `s.yyy`. The AUTO-command `@ap xxx yyy` is given in the GUI by clicking *Files/Append*.

9.4 Customizing the GUI.

9.4.1 Print-button.

The *Misc/Print*-button on the Menu Bar can be customized by editing the file `GuiConsts.h` in directory `auto/07p/include`.

9.4.2 GUI colors.

GUI colors can be customized by creating an X resource file. Two model files can be found in directory `auto/07p/gui`, namely, `Xdefaults.1` and `Xdefaults.2`. To become effective, edit

one of these, if desired, and copy it to `.Xdefaults` in your home directory. Color names can often be found in the system file `/usr/lib/X11/rgb.txt`.

9.4.3 On-line help.

The file `auto/07p/include/GuiGlobal.h` contains on-line help on AUTO-constants and demos. The text can be updated, subject to a modifiable maximum length. On SGI machines this is 10240 bytes, which can be increased, for example, to 20480 bytes, by replacing the line *CC = cc -Wf, -XNl10240 -O* in `auto/07p/gui/Makefile` by *CC = cc -Wf, -XNl20480 -O* On other machines, the maximum message length is the system defined maximum string literal length.

Chapter 10

Description of AUTO-Constants.

10.1 The AUTO-Constants File.

As described in Section 3, if the equations-file is `xxx.f90` then the constants that define the computation are normally expected in the file `c.xxx`. The format of this file is free, with constant=value pairs separated by commas or spaces. Comments start with one of the characters "#" and "!", and run to the end of a line. An example, with default values, is listed below. In real constant files you only need to specify those values that are different from these, but listing all of them allows for easier editing.

Note that this file is not strictly necessary when using the Python interface: you can define all constants inside the scripts if you so wish.

```
# Default AUTO Constants file
e = '', s='', dat='', sv=''
unames = {}, parnames = {}
U = {}, PAR = {}
NDIM=   2, IPS =   1, IRS =   0, ILP =   1
ICP =  [1]
NTST=  20, NCOL=   4, IAD =   3, ISP =   2, ISW = 1, IPLT= 0, NBC= 0, NINT= 0
NMX=    0, NPR=    0, MXBF=  10, IID =   2, ITMX= 9, ITNW= 5, NWTN= 3, JAC= 0
EPSL= 1e-07, EPSU = 1e-07, EPSS = 1e-05
DS  =  0.01, DSMIN= 0.005, DSMAX=   0.1, IADS=   1
NPAR=  36, THL =  {}, THU =  {}
RL0=-1.7976e+308, RL1=1.7976e+308, A0=-1.7976e+308, A1=1.7976e+308,
UZR = {}, UZSTOP = {}, SP = [], STOP = []
IIS = 3, IBR=0, LAB=0, TY=''
NUNSTAB = -1, NSTAB = -1, IEQUIB = 1, ITWIST = 0, ISTART = 5
IREV = [], IFIXED = [], IPSI = []
```

The significance of the AUTO-constants, grouped by function, is described in the sections below. The HomCont constants `NUNSTAB`, `NSTAB`, `IEQUIB`, `ITWIST`, `ISTART`, `IREV`, `IFIXED`, and `IPSI` are explained in Chapter 20. Representative demos that illustrate use of the AUTO-constants are also mentioned.

10.2 Problem Constants.

10.2.1 NDIM

Dimension of the system of equations as specified in the user-supplied routine FUNC.

10.2.2 NBC

The number of boundary conditions as specified in the user-supplied routine BCND.
(Demos exp, kar.)

10.2.3 NINT

The number of integral conditions as specified in the user-supplied routine ICND.
(Demos int, lin, obv.)

10.2.4 NPAR

Maximum parameter number that can be used in all user-supplied routines.

10.2.5 JAC

Used to indicate whether derivatives are supplied by the user or to be obtained by differencing :

- JAC=0 : No derivatives are given by the user. (Most demos use JAC=0.)
- JAC=1 : Derivatives with respect to state- and problem-parameters are given in the user-supplied routines FUNC, BCND, ICND and FOPT, where applicable. This may be necessary for sensitive problems. It is also recommended for computations in which AUTO generates an extended system, for example, when ISW=2. (For ISW see Section 10.8.4.)
  (Demos int, dd2, obt, plp, ops.)
- JAC=-1 : As for JAC=1, but derivatives with respect to problem-parameters may be omitted in FUNC.
  (Demo san.)

10.3 Discretization Constants.

10.3.1 NTST

The number of mesh intervals used for discretization. NTST remains fixed during any particular run, but can be changed when restarting. (For mesh adaption see IAD in Section 10.3.3.) Recommended value of NTST : As small as possible to maintain convergence.
(Demos exp, ab, spb.)

10.3.2 NCOL

The number of Gauss collocation points per mesh interval, ($2 \leq$ NCOL ≤ 7). NCOL remains fixed during any given run, but can be changed when restarting at a previously computed solution. The choice NCOL=4, used in most demos, is recommended. If NDIM is "large" and the solutions "very smooth" then NCOL=2 may be appropriate.

10.3.3 IAD

This constant controls the mesh adaption :

- IAD=0 : Fixed mesh. Normally, this choice should never be used, as it may result in spurious solutions. (Demo ext.)

- IAD>0 : Adapt the mesh every IAD steps along the family. Most demos use IAD=3, which is the strongly recommended value.

When computing "trivial" solutions to a boundary value problem, for example, when all solution components are constant, then the mesh adaption may fail under certain circumstances, and overflow may occur. In such case, try recomputing the solution family with a fixed mesh (IAD=0). Be sure to set IAD back to IAD=3 for computing eventual non-trivial bifurcating solution families.

10.4 Tolerances.

10.4.1 EPSL

Relative convergence criterion for equation parameters in the Newton/Chord method. Most demos use EPSL=10^{-6} or EPSL=10^{-7}, which is the recommended value range.

10.4.2 EPSU

Relative convergence criterion for solution components in the Newton/Chord method. Most demos use EPSU=10^{-6} or EPSU=10^{-7}, which is the recommended value range.

10.4.3 EPSS

Relative arclength convergence criterion for the detection of special solutions. Most demos use EPSS=10^{-4} or EPSS=10^{-5}, which is the recommended value range. Generally, EPSS should be approximately 100 to 1000 times the value of EPSL, EPSU.

10.4.4 ITMX

The maximum number of iterations allowed in the accurate location of special solutions, such as bifurcations, folds, and user output points, by Müller's method with bracketing. The recommended value is ITMX=8, used in most demos.

10.4.5 NWTN

After NWTN Newton iterations the Jacobian is frozen, i.e., AUTO uses full Newton for the first NWTN iterations and the Chord method for iterations NWTN+1 to ITNW. The choice NWTN=3 is strongly recommended and used in most demos. Note that this constant is only effective for ODEs, i.e., for solving the piecewise polynomial collocation equations. For algebraic systems AUTO always uses full Newton.

10.4.6 ITNW

The maximum number of combined Newton-Chord iterations. When this maximum is reached, the step will be retried with half the stepsize. This is repeated until convergence, or until the minimum stepsize is reached. In the latter case the computation of the family is discontinued and a message printed in fort.9. The recommended value is ITNW=5, but ITNW=7 may be used for "difficult" problems, for example, demos spb, chu, plp, etc.

10.5 Continuation Step Size.

10.5.1 DS

AUTO uses pseudo-arclength continuation for following solution families. The pseudo-arclength stepsize is the distance between the current solution and the next solution on a family. By default, this distance includes all state variables (or state functions) and all free parameters. The constant DS defines the pseudo-arclength stepsize to be used for the first attempted step along any family. (Note that if IADS>0 then DS will automatically be adapted for subsequent steps and for failed steps.) DS may be chosen positive or negative; changing its sign reverses the direction of computation. The relation $\texttt{DSMIN} \leq |\ \texttt{DS}\ | \leq \texttt{DSMAX}$ must be satisfied. The precise choice of DS is problem-dependent; the demos use a value that was found appropriate after some experimentation.

10.5.2 DSMIN

This is minimum allowable absolute value of the pseudo-arclength stepsize. DSMIN must be positive. It is only effective if the pseudo-arclength step is adaptive, i.e., if IADS>0. The choice of DSMIN is highly problem-dependent; most demos use a value that was found appropriate after some experimentation. See also the discussion in Section 11.2.

10.5.3 DSMAX

The maximum allowable absolute value of the pseudo-arclength stepsize. DSMAX must be positive. It is only effective if the pseudo-arclength step is adaptive, i.e., if IADS>0. The choice of DSMAX is highly problem-dependent; most demos use a value that was found appropriate after some experimentation. See also the discussion in Section 11.2.

10.5.4 IADS

This constant controls the frequency of adaption of the pseudo-arclength stepsize.

- IADS=0 : Use fixed pseudo-arclength stepsize, i.e., the stepsize will be equal to the specified value of DS for every step. The computation of a family will be discontinued as soon as the maximum number of iterations ITNW is reached. This choice is not recommended. (Demo tim.)

- IADS>0 : Adapt the pseudo-arclength stepsize after every IADS steps. If the Newton/Chord iteration converges rapidly then | DS | will be increased, but never beyond DSMAX. If a step fails then it will be retried with half the stepsize. This will be done repeatedly until the step is successful or until | DS | reaches DSMIN. In the latter case nonconvergence will be signalled. The strongly recommended value is IADS=1, which is used in almost all demos.

10.5.5 THL

By default, the pseudo-arclength stepsize includes all state variables (or state functions) and all free parameters. Under certain circumstances one may want to modify the weight accorded to individual parameters in the definition of stepsize. For this purpose, THL defines the parameters whose weight is to be modified. If THL={} then all weights will have default value 1.0, else one must enter pairs, {*Parameter Index* : *Weight*, ...} .

For example, for the computation of periodic solutions it is recommended that the period not be included in the pseudo-arclength continuation stepsize, in order to avoid period-induced limitations on the stepsize near orbits of infinite period. This exclusion can be accomplished by setting THL={11:0.0}. If THL is not specified this is the default for computing periodic solutions (IPS=2). Most demos that compute periodic solutions use this option; see for example demo ab.

10.5.6 THU

Under certain circumstances one may want to modify the weight accorded to individual state variables (or state functions) in the definition of stepsize. For this purpose, THU defines the number of states whose weight is to be modified. If THU={} then all weights will have default value 1.0, else one must enter pairs, {*State Index* : *Weight*, ...} . At present none of the demos use this option.

10.6 Diagram Limits.

There are five ways to limit the computation of a family :

- By specifying a stopping condition in the list associated with the constant STOP; see Section 10.6.1.

- By specifying parameters and parameter values in the list associated with the constant UZSTOP; see Section 10.9.12.

- By specifying the maximum number of steps, NMX.

- By specifying a negative parameter index in the list associated with the constant UZR; see Section 10.9.11.

- By appropriate choice of the computational window defined by the constants RL0, RL1, A0, and A1. One should always check that the starting solution lies within this computational window, otherwise the computation will stop immediately at the starting point. Most demos do not specify these constants, and use an unbounded window.

10.6.1 STOP

This constant adds stopping conditions. It is specified as a list of bifurcation type strings followed by a number n greater than zero. These strings specify that the contination should stop as soon as the nth bifurcation of the associated type has been reached. Example:

`STOP=['HB3','UZ3']` Stop at the third Hopf bifurcation or third user defined point (see Section 10.9.11), whichever comes first.

10.6.2 NMX

The maximum number of steps to be taken along any family.

10.6.3 RL0

The lower bound on the principal continuation parameter. (This is the parameter which appears first in the ICP list; see Section 10.7.1.).

10.6.4 RL1

The upper bound on the principal continuation parameter.

10.6.5 A0

The lower bound on the principal solution measure. (By default, if IPLT=0, the principal solution measure is the L_2-norm of the state vector or state vector function. See the AUTO-constant IPLT in Section 10.9.10 for choosing another principal solution measure.)

10.6.6 A1

The upper bound on the principal solution measure.

10.7 Free Parameters.

10.7.1 ICP

For each equation type and for each continuation calculation there is a typical ("generic") number of problem parameters that must be allowed to vary, in order for the calculations to be properly posed. The array ICP designates these free parameters. The parameter that appears first in the ICP list is called the "principal continuation parameter". Specific examples and special cases are described below.

10.7.2 Fixed points.

The simplest case is the continuation of a solution family to the system $f(u,p) = 0$, where $f(\cdot,\cdot), u \in \mathrm{R}^n$, cf. Equation (2.1). Such a system arises in the continuation of ODE stationary solutions and in the continuation of fixed points of discrete dynamical systems. There is only one free parameter here.

As a concrete example, consider Run 1 of demo ab, where ICP=[1]. Thus, in this run PAR(1) is designated as the free parameter.

10.7.3 Periodic solutions and rotations.

The continuation of periodic solutions and rotations generically requires two parameters, namely, one problem parameter and the period. For example, in Run 2 of demo ab we have ICP=[1,11]. Thus, in this run, the free parameters are PAR(1) and PAR(11). (Note that AUTO reserves PAR(11) for the period.)

Actually, for periodic solutions, it is sufficient to only specify the index of the free problem parameter, as AUTO will automatically add PAR(11). However, in this case the period will not appear in the screen output and in the fort.7 output-file.

For fixed period orbits one must specify two free problem parameters. For example, in Run 7 of demo pp2, we have ICP=[1,2], with PAR(1) and PAR(2) specified as free problem parameters. The period PAR(11) is fixed in this run. If the period is large then such a continuation provides a simple and effective method for computing a locus of homoclinic orbits.

10.7.4 Folds and Hopf bifurcations.

The continuation of folds for algebraic problems and the continuation of Hopf bifurcations requires two free problem parameters. For example, to continue a fold in Run 3 of demo ab, we have ICP=[1,3], with PAR(1) and PAR(3) specified as free parameters. Note that one must set ISW=2 for computing such loci of special solutions. Also note that in the continuation of folds the principal continuation parameter must be the one with respect to which the fold was located.

10.7.5 Folds and period-doublings.

The continuation of folds, for periodic orbits and rotations, and the continuation of period-doubling and torus bifurcations require two free problem parameters plus the free period. Thus, one would normally specify three parameters. For example, in Run 6 of demo `pen`, where a locus of period-doubling bifurcations is computed for rotations, we have `ICP=[2,3,11]`, with `PAR(2)`, `PAR(3)`, and `PAR(11)` specified as free parameters. Note that one must set `ISW`=2 for computing such loci of special solutions. Also note that in the continuation of folds the principal continuation parameter must be the one with respect to which the fold was located.

Actually, one may only specify the problem parameters, as AUTO will automatically add the period. For example, in Run 3 of demo `plp`, where a locus of folds is computed for periodic orbits, we have `ICP=[4,1]`, with `PAR(4)` and `PAR(1)` specified as free parameters. However, in this case the period will not appear in the screen output and in the `fort.7` output-file.

To continue a locus of folds, period-doubling or torus bifurcations with fixed period, simply specify three problem parameters, not including `PAR(11)`. For torus bifurcations it is also possible to specify four problem parameters (possibly including `PAR(11)`). In that case the angle of the torus (`PAR(12)`) stays fixed.

10.7.6 Boundary value problems.

The simplest case is that of boundary value problems where `NDIM`=`NBC` and where `NINT`=0. Then, generically, one free problem parameter is required for computing a solution family. For example, in demo `exp`, we have `NDIM`=`NBC`=2, `NINT`=0. Thus, in this demo one free parameter is designated, namely `PAR(1)`.

More generally, for boundary value problems with integral constraints, the generic number of free parameters is `NBC` + `NINT`−`NDIM` +1. For example, in demo `lin`, we have `NDIM`=2, `NBC`=2, and `NINT`=1. Thus `ICP=[1,3]`. Indeed, in this demo two free parameters are designated, namely `PAR(1)` and `PAR(3)`.

10.7.7 Boundary value folds.

To continue a locus of folds for a general boundary value problem with integral constraints, set `#ICP`=`NBC`+`NINT`−`NDIM`+2, and specify this number of parameter indices to designate the free parameters.

10.7.8 Optimization problems.

In algebraic optimization problems one must set `ICP`(1)=10, as AUTO uses `PAR(10)` as principal continuation parameter to monitor the value of the objective function. Furthermore, one must designate one free equation parameter in `ICP`(2). Thus, `ICP=[10,2]` in the first run.

Folds with respect to `PAR(10)` correspond to extrema of the objective function. In a second run one can restart at such a fold, with an additional free equation parameter specified in `ICP`(3). Thus, `ICP=[10,2,3]` in the second run.

The above procedure can be repeated. For example, folds from the second run can be continued in a third run with three equation parameters specified in addition to `PAR(10)`. Thus,

#ICP=4 in the third run.

For a simple example see demo opt, where a four-parameter extremum is located. Note that #ICP=5 in each of the four constants-files of this demo, with the indices of PAR(10) and PAR(1)-PAR(4) specified in ICP. Thus, in the first three runs, there are overspecified parameters. However, AUTO will always use the correct number of parameters. Although the overspecified parameters will be printed, their values will remain fixed.

10.7.9 Internal free parameters.

The actual continuation scheme in AUTO may use additional free parameters that are automatically added. The simplest example is the computation of periodic solutions and rotations, where AUTO automatically puts the period, if not specified, in PAR(11). The computation of loci of folds, Hopf bifurcations, and period-doublings also requires additional internal continuation parameters. These will be automatically added, and their indices will be greater than NPAR. Other use depends on IPS: see Section 11.1.

10.7.10 Parameter overspecification.

The number of specified parameter indices is allowed to be be greater than the generic number. In such case there will be "overspecified" parameters, whose values will appear in the screen and fort.7 output, but which are not part of the continuation process. A simple example is provided by demo opt, where the first three runs have overspecified parameters whose values, although constant, are printed.

There is, however, a more useful application of parameter overspecification. In the user-supplied routine PVLS one can define solution measures and assign these to otherwise unused parameters. Such parameters can then be overspecified, in order to print them on the screen and in the fort.7 output. It is important to note that such overspecified parameters must appear at the end of the ICP list, as they cannot be used as true continuation parameters.

For an example of using parameter overspecification for printing user-defined solution measures, see demo pvl. This is a boundary value problem (Bratu's equation) which has only one true continuation parameter, namely PAR(1). Three solution measures are defined in the routine PVLS, namely, the L_2-norm of the first solution component, the minimum of the second component, and the left boundary value of the second component. These solution measures are assigned to PAR(2), PAR(3), PAR(4), and PAR(5), respectively. In the constants-file c.pvl we have #ICP=5, with PAR(1)-PAR(5) specified as parameters. Thus, in this example, PAR(2)-PAR(5) are overspecified. Note that PAR(1) must appear first in the ICP list; the other parameters cannot be used as true continuation parameters.

10.8 Computation Constants.

10.8.1 ILP

- ILP=0 : No detection of folds. This choice is recommended.

- ILP=1 : Detection of folds. To be used if subsequent fold continuation is intended.

10.8.2 SP

This constant controls the detection of bifurcations and adds stopping conditions. It is specified as a list of bifurcation type strings followed by an optional number. If this number is 0, then the detection of this bifurcation is turned off, and if it is missing then the detection is turned on. A number n greater than zero specifies that the contination should stop as soon as the nth bifurcation of this type has been reached. Examples:

- `SP=['LP0']`
  turn off detection of folds.

- `SP=['LP','HB3','BP0','UZ3']`
  turn on the detection of folds and Hopf bifurcations, turn off detection of branch points and stop at the third Hopf bifurcation or third user defined point, whichever comes first.

10.8.3 ISP

This constant controls the detection of Hopf bifurcations, branch points, period-doubling bifurcations, and torus bifurcations.

- `ISP=0` : This setting disables the detection of Hopf bifurcations, branch points, period-doubling bifurcations, and torus bifurcations and the computation of Floquet multipliers.

- `ISP=1` : Branch points and Hopf bifurcations are detected for algebraic equations. Branch points, period-doubling bifurcations and torus bifurcations are not detected for periodic solutions and boundary value problems. However, Floquet multipliers are computed.

- `ISP=2` : This setting enables the detection of all special solutions. For periodic solutions and rotations, the choice `ISP`=2 should be used with care, due to potential inaccuracy in the computation of the linearized Poincaré map and possible rapid variation of the Floquet multipliers. The linearized Poincaré map always has a multiplier $z = 1$. If this multiplier becomes inaccurate then the automatic detection of secondary periodic bifurcations will be discontinued and a warning message will be printed in `fort.9`. See also Section 11.4.

- `ISP=3` : Hopf bifurcations will not be detected. Branch points will be detected, and AUTO will monitor the Floquet multipliers. Period-doubling and torus bifurcations will go undetected. This option is useful for certain problems with non-generic Floquet behavior.

- `ISP=4` : Branch points and Hopf bifurcations are detected for algebraic equations. Branch points are not detected for periodic solutions and boundary value problems. AUTO will monitor the Floquet multipliers, and period-doubling and torus bifurcations will be detected.

10.8.4 ISW

This constant controls branch switching at branch points for the case of differential equations. Note that branch switching is automatic for algebraic equations.

- ISW=1 : This is the normal value of ISW.

- ISW=−1 : If IRS is the label of a branch point or a period-doubling bifurcation then branch switching will be done. For period doubling bifurcations it is recommended that NTST be increased. For examples see Run 2 and Run 3 of demo lor, where branch switching is done at period-doubling bifurcations, and Run 2 and Run 3 of demo bvp, where branch switching is done at a transcritical branch point.

- ISW=2 : If IRS is the label of a fold, a Hopf bifurcation point, a period-doubling, a torus bifurcation, or, in a non-generic (symmetric) system, a branch point then a locus of such points will be computed. An additional free parameter must be specified for such continuations; see also Section 10.7.

- ISW=3 : If IRS is the label of a branch point in a generic (non-symmetric) system then a locus of such points will be computed. Two additional free parameters must be specified for such continuations; see also Section 10.7.

10.8.5 MXBF

This constant, which is effective for algebraic problems only, sets the maximum number of bifurcations to be treated. Additional branch points will be noted, but the corresponding bifurcating families will not be computed. If MXBF is positive then the bifurcating families of the first MXBF branch points will be traced out in both directions. If MXBF is negative then the bifurcating families of the first | MXBF | branch points will be traced out in only one direction.

10.8.6 s

This constant sets the name of the solution file from which the computation is to be restarted, instead of fort.3: if s='xxx' then the name of the restart file is s.xxx.

10.8.7 dat

This constant, where dat='xxx', sets the name of a user-supplied ASCII data file xxx.dat, from which the contination is to be restarted. AUTO automatically sets the period in PAR(11). Other parameter values must be set in STPNT. (When necessary, PAR(11) may also be redefined there.)

The first column in the data file denotes the time, which does *not* need to be rescaled to the interval $[0, 1]$, and further columns the coordinates of the solution. The constant IRS must be set to 0.

(Demos lor, pen.)

10.8.8 U

This constant, where U={i1:x1,i2:x2}, changes the value of U(i1) to x1, U(i2) to x2, and so on, with respect to the solution to start from. This is only valid for restarting from algebraic problems or a constant-in-time solution.

10.8.9 PAR

This constant, where PAR={i1:x1,i2:x2}, changes the value of PAR(i1) to x1, PAR(i2) to x2, and so on, with respect to the solution to start from.

10.8.10 IRS

This constant sets the label of the solution where the computation is to be restarted.

- IRS=0 : This setting is typically used in the first run of a new problem. In this case a starting solution must be defined in the user-supplied routine STPNT. For representative examples of analytical starting solutions see demos ab and frc. For starting from unlabeled numerical data see the dat command above, and demos lor and pen.

- IRS>0 : Restart the computation at the previously computed solution with label IRS. This solution is normally expected to be in the current data-file s.xxx; see also the @r and @R commands in Section 5. Various AUTO-constants can be modified when restarting.

- IRS<0 : Restart the computation at the -IRSth computed solution in the restart file. This is especially useful if you do not want to look up label numbers and know for sure that the solution to continue from is at a fixed position.

- IRS='XYn' : Restart the computation at the nth label of type XY in the restart file, for instance 'HB12' to restart at the twelfth Hopf bifurcation.

10.8.11 TY

This constant modifies the type from the restart solution. This is sometimes useful in conservative or extended systems, declaring a regular point to be a Hopf bifurcation point (TY='HB') or a branch point (TY='BP'). Use TY='HB4' to copy the period of the emanating periodic orbit from PAR(4) (for example set in the routine PVLS in the equations file) to PAR(11). (Demo r3b.)

10.8.12 IPS

This constant defines the problem type :

- IPS=0 : An algebraic bifurcation problem. Hopf bifurcations will not be detected and stability properties will not be indicated in the fort.7 output-file.

- IPS=1 : Stationary solutions of ODEs with detection of Hopf bifurcations. The sign of PT, the point number, in fort.7 is used to indicate stability : $-$ is stable , $+$ is unstable. (Demo ab.)

- IPS=$-$1 : Fixed points of the discrete dynamical system $u^{(k+1)} = f(u^{(k)}, p)$, with detection of Hopf bifurcations. The sign of PT in fort.7 indicates stability : $-$ is stable , $+$ is unstable. (Demo dd2.)

- IPS=−2 : Time integration using implicit Euler. The AUTO-constants DS, DSMIN, DSMAX, and ITNW, NWTN control the stepsize. In fact, pseudo-arclength is used for "continuation in time". Note that the time discretization is only first order accurate, so that results should be carefully interpreted. Indeed, this option has been included primarily for the detection of stationary solutions, which can then be entered in the user-supplied routine STPNT. (Demo ivp.)

- IPS=2 : Computation of periodic solutions. Starting data can be a Hopf bifurcation point (Run 2 of demo ab), a periodic orbit from a previous run (Run 4 of demo pp2), an analytically known periodic orbit (Run 1 of demo frc), or a numerically known periodic orbit (Demo lor). The sign of PT in fort.7 is used to indicate stability : − is stable , + is unstable or unknown.

- IPS=4 : A boundary value problem. Boundary conditions must be specified in the user-supplied routine BCND and integral constraints in ICND. The AUTO-constants NBC and NINT must be given correct values. (Demos exp, int, kar.)

- IPS=5 : Algebraic optimization problems. The objective function must be specified in the user-supplied routine FOPT. (Demo opt.)

- IPS=7 : A boundary value problem with computation of Floquet multipliers. This is a very special option; for most boundary value problems one should use IPS=4. Boundary conditions must be specified in the user-supplied routine BCND and integral constraints in ICND. The AUTO-constants NBC and NINT must be given correct values.

- IPS=9 : This option is used in connection with the HomCont algorithms described in Chapters 20-26 for the detection and continuation of homoclinic bifurcations. (Demos san, mtn, kpr, cir, she, rev.)

- IPS=11 : Spatially uniform solutions of a system of parabolic PDEs, with detection of traveling wave bifurcations. The user need only define the nonlinearity (in routine FUNC), initialize the wave speed in PAR(10), initialize the diffusion constants in PAR(15,16,···), and set a free equation parameter in ICP(1). (Run 2 of demo wav.)

- IPS=12 : Continuation of traveling wave solutions to a system of parabolic PDEs. Starting data can be a Hopf bifurcation point from a previous run with IPS=11, or a traveling wave from a previous run with IPS=12. (Run 3 and Run 4 of demo wav.)

- IPS=14 : Time evolution for a system of parabolic PDEs subject to periodic boundary conditions. Starting data may be solutions from a previous run with IPS=12 or 14. Starting data can also be specified in STPNT, in which case the wave length must be specified in PAR(11), and the diffusion constants in PAR(15,16,···). AUTO uses PAR(14) for the time variable. DS, DSMIN, and DSMAX govern the pseudo-arclength continuation in the space-time variables. Note that the time discretization is only first order accurate, so that results should be carefully interpreted. Indeed, this option is mainly intended for the detection of stationary waves. (Run 5 of demo wav.)

- IPS=15 : Optimization of periodic solutions. The integrand of the objective functional must be specified in the user supplied routine FOPT. Only PAR(1-9) should be used for problem parameters. PAR(10) is the value of the objective functional, PAR(11) the period, PAR(12) the norm of the adjoint variables, PAR(14) and PAR(15) are internal optimality variables. PAR(21-29) and PAR(31) are used to monitor the optimality functionals associated with the problem parameters and the period. Computations can be started at a solution computed with IPS=2 or IPS=15. For a detailed example see demo ops.

- IPS=16 : This option is similar to IPS=14, except that the user supplies the boundary conditions. Thus this option can be used for time-integration of parabolic systems subject to user-defined boundary conditions. For examples see the first runs of demos pd1, pd2, and bru. Note that the space-derivatives of the initial conditions must also be supplied in the user supplied routine STPNT. The initial conditions must satisfy the boundary conditions. This option is mainly intended for the detecting stationary solutions.

- IPS=17 : This option can be used to continue stationary solutions of parabolic systems obtained from an evolution run with IPS=16. For examples see the second runs of demos pd1 and pd2.

10.9 Output Control.

10.9.1 unames

This constant, where unames={i1:s1,i2:s2}, changes the names in all output from U(i1) to s1, from U(i2) to s2, and so on. You can also refer to these strings, instead of the corresponding indices, in the constants U and THU.

10.9.2 parnames

This constant, where parnames={i1:s1,i2:s2}, changes the names in all output from PAR(i1) to s1, from PAR(i2) to s2, and so on. You can also refer to these strings, instead of the corresponding indices, in the constants ICP, THL, and UZR.

10.9.3 e

This constant, where e='xxx', is only for use by post-processors: it denotes the name of the equations file and is stored in the bifurcation diagram file (fort.7), so restarts in the Python interface are possible without needing to specify the equations file.

10.9.4 sv

This constant specifies a string to write the output to instead of fort.7, fort.8, and fort.9: if sv='xxx', then the output files are b.xxx, s.xxx, and d.xxx.

10.9.5 NPR

This constant can be used to regularly write `fort.8` plotting and restart data. IF `NPR`>0 then such output is written every `NPR` steps. IF `NPR`=0 or if `NPR`≥`NMX` then no such output is written. Note that special solutions, such as branch points, folds, end points, etc., are always written in `fort.8`. Furthermore, one can specify parameter values where plotting and restart data is to be written; see Section 10.9.11. For these reasons, and to limit the output volume, it is recommended that `NPR` output be kept to a minimum.

10.9.6 IBR

This constant specifies the initial branch number `BR` that is used. The default `IBR=0` means that that this number is determined automatically.

10.9.7 LAB

This constant specifies the initial label number `LAB` that is used. The default `LAB=0` means that that this number is determined automatically. Using `LAB=1` means you do not need to relabel after a non-appended continuation if this is desired.

10.9.8 IIS

This constant controls the amount of information printed in `fort.8` : the greater `IIS` the more solutions contain the corresponding vector giving the direction of the branch. The direction of the branch is necessary for restart points when switching branches, but make the solution file almost two times bigger than necessary when switching branches is never performed from solutions in this file.

- `IIS=0` : The direction of the branch is never provided.
- `IIS=1` : The direction of the branch is only provided at special points from which branch switching can be performed (types LP (boundary value problems only), BP, PD, TR).
- `IIS=2` : The direction of the branch is provided at all special points but not at regular points without a type label.
- `IIS=3` : The direction of the branch is always provided. This is the default setting.

10.9.9 IID

This constant controls the amount of diagnostic output printed in `fort.9` : the greater `IID` the more detailed the diagnostic output.

- `IID=0` : No diagnostic output.
- `IID=1` : Minimal diagnostic output. This setting is not recommended.
- `IID=2` : Regular diagnostic output. This is the recommended value of `IID`.

- IID=3 : This setting gives additional diagnostic output for algebraic equations, namely the Jacobian and the residual vector at the starting point. This information, which is printed at the beginning of fort.9, is useful for verifying whether the starting solution in STPNT is indeed a solution.

- IID=4 : This setting gives additional diagnostic output for differential equations, namely the reduced system and the associated residual vector. This information is printed for every step and for every Newton iteration, and should normally be suppressed. In particular it can be used to verify whether the starting solution is indeed a solution. For this purpose, the stepsize DS should be small, and one should look at the residuals printed in the fort.9 output-file. (Note that the first residual vector printed in fort.9 may be identically zero, as it may correspond to the computation of the starting direction. Look at the second residual vector in such case.) This residual vector has dimension NDIM+NBC+NINT+1, which accounts for the NDIM differential equations, the NBC boundary conditions, the NINT user-defined integral constraints, and the pseudo-arclength equation. For proper interpretations of these data one may want to refer to the solution algorithm for solving the collocation system, as described in Doedel, Keller & Kernévez (1991*b*).

- IID=5 : This setting gives very extensive diagnostic output for differential equations, namely, debug output from the linear equation solver. This setting should not normally be used as it may result in a huge fort.9 file. It gives incomplete results when used in combination with MPI parallellization.

10.9.10 IPLT

This constant allows redefinition of the principal solution measure, which is printed as the second (real) column in the screen output and in the fort.7 output-file :

- If IPLT = 0 then the L_2-norm is printed. Most demos use this setting. For algebraic problems, the standard definition of L_2-norm is used. For differential equations, the L_2-norm is defined as

$$\sqrt{\int_0^1 \sum_{k=1}^{NDIM} U_k(x)^2 \, dx} \; .$$

  Note that the interval of integration is $[0, 1]$, the standard interval used by AUTO. For periodic solutions the independent variable is transformed to range from 0 to 1, before the norm is computed. The AUTO-constants THL(*) and THU(*) (see Section 10.5.5 and Section 10.5.6) affect the definition of the L_2-norm.

- If $0 <$ IPLT $\leq$ NDIM then the maximum of the IPLT'th solution component is printed.

- If $-$NDIM $\leq$ IPLT <0 then the minimum of the IPLT'th solution component is printed. (Demo fsh.)

- If NDIM $<$ IPLT $\leq$ 2*NDIM then the integral of the (IPLT$-$NDIM)'th solution component is printed. (Demos exp, lor.)

- If 2*NDIM $<$ IPLT $\leq$ 3*NDIM then the L_2-norm of the (IPLT−NDIM)'th solution component is printed. (Demo frc.)

Note that for algebraic problems the maximum and the minimum are identical. Also, for ODEs the maximum and the minimum of a solution component are generally much less accurate than the L_2-norm and component integrals. Note also that the routine PVLS provides a second, more general way of defining solution measures; see Section 10.7.10.

10.9.11 UZR

This constant allows the setting of parameter values at which labeled plotting and restart information is to be written in the fort.8 output-file. Optionally, it also allows the computation to terminate at such a point.

- Set UZR={} if no such output is needed. Many demos use this setting.
- Else one must enter pairs, {*Parameter-Index* : *Parameter-Value*, ...} ,
  or indices with lists of values, {*Parameter-Index* :

  $$Parameter - Value, ...$$

  , ...} , to designate the parameters and the parameter values at which output is to be written. For examples see demos exp, int, and fsh.
- If such a parameter index is preceded by a minus sign then the computation will terminate at such a solution point. See also STOP in Section 10.6.1 above and UZSTOP in Section 10.9.12 below for alternative termination methods. (Demos pen and bru.)

Note that fort.8 output can also be written at selected values of overspecified parameters. For an example see demo pvl. For details on overspecified parameters see Section 10.7.10.

10.9.12 UZSTOP

This constant specifies parameter values in the same way as UZR in Section 10.9.11 above, but the computation will always terminate if any solution point that is specified is encountered.

10.10 Quick reference

e, s, dat, sv	Define file names: equation prefix (.f,.f90,.c), restart solution suffix (s.), user data prefix (.dat), output suffix (b.,s.,d.)
unames, parnames	Dictionary (mapping) of U(*) and PAR(*) to user-defined names
NDIM	Problem dimension
IPS	Problem type; 0=AE, 1=FP(ODEs), -1=FP(maps), 2=PO, -2=IVP, 4=BVP, 7=BVP with Floquet multipliers, 5=algebraic optimization problem, 15=optimization of periodic solutions
IRS, TY	Start solution label, start solution type
ILP	Fold detection; 1=on, 0=off
ICP	Continuation parameters
NTST	# mesh intervals
NCOL	# collocation points
IAD	Mesh adaption every IAD steps; 0=off
ISP	Bifurcation detection; 0=off, 1=BP(FP), 3=BP(PO,BVP), 2=all
ISW	Branch switching; 1=normal, -1=switch branch (BP, HB, PD), 2=switch to two-parameter continuation (LP, BP, HB, TR) 3=switch to three-parameter continuation (BP)
IPLT	Select principal solution measure
NBC	# boundary conditions
NINT	# integral conditions
NMX	Maximum number of steps
RL0, RL1	Parameter interval $RL0 \leq \lambda \leq RL1$
A0, A1	Interval of principal solution measure $A0 \leq \|\cdot\| \leq A1$
NPR	Print and save restart data every NPR steps
MXBF	Automatic branch switching for the first MXBF bifurcation points if IPS=0, 1
IBR, LAB	Set initial branch and label number; 0=automatic
IIS	Control solution output of branch direction vector; 0=never, 3=always
IID	Control diagnostic output; 0=none, 1=little, 2=normal, 4=extensive
ITMX	Maximum # of iterations for locating special solutions/points
ITNW	Maximum # of correction steps
NWTN	Corrector uses full newton for NWTN steps
JAC	User defines derivatives; 0=no, 1=yes
EPSL, EPSU, EPSS	Convergence criterion: parameters, solution components, special points
DS	Start step size
DSMIN, DSMAX	Step size interval $\texttt{DSMIN} \leq h \leq \texttt{DSMAX}$
IADS	Step size adaption every IADS steps; 0=off
NPAR	Maximum number of parameters
THL, THU	list of parameter and solution weights
UZR, UZSTOP	list of values for user defined output
SP, STOP	list of bifurcations to check and bifurcation stop conditions
NUNSTAB, NSTAB	HomCont: unstable and stable manifold dimensions
IEQUIB, ITWIST, ISTART	HomCont: control solution types adjoint, starting
IREV, IFIXED, IPSI	HomCont: control reversibility, fixed parameters, test functions

Chapter 11

Notes on Using AUTO.

11.1 Restrictions on the Use of PAR.

The array PAR in the user-supplied routines is available for equation parameters that the user wants to vary at some point in the computations. In any particular computation the free parameter(s) must be designated in ICP; see Section 10.7. The following restrictions apply :

- The default maximum number of parameters, NPAR has a pre-defined value in auto/07p/include/auto.h, of NPARX=36. Any change NPARX must be followed by recompilation of AUTO.

- Generally one should avoid certain parameters for equation parameters, as AUTO may need those internally, as follows:

  IPS=**0,4:** No additional parameters with indices less than or equal to NPAR are reserved.

  IPS=**−1,1,2,7:** AUTO reserves PAR(11) to store the period for stationary solutions at Hopf bifurcations only and continuously for periodic orbits. For IPS=2 and IPS=7, AUTO also reserves PAR(12) to store the angle of the torus at torus bifurcations.

  IPS=**−2:** The integration time is stored in PAR(14).

  IPS=**5:** The value of the objective functional is stored in PAR(10).

  IPS=**9:** The length in time of the truncated homoclinic or heteroclinic orbit is stored in PAR(11). For the adjoint variational equations, PAR(10) is used. The equilibria are stored in PAR(11) to PAR(11+NDIM-1)/PAR(11+2*NDIM-1) (homoclinic/heteroclinic). Test function values may be stored in PAR(21) to PAR(36). Homoclinic branch switching uses PAR(20) and higher to store time intervals and gap sizes.

  IPS=**11,12:** The wave speed is in PAR(10), and the diffusion constants in PAR(15,16,$\cdots$). The period (for periodic solutions and at Hopf bifurcations) is stored in PAR(11).

  IPS=**14,16:** AUTO uses PAR(14) for the time variable, and the diffusion constants are in PAR(15,16,$\cdots$). The period is stored in PAR(11). The previous time for each step is stored in PAR(12).

  IPS=**17:** The diffusion constants are in PAR(15,16,$\cdots$). The period is stored in PAR(11).

IPS=15: Only PAR(1-9) should be used for problem parameters. PAR(10) is the value of the objective functional, PAR(11) the period, PAR(12) the norm of the adjoint variables, PAR(14) and PAR(15) are internal optimality variables. PAR(21-29) and PAR(31) are used to monitor the optimality functionals associated with the problem parameters and the period.

11.2 Efficiency.

In AUTO, efficiency has at times been sacrificed for generality of programming. This applies in particular to computations in which AUTO generates an extended system, for example, computations with ISW=2. However, the user has significant control over computational efficiency, in particular through judicious choice of the AUTO-constants DS, DSMIN, and DSMAX, and, for ODEs, NTST and NCOL. Initial experimentation normally suggests appropriate values.

Slowly varying solutions to ODEs can often be computed with remarkably small values of NTST and NCOL, for example, NTST=5, NCOL=2. Generally, however, it is recommended to set NCOL=4, and then to use the "smallest" value of NTST that maintains convergence.

The choice of the pseudo-arclength stepsize parameters DS, DSMIN, and DSMAX is highly problem dependent. Generally, DSMIN should not be taken too small, in order to prevent excessive step refinement in case of non-convergence. It should also not be too large, in order to avoid instant non-convergence. DSMAX should be sufficiently large, in order to reduce computation time and amount of output data. On the other hand, it should be sufficiently small, in order to prevent stepping over bifurcations without detecting them. For a given equation, appropriate values of these constants can normally be found after some initial experimentation.

The constants ITNW, NWTN, THL, EPSU, EPSL, EPSS also affect efficiency. Understanding their significance is therefore useful; see Section 10.4 and Section 10.5. Finally, it is recommended that initial computations be done with ILP=0; no fold detection; and ISP=1; no bifurcation detection for ODEs.

11.3 Correctness of Results.

AUTO-computed solutions to ODEs are almost always structurally correct, because the mesh adaption strategy, if IAD>0, safeguards to some extent against spurious solutions. If these do occur, possibly near infinite-period orbits, the unusual appearance of the solution family typically serves as a warning. Repeating the computation with increased NTST is then recommended.

11.4 Bifurcation Points and Folds.

It is recommended that the detection of folds and bifurcation points be initially disabled. For example, if an equation has a "vertical" solution family then AUTO may try to locate one fold after another.

Generally, degenerate bifurcations cannot be detected. Furthermore, bifurcations that are close to each other may not be noticed when the pseudo-arclength step size is not sufficiently small. Hopf bifurcation points may go unnoticed if no clear crossing of the imaginary axis takes

place. This may happen when there are other real or complex eigenvalues near the imaginary axis and when the pseudo-arclength step is large compared to the rate of change of the critical eigenvalue pair. A typical case is a Hopf bifurcation close to a fold. Similarly, Hopf bifurcations may go undetected if switching from real to complex conjugate, followed by crossing of the imaginary axis, occurs rapidly with respect to the pseudo-arclength step size. Secondary periodic bifurcations may not be detected for similar reasons. In case of doubt, carefully inspect the contents of the diagnostics file `fort.9`.

11.5 Floquet Multipliers.

AUTO extracts an approximation to the linearized Poincaré map from the Jacobian of the linearized collocation system that arises in Newton's method. This procedure is very efficient; the map is computed at negligible extra cost. The linear equations solver of AUTO is described in Doedel, Keller & Kernévez (1991*b*). The actual Floquet multiplier solver was written by Fairgrieve (1994). For a detailed description of the algorithm see Fairgrieve & Jepson (1991).

For periodic solutions, the exact linearized Poincaré map always has a multiplier $z = 1$. A good accuracy check is to inspect this multiplier in the diagnostics output-file `fort.9`. If this multiplier becomes inaccurate then the automatic detection of potential secondary periodic bifurcations (if `ISP`=2) is discontinued and a warning is printed in `fort.9`. It is strongly recommended that the contents of this file be habitually inspected, in particular to verify whether solutions labeled as BP or TR (cf. Table 6.1) have indeed been correctly classified.

11.6 Memory Requirements.

The run-time memory requirements depend on the values the user set in the constants file and are roughly proportional to the value $\texttt{NTST} \times (\texttt{NDIM} \times (\texttt{NCOL}+1)+\texttt{NBC}) \times (\texttt{NDIM} \times \texttt{NCOL}+\texttt{NINT}+1)$.

Chapter 12

AUTO Demos : Tutorial.

12.1 Introduction.

The directory `auto/07p/demos` has a large number of subdirectories, for example `ab`, `pp2`, `exp`, etc., each containing all necessary files for certain illustrative calculations. Each subdirectory, say `xxx`, corresponds to a particular equation and contains one equations-file xxx.{f90,f,c} and one or more constants-files c.xxx.i, one for each successive run of the demo. You also find Python script files xxx.auto and clean.auto: the command auto xxx.auto runs the demo, and the command auto clean.auto deletes all generated files. To see how the equations have been programmed, inspect the equations-file. To understand in detail how AUTO is instructed to carry out a particular task, inspect the appropriate constants-file and Python script. In this chapter we describe the tutorial demo `cusp` in detail. A brief description of other demos is given in later chapters.

12.2 cusp : A Tutorial Demo.

This demo illustrates the computation of stationary solutions, locating saddle-node bifurcations of these solutions, and the continuation of a saddle-node bifurcation in two parameters.
The cusp normal form equation is given by

$$\dot{x} = \mu + \lambda x - x^3. \tag{12.1}$$

12.3 Copying the Demo Files.

The commands listed in Table 12.1 will copy the demo files to your work directory.

Unix-COMMAND	ACTION
`auto`	start the AUTO-07p Command Line User Interface
AUTO-COMMAND	ACTION
`cd`	go to main directory (or other directory).
`mkdir cusp`	create an empty work directory. Note: the '!' is used to signify a command which is sent to the shell.
`cd cusp`	change to the work directory.
`demo('cusp')`	copy the demo files to the work directory.

Table 12.1: Copying the demo `cusp` files.

Typing `ls` reveals the existence of 5 files:

1. `cusp.f90`: This file contains the differential equations and the initial values. If you inspect it, you will see that only two routines are used. The subroutine `FUNC` specifies the actual differential equation. The routine `STPNT` gives AUTO the initial values of `PAR(1)`$= \lambda$ and `PAR(2)`$= \mu$, which are 1.0 and 0.0, and the initial value of x, which is 0. For your own models you would generally copy another equation file and then only change the pieces that actually define the equation.

2. `c.cusp`: The initial computational constants are stored in this file. Most importantly, you see that the dimension of the problem (`NDIM`) is set to 1, and the problem type `IPS` is set to 1 to specify continuation of a stationary solution. The constants given by `ICP` specify the parameters that are used for the continuation. In this case these are 'mu' for μ and 'lambda' for λ, which correspond to the indices 2 and 1, respectively, using the constant `parnames`. Since initially we only really continue in one parameter (μ), the second parameter λ is *overspecified*. Another important constant is the initial step size `DS`: as it is positive, we initally continue in the positive μ direction.

3. `cusp.auto`: A script with Python CLUI commands that steer the calculation.

4. `clean.auto`: A script that cleans the directory of all generated files.

5. `autorc`: A file that contains default settings for the 2-dimensional plotting tool PyPLAUT.

6. `plaut04.rc`: A file that contains default settings for the 3-dimensional plotting tool PLAUT04.

12.4 Executing all Runs Automatically.

To execute all prepared runs of demo cusp, simply type the command given in Table 12.2.

AUTO-COMMAND	ACTION
`demofile('cusp.auto')`	execute all runs of demo cusp interactively

Table 12.2: Executing all runs of demo cusp.

The command in Table 12.2 begins a tutorial which will proceed one step each time the user presses a key. Each step consists of a single AUTO command preceded by instructions as to what action the command performs. The tutorial script cusp.auto performs the demo by reading in a single AUTO constants file and then interactively modifying it to perform each of the demo. The essential commands in cusp.auto are given in Table 12.4.

Note that there are four separate runs, where each `run` command performs a run. In the first run, a branch of stationary solutions is traced out. Along it, one fold (LP) (limit point, or in this case, a saddle-node bifurcation) is located. The free parameter is μ. The other parameter λ remains fixed in this run. Note also that only special, labeled solution points are printed on the screen. Detailed results are saved in the Python variable `mu`.

The second run does the same thing but now in the negative direction of μ, i.e., backwards instead of forwards. The backwards continuation is appended to the forwards continuation in the data-files. Afterwards we perform a relabelling to make sure that we have unique labels for each special solution. Next the relabelled result is saved to the data-files b.mu, s.mu, and d.mu.

The results are then plotted on the screen. Pressing the enter key at the command line causes an automatic μ vs. x display that shows the two fold points at labels 2 and 7.

In the third run, the fold detected in the first run is followed in the two parameters μ and λ. We know that label 2, with solution `mu(2)` is the right solution to start from. However,

we did not know this number in advance, and moreover, in sensitive cases, it can be different on different computer types. Another way to specify the starting label is to use the notation `mu('LP1')`: this specifies the first LP-labelled solution of all solutions in `mu`.

Furthermore the command that accomplishes this must change the constant `ISW` of the constants file: it must be set to 2 to cause a two-parameter continuation.

The fourth run continues this branch in opposite direction. The detailed results of these continuations are saved in the data-files b.cusp, s.cusp, and d.cusp. Finally, a plot of the cusp is produced.

The numerical results are given below in somewhat abbreviated form. Some differences in output are to be expected on different machines. This does not mean that the results have different accuracy, but simply that arithmetic differences have accumulated from step to step, possibly leading to different step size decisions.

Next, reset the work directory, by typing the command given in Table 12.3.

AUTO-COMMAND	ACTION
`clean()`	remove temporary files of demo cusp
`delete('mu')`	remove 'mu' data-files of demo cusp
`delete('cusp')`	remove 'cusp' data-files of demo cusp

Table 12.3: Cleaning the demo cusp work directory.

```
# Run forwards

  BR    PT  TY  LAB       mu          L2-NORM          x            lambda
   1     1  EP    1   0.00000E+00   0.00000E+00   0.00000E+00   1.00000E+00
   1    14  LP    2   3.84900E-01   5.77360E-01  -5.77360E-01   1.00000E+00
   1    20        3   1.26582E-01   9.29410E-01  -9.29410E-01   1.00000E+00
   1    40        4  -1.38347E+00   1.40803E+00  -1.40803E+00   1.00000E+00
   1    47  UZ    5  -1.99999E+00   1.52138E+00  -1.52138E+00   1.00000E+00

# Run backwards

  BR    PT  TY  LAB       mu          L2-NORM          x            lambda
   1     1  EP    1   0.00000E+00   0.00000E+00   0.00000E+00   1.00000E+00
   1    14  LP    2  -3.84900E-01   5.77360E-01   5.77360E-01   1.00000E+00
   1    20        3  -1.26582E-01   9.29410E-01   9.29410E-01   1.00000E+00
   1    40        4   1.38347E+00   1.40803E+00   1.40803E+00   1.00000E+00
   1    47  UZ    5   1.99999E+00   1.52138E+00   1.52138E+00   1.00000E+00

# Forward continuation of the first fold in two parameters

  BR    PT  TY  LAB       mu          L2-NORM          x            lambda
   2    20       11   1.09209E+00   8.17354E-01  -8.17354E-01   2.00420E+00
   2    34  UZ   12   1.99995E+00   9.99991E-01  -9.99991E-01   2.99995E+00
```

```
# Backward continuation of the fold in two parameters

  BR    PT  TY  LAB       mu          L2-NORM          x            lambda
   2    20       11   5.42543E-02   3.00470E-01  -3.00470E-01   2.70847E-01
   2    29  CP   12  -2.02768E-12   1.00472E-04   1.00472E-04   3.02837E-08
   2    40       13  -9.09414E-02   3.56925E-01   3.56925E-01   3.82187E-01
   2    60       14  -5.73716E-01   6.59512E-01   6.59512E-01   1.30487E+00
   2    80       15  -1.68023E+00   9.43582E-01   9.43582E-01   2.67104E+00
   2    85  UZ   16  -1.99995E+00   9.99992E-01   9.99992E-01   2.99995E+00
```

The CLUI was used to generate the constants file at runtime. In the example below, the constant file c.cusp will be read in, and the CLUI will be used to make the appropriate changes to perform the calculation.

AUTO-COMMAND	ACTION
`cusp = load('cusp')`	load the problem definition cusp
`mu = run(cusp)`	execute the run
`mu = mu + run(cusp,DS='-')`	execute the run backwards and append the results to `mu`
`mu = rl(mu)`	relabel solutions in mu
`save(mu, 'mu')`	save the results in the files b.mu, s.mu, and d.mu
`lp1 = load(mu('LP1'), ISW=2)`	use the first fold (LP) in `mu` as the restart solution, and change ISW to 2.
`cusp = run(lp1)`	execute the third run of demo cusp
`cusp = cusp + run(lp1,DS='-')`	execute the fourth run of demo cusp
`save(cusp,'cusp')`	save the results in the files b.cusp, s.cusp, and d.cusp

Table 12.4: Selected runs of demo cusp.

12.5 Plotting the Results with AUTO.

The bifurcation diagram computed in the runs above was stored in the files b.mu and b.cusp, while each labeled solution is fully stored in s.mu and s.cusp. To use AUTO to graphically inspect these data-files. type the AUTO-command given in Table 12.5. The saved plots are shown in Figure 12.1 and in Figure 12.2.

Figure 12.1 shows the bifurcation diagrams for the first run, and Figure 12.2 for the second run.

The plotting window consists of a menubar at the top, a plotting area, and a control panel with four control widgets at the bottom. By default the first two columns in the bifurcation diagram output are plotted against each other. To obtain a μ versus x bifurcation diagram you need to plot column 'mu' versus column 'x'. You can do that by changing the "Y" box to say "[x]", either by typing it there, by using the menu obtained by clicking the downwards

facing triangle or by using a scripted command as used in cusp.auto. You can also change the mode of the plotting tool from "bifurcation" to "solution". This is accomplished by clicking on the widget marked "Type" on the bottom control panel and setting it from "bifurcation" to "solution". In the plotting window will appear a plot of the first labeled solution, in this case just a point. You can plot all points by changing the "Label" to "[1,2,3,4,5,6,7,8,9,10]".

The plotting tool can also be used to create Postscript files from plots by selecting the "File" on the menubar and then selecting the "Save Postscript..." from the drop down menu. This will bring up a dialog into which the user can enter the filename of the postscript file to save the plot in. When using matplotlib you can also click on the floppy disk icon to save using a variety of file formats. Further information on the plotting tool can be found in Section 4.11.

AUTO-COMMAND	ACTION
`plot(mu)`	run AUTO to graph the contents of `mu`
`plot("mu")`	run AUTO to graph the contents of b.mu and s.mu; (this is the same as `plot(loadbd("mu"))`)

Table 12.5: Commands for plotting or the bifurcation diagram and solutions of the Python variable `mu`, and the files b.mu and s.mu

12.6 Plotting the Results with AUTO in 3D.

Whilst not very useful for this simple example, you can also plot your results in 3D, using the `plot3` command (PLAUT04), for example `plot3('mu')`. Unlike the PyPLAUT tool by default, this shows the stable and unstable parts: blue is stable, and red is unstable. You can also spin the bifurcation diagram around and zoom in using the mouse.

12.7 Exporting the Results for different plotters.

It is often useful to use other plotting programs or general-purpose tools to work with AUTO's data. The "writeRawFilename" method (see also Section 4.9) can be used for this. In this tutorial we can for instance export the bifurcation diagram using
`cusp.writeRawFilename('cusp.dat')` , and then use the command `plot 'cusp.dat' using 1:4 wi li` to plot the bifurcation diagram in GNUPlot.

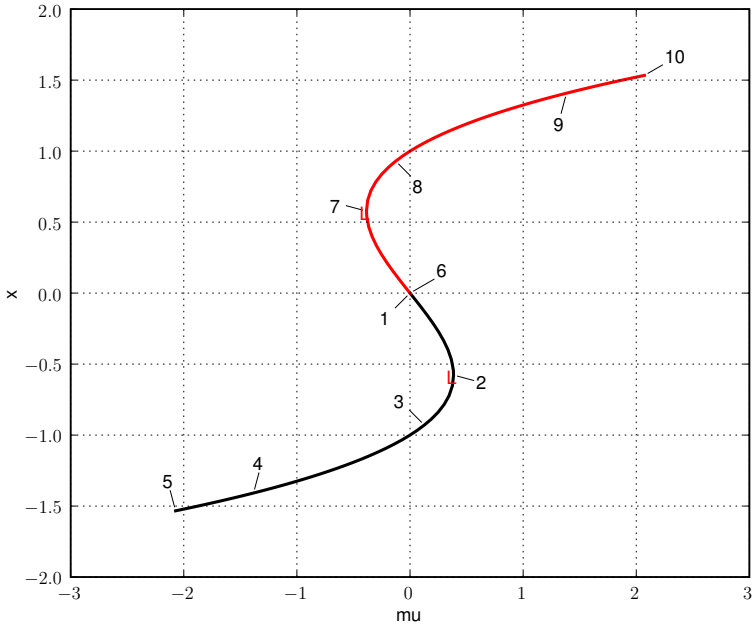


Figure 12.1: The first bifurcation diagram of demo cusp.

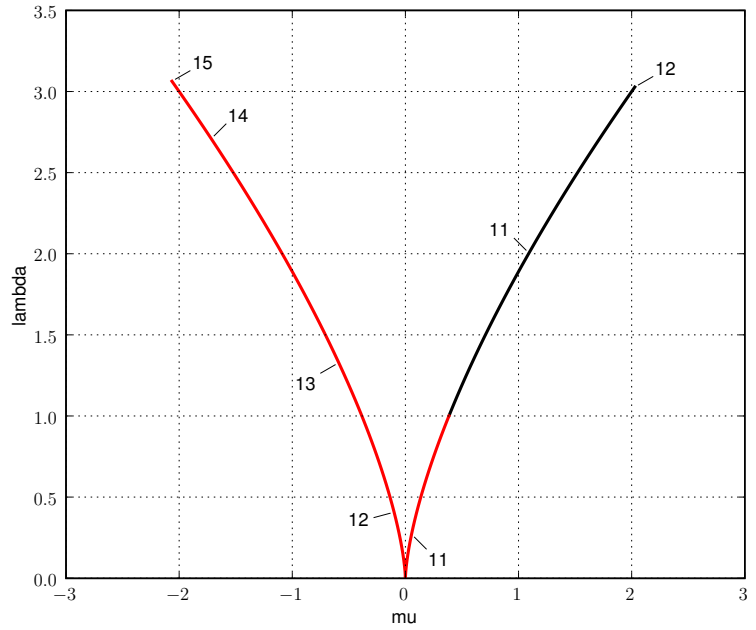


Figure 12.2: The second bifurcation diagram of demo cusp.

12.8 ab : A Programmed Demo.

This demo illustrates the computation of stationary solutions, Hopf bifurcations and periodic solutions. The equations, that model an A $\rightarrow$ B reaction, are those from Uppal, Ray & Poore (1974), namely

$$\begin{aligned} u_1' &= -u_1 + p_1(1-u_1)e^{u_2}, \\ u_2' &= -u_2 + p_1 p_2(1-u_1)e^{u_2} - p_3 u_2. \end{aligned} \tag{12.2}$$

This demo is fully scripted, see Table 12.6.

AUTO-COMMAND	ACTION
`mkdir ab`	create an empty work directory
`cd ab`	change directory
`demo('ab')`	copy the demo files to the work directory
`auto('ab.auto')`	run the demo

Table 12.6: Commands for running demo `ab`.

If you look at the file `ab.auto` you see that the script computes a stationary solution family for certain values of p_2, and that a periodic orbit family is computed for each Hopf bifurcation that was found in the stationary solution families.

Chapter 13

AUTO Demos : Fixed points.

13.1 enz : Stationary Solutions of an Enzyme Model.

The equations, that model a two-compartment enzyme system (Kernévez (1980)), are given by

$$\begin{aligned} s_1' &= (s_0 - s_1) + (s_2 - s_1) - \rho R(s_1), \\ s_2' &= (s_0 + \mu - s_2) + (s_1 - s_2) - \rho R(s_2), \end{aligned} \tag{13.1}$$

where

$$R(s) = \frac{s}{1 + s + \kappa s^2}.$$

The free parameter is s_0. Other parameters are fixed. This equation is also considered in Doedel, Keller & Kernévez (1991a).

AUTO-COMMAND	ACTION
`mkdir enz`	create an empty work directory
`cd enz`	change directory
`demo('enz')`	copy the demo files to the work directory
`run('enz')`	compute stationary solution families
`save('enz')`	save output-files as b.enz, s.enz, d.enz

Table 13.1: Python commands for running demo enz.

AUTO-COMMAND	ACTION
`mkdir enz`	create an empty work directory
`cd enz`	change directory
`@dm enz`	copy the demo files to the work directory
`@r enz`	compute stationary solution families
`@sv enz`	save output-files as b.enz, s.enz, d.enz

Table 13.2: Shell commands for running demo enz.

13.2 dd2 : Fixed Points of a Discrete Dynamical System.

This demo illustrates the computation of a solution family and its bifurcating families for a discrete dynamical system. Also illustrated are the continuation of period-doubling bifurcations, and branch switching at such points. The equations, a discrete predator-prey system, are

$$\begin{aligned} u_1^{k+1} &= p_1 u_1^k (1 - u_1^k) - p_2 u_1^k u_2^k, \\ u_2^{k+1} &= (1 - p_3) u_2^k + p_2 u_1^k u_2^k. \end{aligned} \tag{13.2}$$

In the first, third, and fourth run p_1 is free. In the second run, both p_1 and p_2 are free. The remaining equation parameter, p_3, is fixed in both runs.

AUTO-COMMAND	ACTION

`mkdir dd2`
`cd dd2`
`demo('dd2')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='dd2',c='dd2')` `save('dd2')`	1st run; fixed point solution branches save output-files as b.dd2, s.dd2, d.dd2

`run(r1("PD1"),ICP=["p1","p2"],ISW=2)`
 	
`save('pd')`
	2nd run; a locus of period-doubling bifurcations.
	save output-files as b.pd, s.pd, d.pd

`r3=run(r1("PD1"),ISW=-1)` `append('dd2')`	3rd run; the bifurcating period-2 orbit. append output-files to b.dd2, s.dd2, d.dd2
`run(r3("PD1"))` `append('dd2')`	4th run; the bifurcation period-4 orbit. append output-files to b.dd2, s.dd2, d.dd2

Table 13.3: Commands for running demo dd2.

13.3 log : The Logistic Map.

This demo shows 5 subsequent periodic doublings in the logistic map

$$x^{k+1} = \mu x(1-x), \tag{13.3}$$

and approximates the Feigenbaum constant. The script log.auto shows a Python loop in which values of μ for subsequent period-doubling bifurcations are compared.

AUTO-COMMAND	ACTION
`mkdir log`	create an empty work directory
`cd log`	change directory
`demo('log')`	copy the demo files to the work directory
`auto('log.auto')`	run the script log.auto
`plot('log')`	plot the bifurcation diagram

Table 13.4: Commands for running demo log.

13.4 hen: The Hénon Map.

In this demo, a two-parameter bifurcation analysis of the Hénon map

$$\begin{aligned} x^{k+1} &= y, \\ y^{k+1} &= \alpha - \beta x - y^2, \end{aligned} \tag{13.4}$$

is performed. This demo features the detection and continuation of Naimark-Sacker, period-doubling, and fold bifurcations in two parameters. On these codimension-one bifurcation curves certain codimension-two bifurcations are detected: the 1:1 (R1), 1:2 (R2), 1:3 (R3), and 1:4 (R4) resonance and fold-flip (LPD) bifurcation points. After running the script hen.auto, the results can be plotted using `plot('hen')` or `@pp hen`.

AUTO-COMMAND	ACTION
`mkdir hen`	create an empty work directory
`cd hen`	change directory
`demo('hen')`	copy the demo files to the work directory
`r1 = run('hen')`	fixed point solution branch for β ($\alpha = 1$) (detects a period-doubling (PD) and a Naimark-Sacker (TR) bifurcation)
`save('beta')`	save output-files as b.beta, s.beta, d.beta
`run(r1("TR1"),ICP=['alpha','beta'], ISW=2,ILP=0,STOP=['R11','R21'])`	continue the TR bifurcation in two parameters until a 1:1 or 1:2 resonance is found
`save('hen')`	save output-files as b.hen, s.hen, d.hen
`run(DS='-')`	compute last continuation the opposite way
`append('hen')`	append output-files to b.hen, s.hen, d.hen
`run(r1("PD1"),ICP=['alpha','beta'], ISW=2,ILP=0)`	continue the PD bifurcation in two parameters
`append('hen')`	append output-files as b.hen, s.hen, d.hen
`run(DS='-')`	compute last continuation the opposite way
`append('hen')`	append output-files to b.hen, s.hen, d.hen
`r4=run(c='hen',ICP=['alpha'], DS='-',STOP=['LP1'])`	fixed point solution branch for α ($\beta = 1$) (detects and stops at a fold (LP))
`save('alpha')`	save output-files as b.alpha, s.alpha, d.alpha
`run(r4("LP1"),ICP=['alpha','beta'], ISW=2,ILP=0)`	continue the LP bifurcation in two parameters
`append('hen')`	append output-files as b.hen, s.hen, d.hen
`run(DS='-')`	compute last continuation the opposite way
`append('hen')`	append output-files to b.hen, s.hen, d.hen
`merge('hen')`	join all forward and backward branches into single branches
`relabel('hen')`	make all labels unique

Table 13.5: Commands for running demo hen.

Chapter 14

AUTO Demos : Periodic solutions.

14.1 lrz : The Lorenz Equations.

This demo computes two symmetric homoclinic orbits in the Lorenz equations

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= \rho x - y - xz, \\ z' &= xy - \beta z. \end{aligned} \tag{14.1}$$

Here ρ is the free parameter, and $\beta = 8/3$, $\sigma = 10$. The two homoclinic orbits correspond to the final, large period orbits on the two periodic solution families.

AUTO-COMMAND	ACTION

`mkdir lrz`
`cd lrz`
`demo('lrz')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`lrz=run(e='lrz',c='lrz')` `save('lrz')`	compute stationary solutions save all output to b.lrz, s.lrz, d.lrz

`run(lrz('HB1'),IPS=2,ICP=['rho',`
`'PERIOD'],NMX=35,NPR=2,DS=0.5)`
`append('lrz')`
	compute periodic solutions; the final orbit
	is near-homoclinic.
	append all output to b.lrz, s.lrz, d.lrz

`run(lrz('HB2'),IPS=2,ICP=['rho',`
`'PERIOD'],NMX=35,NPR=2,DS=0.5)`
`append('lrz')`
	compute the symmetric periodic solution family
	
	append all output to b.lrz, s.lrz, d.lrz

Table 14.1: Commands for running demo lrz.

14.2 abc : The A → B → C Reaction.

This demo illustrates the computation of stationary solutions, Hopf bifurcations and periodic solutions in the A → B → C reaction (Doedel & Heinemann (1983)).

$$\begin{aligned} u_1' &= -u_1 + p_1(1-u_1)e^{u_3}, \\ u_2' &= -u_2 + p_1 e^{u_3}(1-u_1-p_5u_2), \\ u_3' &= -u_3 - p_3u_3 + p_1p_4e^{u_3}(1-u_1+p_2p_5u_2), \end{aligned} \tag{14.2}$$

with $p_2 = 1$, $p_3 = 1.55$, $p_4 = 8$, and $p_5 = 0.04$. The free parameter is p_1.

The equations, as programmed in the equations-file `abc.f90`, appear in Table 14.2. The starting point, an equilibrium of the equations, is also defined in the equations-file `abc.f90`, as shown in Table 14.3. (The equations-file `abc.f90` also contains the skeletons of some other routines, which must be supplied, but which are not used in this application.)

A more advanced version, that continues branch points in three parameters is provided by the demo `abcb`.

In the constants-file (`c.abc.1`) for the first run, as shown in Table 14.4, we note the following:

- `IPS=1` : a family of stationary solutions is computed.
- `IRS=0` : the starting point defined in `STPNT` is to be used (see Table 14.3).
- `ICP=[1]` : the continuation parameter is PAR(1)
- `UZR={-1:0.4}` : there is one user output point, namely at `PAR(1)=0.4`. Moreover, since the index ("`-1`") in the last line of the constants-file `c.abc.1` is negative, the calculation will terminate when the calculation reaches the value `PAR(1)=0.4`..

In the constants-file (`c.abc.2`) for the second run, as shown in Table 14.5, we note that:

- `IPS=2` : a family of periodic solutions is computed.
- `IRS=2` : the starting point is the solution with label 2, (a Hopf bifurcation point), to be read from the solutions-file (here `s.abc`).
- `ICP=[1,11]` : there are two continuation parameters (namely `PAR(1)`, and the period, `PAR(11)`).
- `UZR={-1:0.25}` : there is one user output point, now at `PAR(1)=0.25`, where the calculation is to terminate, since the index ("`-1`") is negative.

```
      SUBROUTINE FUNC(NDIM,U,ICP,PAR,IJAC,F,DFDU,DFDP)
!     ---------- ----

      IMPLICIT NONE
      INTEGER, INTENT(IN) :: NDIM, ICP(*), IJAC
      DOUBLE PRECISION, INTENT(IN) :: U(NDIM), PAR(*)
      DOUBLE PRECISION, INTENT(OUT) :: F(NDIM)
      DOUBLE PRECISION, INTENT(INOUT) :: DFDU(NDIM,NDIM), DFDP(NDIM,*)

      DOUBLE PRECISION X1,X2,X3,D,ALPHA,BETA,B,S,E,X1C

       X1=U(1)
       X2=U(2)
       X3=U(3)

       D=PAR(1)
       ALPHA=PAR(2)
       BETA=PAR(3)
       B=PAR(4)
       S=PAR(5)

       E=DEXP(X3)
       X1C=1-X1

       F(1)=-X1 + D*X1C*E
       F(2)=-X2 + D*E*(X1C - S*X2)
       F(3)=-X3 - BETA*X3 + D*B*E*(X1C + ALPHA*S*X2)

      END SUBROUTINE FUNC
```

Table 14.2: The equations for demo `abc`, as defined in the equations-file `abc.f90`.

```
      SUBROUTINE STPNT(NDIM,U,PAR,T)
!     ---------- -----

      IMPLICIT NONE
      INTEGER, INTENT(IN) :: NDIM
      DOUBLE PRECISION, INTENT(INOUT) :: U(NDIM),PAR(*)
      DOUBLE PRECISION, INTENT(IN) :: T

       PAR(1)=0.0
       PAR(2)=1.0
       PAR(3)=1.55
       PAR(4)=8.
       PAR(5)=0.04

       U(1)=0.
       U(2)=0.
       U(3)=0.

      END SUBROUTINE STPNT
```

Table 14.3: The starting solution for demo `abc`, as defined in the equations-file `abc.f90`.

```
NDIM=   3, IPS =   1, IRS =   0, ILP =   1
ICP =  [1]
NTST=  15, NCOL=   4, IAD =   3, ISP =   1, ISW = 1, IPLT= 0, NBC= 0, NINT= 0
NMX=  130, NPR=  200, MXBF=  10, IID =   2, ITMX= 8, ITNW= 5, NWTN= 3, JAC= 0
EPSL= 1e-07, EPSU = 1e-07, EPSS =0.0001
DS  =  0.02, DSMIN= 0.001, DSMAX=   0.1, IADS=   1
NPAR = 5, THL =  {11: 0.0}, THU =  {}
UZR =  {1: 0.4}, STOP = ['UZ1']
```

Table 14.4: The constants-file `c.abc.1` for Run 1 (stationary solutions) of demo `abc`.

```
NDIM=   3, IPS =   2, IRS =   2, ILP =   1
ICP =  [1, 11]
NTST=  25, NCOL=   4, IAD =   3, ISP =   1, ISW = 1, IPLT= 0, NBC= 0, NINT= 0
NMX=  200, NPR=  200, MXBF=  10, IID =   2, ITMX= 8, ITNW= 5, NWTN= 3, JAC= 0
EPSL= 1e-07, EPSU = 1e-07, EPSS =0.0001
DS  =  0.02, DSMIN= 0.001, DSMAX=   0.1, IADS=   1
NPAR = 5, THL =  {11: 0.0}, THU =  {}
UZR =  {1: 0.25}, STOP = ['UZ1']
```

Table 14.5: The constants-file `c.abc.2` for Run 2 (periodic orbits) of demo `abc`.

COMMAND	ACTION
`mkdir abc`	create an empty work directory
`cd abc`	change directory
`@dm abc`	copy the demo files to the work directory
`@R abc 1`	compute the stationary solution family with four Hopf bifurcations
`@sv abc`	save output-files as `b.abc, s.abc, d.abc`
`@R abc 2`	compute a family of periodic solutions from the first Hopf point
`@ap abc`	append the output-files to `b.abc, s.abc, d.abc`
`@R abc 3`	compute a family of periodic solutions from the second Hopf point
`@ap abc`	append the output-files to `b.abc, s.abc, d.abc`
`@R abc 4`	compute a family of periodic solutions from the third Hopf point
`@ap abc`	append the output-files to `b.abc, s.abc, d.abc`
`@R abc 5`	compute a family of periodic solutions from the fourth Hopf point
`@ap abc`	append the output-files to `b.abc, s.abc, d.abc`

Table 14.6: Unix Commands for running demo `abc`.

```
abc=run(e='abc',c='abc.1')
abc=abc+run(abc('HB1'),c='abc.2')
abc=abc+run(abc('HB2'),c='abc.3')
abc=abc+run(abc('HB3'),c='abc.4')
abc=abc+run(abc('HB4'),c='abc.5')
save(abc,'abc')
```

Table 14.7: Python Commands for running demo `abc`.

```
abc=run(e='abc',c='abc.1')
for solution in abc('HB'):
    abc=abc+run(solution,c='abc.2')
abc=rl(abc)
save(abc,'abc')
```

Table 14.8: Python Program for running demo `abc`.

14.3 pp2 : A 2D Predator-Prey Model.

This demo illustrates the computation of families of stationary solutions, including bifurcating stationary families, as well as the detection of a Hopf bifurcation. The first run computes the families of stationary solutions, bounded by $0 \leq p_1 \leq 1$ and $u_1 \geq -0.25$. Then the script pp2.auto scans the first run for Hopf bifurcations, finds one, and computes the family of periodic solutions that emanates from the Hopf bifurcation. This family terminates in a heteroclinic orbit. The continuation is configured to stop if the period `PAR(11)`$= 36$, when the heteroclinic orbit is very close.

The equations, which model a predator-prey system with harvesting, are

$$\begin{aligned} u_1' &= p_2 u_1(1-u_1) - u_1 u_2 - p_1(1-e^{-p_3 u_1}), \\ u_2' &= -u_2 + p_4 u_1 u_2. \end{aligned} \tag{14.3}$$

Here p_1 (quota) is the principal continuation parameter, while $p_2 = p_4 = 3$ and $p_3 = 5$, are fixed. The variables u_1 and u_2 denote prey and predator, for instance fish and sharks. The use of PLAUT is also illustrated. The saved plots are shown in Figure 14.1 and Figure 14.2. You can obtain similar figures using the Python CLUI's plot command and using PLAUT04.

COMMAND	ACTION

`mkdir pp2`
`cd pp2`
`@dm pp2`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`auto pp2.auto` or `auto('pp2.auto')`	Run the script pp2.auto

Table 14.9: Commands for running demo `pp2`.

AUTO-COMMAND	ACTION
`@p pp2` or `@pp pp2`	run PLAUT or PyPLAUT to graph the contents of `b.pp2` and `s.pp2`;
PLAUT/PyPLAUT-COMMAND	ACTION
`d2`	set convenient defaults
`ax`	select axes
`1 3`	select real columns 1 and 3 in `b.pp2`
`bd0`	plot the bifurcation diagram; $max\ u_1$ versus p_1
`d1`	choose other default settings
`bd`	get blow-up of current bifurcation diagram
`0 1 -0.25 1`	enter diagram limits
`sav`	save plot (see Figure 14.1)
`fig.1` or `fig1.eps`	upon prompt, enter a new file name, e.g., `fig.1` or `fig.eps`
`cl`	clear the screen
`2d`	enter 2D mode, for plotting labeled solutions
`11 15 19 23`	select these labeled orbits in `s.pp2`
`d`	default orbit display; u_1 versus time
`1 3`	select columns 1 and 3 in `s.pp2`
`d`	display the orbits; u_2 versus time
`2 3`	select columns 2 and 3 in `s.pp2`
`d`	phase plane display; u_2 versus u_1
`sav`	save plot (see Figure 14.2)
`fig.2` or `fig2.eps`	upon prompt, enter a new file name
`ex`	exit from 2D mode
`end`	exit from PLAUT/PyPLAUT

Table 14.10: Plotting commands for demo `pp2`.

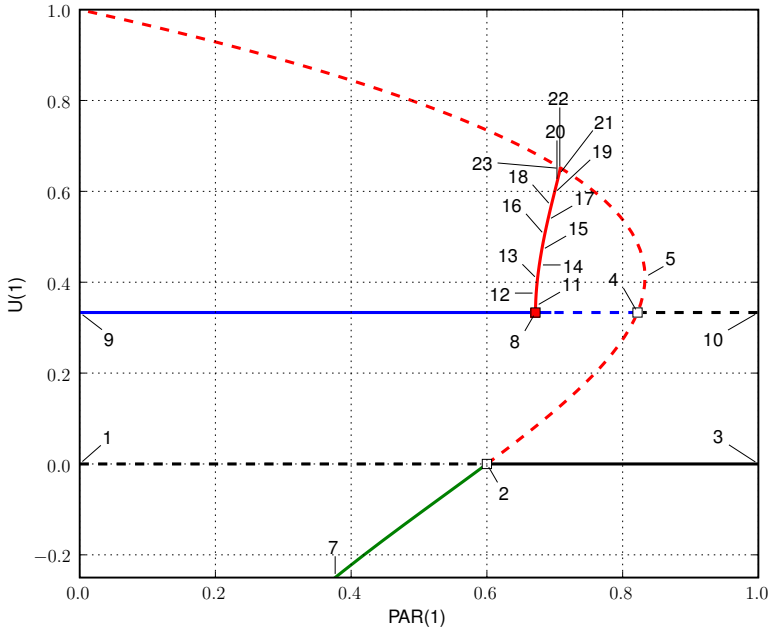


Figure 14.1: The bifurcation diagram of demo `pp2`.

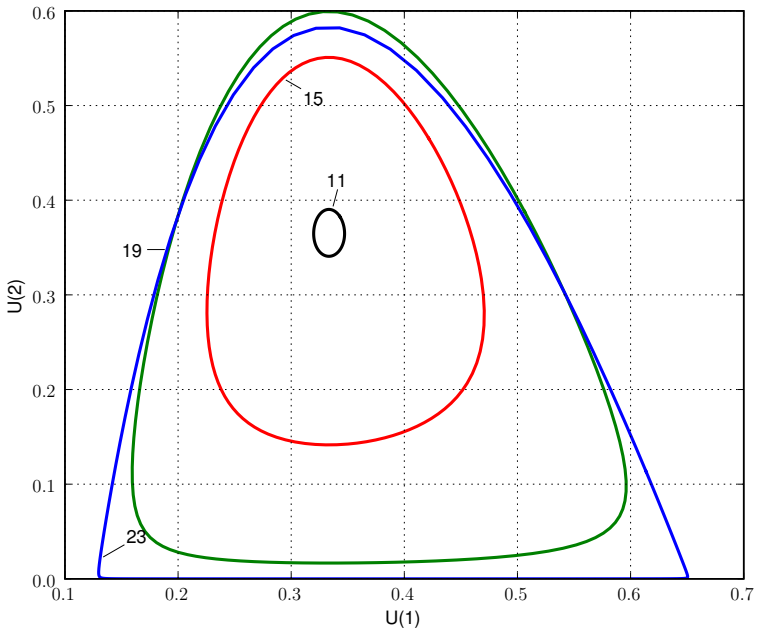


Figure 14.2: The phase plot of solutions 11, 15, 19, and 23 in demo `pp2`.

14.4 lor : Starting an Orbit from Numerical Data.

This demo illustrates how to start the computation of a family of periodic solutions from numerical data obtained, for example, from an initial value solver. As an illustrative application we consider the Lorenz equations

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= \rho x - y - xz, \\ z' &= xy - \beta z. \end{aligned} \tag{14.4}$$

Numerical simulations with a simple initial value solver show the existence of a stable periodic orbit when $\rho = 280$, $\beta = 8/3$, $\sigma = 10$. Numerical data representing one complete periodic oscillation are contained in the file lor.dat. Each row in lor.dat contains four real numbers, namely, the time variable t, x, y and z. The correponding parameter values are defined in the user-supplied subroutine STPNT. The AUTO constant `dat='lor'` then allows for using the data in lor.dat where we also specify `IRS=0`. The mesh will be suitably adapted to the solution, using the number of mesh intervals `NTST` and the number of collocation point per mesh interval `NCOL` specified in the constants-file c.lor.1.

AUTO-COMMAND	ACTION

`mkdir lor`
`cd lor`
`demo('lor')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`lor=run('lor',c='lor.1')`	compute a solution family, restart from lor.dat save to bifurcation diagram object `lor`
`pd=run(lor('PD1'),c='lor.2')`	switch branches at a period-doubling detected in the first run. Constants changed : `IRS, ISW, NTST`
`save(lor+pd,'lor')`	save the two runs to b.lor, s.lor, d.lor

Table 14.11: Commands for running demo lor.

14.5 frc : A Periodically Forced System.

This demo illustrates the computation of periodic solutions to a periodically forced system. In AUTO this can be done by adding a nonlinear oscillator with the desired periodic forcing as one of the solution components. An example of such an oscillator is

$$\begin{aligned} x' &= x + \beta y - x(x^2 + y^2), \\ y' &= -\beta x + y - y(x^2 + y^2), \end{aligned} \tag{14.5}$$

which has the asymptotically stable solution $x = sin(\beta t)$, $y = cos(\beta t)$. We couple this oscillator to the Fitzhugh-Nagumo equations :

$$\begin{aligned} v' &= \big(F(v) - w\big)/\epsilon, \\ w' &= v - dw - \big(b + r\sin(\beta t)\big), \end{aligned} \tag{14.6}$$

by replacing $\sin(\beta t)$ by x. Above, $F(v) = v(v-a)(1-v)$ and a, b, ϵ and d are fixed. The first run is a homotopy from $r = 0$, where a solution is known analytically, to $r = 0.2$. Part of the solution family with $r = 0.2$ and varying β is computed in the second run. For detailed results see Alexander, Doedel & Othmer (1990).

AUTO-COMMAND	ACTION

`mkdir frc`
`cd frc`
`demo('frc')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='frc',c='frc')` `save(r1,'0')`	homotopy to $r = 0.2$ save output-files as b.0, s.0, d.0

`r2=run(r1('UZ1'),ICP=[5,11],`
`NMX=20,DS=-0.5,DSMAX=5.0)`
`save(r2,'frc')`
	compute solution family; restart from `r1`.
	
	save output-files as b.frc, s.frc, d.frc

Table 14.12: Commands for running demo frc.

14.6 ppp : Continuation of Hopf Bifurcations.

This demo illustrates the continuation of Hopf bifurcations in a 3-dimensional predator prey model (Doedel (1984)). This curve contain branch points, where one locus of Hopf points bifurcates from another locus of Hopf points, and generalized Hopf (Bautin) bifurcations (GH), where the Hopf bifurcation changes from sub- to supercritical. The diagnostics file d.hb can be inspected to see where the Hopf bifurcation is subcritical and where it is supercritical. The equations are

$$\begin{aligned} u_1' &= u_1(1-u_1) - p_4 u_1 u_2, \\ u_2' &= -p_2 u_2 + p_4 u_1 u_2 - p_5 u_2 u_3 - p_1(1-e^{-p_6 u_2}) \\ u_3' &= -p_3 u_3 + p_5 u_2 u_3. \end{aligned} \tag{14.7}$$

Here $p_2 = 1/4$, $p_3 = 1/2$, $p_4 = 3$, $p_5 = 3$, $p_6 = 5$, and p_1 is the free parameter. In the continuation of Hopf points the parameter p_4 is also free.

AUTO-COMMAND	ACTION

`mkdir ppp`
`cd ppp`
`demo('ppp')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`ppp=run(e='ppp',c='ppp')`	compute stationary solutions; detect Hopf bifurcations

`ppp=ppp+run(ppp("HB2"),IPS=2,ICP=[1,11],`
`ILP=0,NMX=15,NPR=50,DS=0.1,DSMAX=0.5)`
`save(ppp,'ppp')`
	compute a family of periodic solutions
	
	save the output to b.ppp, s.ppp, d.ppp

`hb = run(ppp("HB2"),ICP=[1,4],ILP=0,`
`ISW=2,NMX=100,RL1=0.58,DSMAX=0.1)`
`save(hb,'hb')`
	compute Hopf bifurcation curves
	
	save the output-files as b.hb, s.hb, d.hb

Table 14.13: Commands for running demo ppp.

14.7 plp : Fold Continuation for Periodic Solutions.

This demo, which corresponds to computations in Doedel, Keller & Kernévez (1991*a*), shows how one can continue folds on a family of periodic solutions in two parameters. The calculation of a locus of Hopf bifurcations is also included. The equations, that model a one-compartment activator-inhibitor system (Kernévez (1980)), are given by

$$\begin{aligned} s' &= (s_0 - s) - \rho R(s,a), \\ a' &= \alpha(a_0 - a) - \rho R(s,a), \end{aligned} \tag{14.8}$$

where

$$R(s,a) = \frac{sa}{1 + s + \kappa s^2}, \qquad \kappa > 0.$$

The free parameter is ρ. In the Hopf and fold continuations the parameter s_0 is also free. The computed loci of Hopf points and folds suggest the existence of *isolas* of periodic solutions. The computation of one such isola is also included in this demo. All calculations can be carried out by running the Python script plp.auto included in the demo.

14.8 ph1 : Phase-Shifting using Continuation.

This demo, which uses the activator-inhibitor model in Doedel, Keller & Kernévez (1991*a*), shows how one can phase-shift a periodic solution. This can be useful in applications, for example when one wants a component of a periodic solution to have a specific value at time 0. The equations are given by

$$\begin{aligned} s' &= (s_0 - s) - \rho R(s,a) \ , \\ a' &= \alpha(a_0 - a) - \rho R(s,a) \ , \end{aligned} \tag{14.9}$$

where

$$R(s,a) = \frac{sa}{1 + s + \kappa s^2} \ .$$

The first two runs compute a family of stationary solutions and a bifurcating family of periodic solutions. The free problem parameter in these runs is ρ. The results are saved in the files b.sa, s.sa, and d.sa. The third run starts at a specified periodic solution in s.sa, namely, the solution with label 6, and phase-shifts this solution in time until $s(0) = 30$. The above sequence of calculations can be carried out by running the Python script ph1.auto included in the demo.

The basic idea for doing the phase shift in the third run is to drop the integral phase condition, which is automatically added when the AUTO-constant IPS has value 2. For this purpose the third run uses the value 4 for IPS, as specified in c.ph1, in which case the periodicity conditions must be specified explicitly in the subroutine BCND in the equations-file ph1.f90. Also, the interval of periodicity must be scaled explicitly to the interval $[0,1]$, which introduces the period T as an explicit parameter in the differential equations. Note that no integral phase condition is specified in ICND. The problem formulation in ph1.f90 is therefore

$$\begin{aligned} s' &= T[(s_0 - s) - \rho R(s,a)] \ , \\ a' &= T[\alpha(a_0 - a) - \rho R(s,a)] \ , \end{aligned} \tag{14.10}$$

with boundary conditions

$$\begin{aligned} s(0) - s(1) &= 0 \ , \\ a(0) - a(1) &= 0 \ . \end{aligned} \tag{14.11}$$

Note that the AUTO parameter PAR(9), defined in the subroutine PVLS in ph1.f90, is used to monitor the value of $s(0)$. Since there are two constraints, the third run requires only one free parameter, namely T (PAR(11)). Note that, to numerical accuracy, T does not change during this run. Alternatively one can use the free parameter ρ (PAR(4)) in the third run. In this case, to numerical accuracy, ρ does not change during the run. The third run terminates when PAR(9) reached the value 30, as specified in the equations-file c.ph1.

14.9 pp3 : Periodic Families and Loci of Hopf Points.

This demo illustrates the computation of stationary solution families that contain Hopf bifurcations, and the computation of the emanating families of periodic solutions. In this example the periodic solution families intersect at a secondary bifurcation point (a branch point). It it also shown how to compute a locus of Hopf bifurcation points in two parameters. (In this example the locus contains branch points, which lead to another locus!)

The equations, which model a 3D predator-prey system with harvesting (Doedel (1984)), are

$$\begin{aligned} u_1' &= u_1(1-u_1) - p_4 u_1 u_2, \\ u_2' &= -p_2 u_2 + p_4 u_1 u_2 - p_5 u_2 u_3 - p_1(1 - e^{-p_6 u_2}) \\ u_3' &= -p_3 u_3 + p_5 u_2 u_3. \end{aligned} \tag{14.12}$$

The free parameter is p_1, while the other parameters are fixed, namely $p_2 = 0.25$, $p_3 = 0.5$, $p_4 = 4$, $p_5 = 3$, and $p_6 = 5$. However, both p_1 and p_4 are free in the computation of loci of Hopf points.

The script in pp3.auto first computes the bifurcation diagram involving the stationary solutions. It finds four Hopf bifurcations. A periodic orbit family is computed from each of these four Hopf bifurcations. Then the second Hopf bifurcation from the first run is continued in two parameters, also producing the other locus.

COMMAND	ACTION

`mkdir pp3`
`cd pp3`
`@dm pp3`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`auto pp3.auto` or `auto('pp3.auto')`	Run the script pp3.auto

Table 14.14: Commands for running demo pp3.

AUTO-COMMAND	ACTION
`@p pp3` or `@pp pp3`	run PLAUT/PyPLAUT to graph the contents of `b.pp3` and `s.pp3`;
PLAUT/PyPLAUT-COMMAND	ACTION
`d2`	set convenient defaults
`ax`	select axes
`1 3`	select real columns 1 and 3 in `b.pp3`
`bd0`	plot the bifurcation diagram; $max\ u_1$ versus p_1
`bd`	get blow-up of current bifurcation diagram
`0 0.6 0 1.2`	enter diagram limits
`d1`	choose other default settings (with labels)
`bd`	another blow-up of the bifurcation diagram
`0 0.6 0 0.75`	enter diagram limits
`d2`	set defaults
`2d`	enter 2D mode, for plotting labeled solutions
`13 14 15`	select these orbits from `s.pp3`
`d`	default orbit display; u_1 versus time
`2 3`	select columns 2 and 3 in `s.pp3`
`d`	display the orbits; u_2 versus u_1
`2d`	enter 2D mode, for plotting labeled solutions
`16 17 18 19`	select these orbits
`d`	default orbit display; u_1 versus time
`2 3`	select columns 2 and 3 in `s.pp3`
`d`	phase plane display; u_2 versus u_1
`2 4`	select columns 2 and 4 in `s.pp3`
`d`	phase plane display; u_3 versus u_1
`ex`	exit from 2D mode
`end`	exit from PLAUT

Table 14.15: Plotting commands for demo `pp3`.

AUTO-COMMAND	ACTION
`@p hb` or `@pp hb`	run PLAUT/PyPLAUT to graph the contents of `b.hb` and `s.hb`;
PLAUT/PyPLAUT-COMMAND	ACTION
`d0`	set defaults
`ax`	select axes
`1 6`	select real columns 1 and 6 in `b.hb`
`bd0`	plot the bifurcation diagram; p_4 versus p_1
`end`	exit from PLAUT

Table 14.16: Plotting the Hopf loci for demo `pp3`.

14.10 tor : Detection of Torus Bifurcations.

This demo uses a model in Freire, Rodríguez-Luis, Gamero & Ponce (1993) to illustrate the detection of a torus bifurcation. It also illustrates branch switching at a secondary periodic bifurcation with double Floquet multiplier at $z = 1$. The computational results also include folds, homoclinic orbits, and period-doubling bifurcations. Their continuation is not illustrated here; see instead the demos plp, pp2, and pp3, respectively. The equations are

$$\begin{aligned} x'(t) &= \big[-(\beta+\nu)x + \beta y - a_3 x^3 + b_3(y-x)^3\big]/r, \\ y'(t) &= \beta x - (\beta+\gamma)y - z - b_3(y-x)^3, \\ z'(t) &= y, \end{aligned} \tag{14.13}$$

where $\gamma = -0.6$, $r = 0.6$, $a_3 = 0.328578$, and $b_3 = 0.933578$. Initially $\nu = -0.9$ and $\beta = 0.5$.

AUTO-COMMAND	ACTION

`mkdir tor`
`cd tor`
`demo('tor')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='tor',c='tor')`	1st run; compute a stationary solution family with Hopf bifurcation
`r2=run(r1("HB1"),IPS=2,ICP=[1,11])`	compute a family of periodic solutions; restart from `r1`.
`r3=run(r2("BP1"),ISW=-1,NMX=90)`	compute a bifurcating family of periodic solutions; restart from `r2`.
`save(r1+r2+r3,'1')`	save output to b.1, s.1, d.1

Table 14.17: Commands for running demo tor.

14.11 pen : Rotations of Coupled Pendula.

This demo illustrates the computation of rotations, i.e., solutions that are periodic, modulo a phase gain of an even multiple of π. AUTO checks the starting data for components with such a phase gain and, if present, it will automatically adjust the computations accordingly. The model equations, a system of two coupled pendula, (Doedel, Aronson & Othmer (1991)), are given by

$$\begin{aligned} \phi_1'' + \epsilon\phi_1' + \sin\phi_1 &= I + \gamma(\phi_2 - \phi_1), \\ \phi_2'' + \epsilon\phi_2' + \sin\phi_2 &= I + \gamma(\phi_1 - \phi_2), \end{aligned} \tag{14.14}$$

or, in equivalent first order form,

$$\begin{aligned} \phi_1' &= \psi_1, \\ \phi_2' &= \psi_2, \\ \psi_1' &= -\epsilon\psi_1 - \sin\phi_1 + I + \gamma(\phi_2 - \phi_1), \\ \psi_2' &= -\epsilon\psi_2 - \sin\phi_2 + I + \gamma(\phi_1 - \phi_2). \end{aligned} \tag{14.15}$$

Throughout $\gamma = 0.175$. Initially, $\epsilon = 0.1$ and $I = 0.4$.

Numerical data representing one complete rotation are contained in the file pen.dat. Each row in pen.dat contains five real numbers, namely, the time variable t, ϕ_1, ϕ_2, ψ_1 and ψ_2. The correponding parameter values are defined in the user-supplied subroutine **STPNT**.

Actually, in this example, a scaled time variable t is given in pen.dat. For this reason the period (`PAR(11)`) is also set in **STPNT**. Normally AUTO would automatically set the period according to the data in pen.dat.

The AUTO-constant `dat='pen'` in c.pen.1 causes AUTO to start from the data in pen.dat. The mesh will be suitably adapted to the solution, using the number of mesh intervals `NTST` and the number of collocation point per mesh interval `NCOL` specified in the constants-file c.pen.1.

The first run, with I as free problem parameter, starts from the solution (here `IRS=0`) in pen.dat. A period-doubling bifurcation is located, and the period-doubled family is computed in the second run. Two branch points are located, and the bifurcating families are traced out in the third and fourth run, respectively. The fifth run generates starting data for the subsequent computation of a locus of period-doubling bifurcations. The actual computation is done in the sixth run, with ϵ and I as free problem parameters.

AUTO-COMMAND	ACTION

`mkdir pen`
`cd pen`
`demo('pen')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`pen=run('pen',c='pen.1')`	locate a period doubling bifurcation; restart from pen.dat
`pen=pen+run(pen('PD1'),c='pen.2')`	a family of period-doubled (and out-of-phase) rotations. Constants changed : `IPS, NTST, ISW, NMX` append output to bifurcation diagram object pen
`pen=pen+run(pen('BP1'),c='pen.3')`	a secondary bifurcating family (without bifurcation detection). Constants changed : `IRS, ISP` append output to bifurcation diagram object pen
`pen=pen+run(pen('BP2'),c='pen.4')`	another secondary bifurcating family (without bifurcation detection). Constants changed : `IRS` append output to bifurcation diagram object pen
`save(pen,'pen')`	save pen to output-files b.pen, s.pen, d.pen
`t=run(pen('PD1'),c='pen.5')`	generate starting data for period doubling continuation. Constants changed : `IRS, ICP, ICP, ISW, NMX`
`pd=run(t,sv='pd')`	compute a locus of period doubling bifurcations; restart from `t`. Constants changed : `IRS` save output-files as b.pd, s.pd, d.pd

Table 14.18: Commands for running demo pen.

14.12 chu : A Non-Smooth System (Chua's Circuit).

Chua's circuit is one of the simplest electronic devices to exhibit complex behavior. For related calculations see Khibnik, Roose & Chua (1993). The equations modeling the circuit are

$$\begin{aligned} u_1' &= \alpha\left[\ u_2 - h(u_1)\ \right]\ , \\ u_2' &= u_1 - u_2 + u_3\ , \\ u_3' &= -\beta\ u_2\ , \end{aligned} \tag{14.16}$$

where

$$h(x) = a_1 x + \frac{1}{2}\ (a_0 - a_1)\ \left\{|\ x+1\ | - |\ x-1\ |\right\}\ ,$$

and where we take $\beta = 14.3$, $a_0 = -1/7$, $a_1 = 2/7$.

Note that $h(x)$ is not a smooth function, and hence the solution to the equations may have non-smooth derivatives. However, for the orthogonal collocation method to attain its optimal accuracy, it is necessary that the solution be sufficiently smooth. Moreover, the adaptive mesh selection strategy will fail if the solution or one of its lower order derivatives has discontinuities. For these reasons we use the smooth approximation

$$|\ x\ | \ \approx\ \frac{2x}{\pi}\ \arctan(Kx),$$

which get better as K increases. In the numerical calculations below we use $K = 10$. The free parameter is α.

AUTO-COMMAND	ACTION

`mkdir chu`
`cd chu`
`demo('chu')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='chu',c='chu')`	1st run; stationary solutions
`r2=run(r1("HB1"),IPS=2,ICP=[1,11])`	2nd run; periodic solutions, with detection of period-doubling.
`save(r1+r2,'chu')`	save all output to b.chu, s.chu, d.chu

Table 14.19: Commands for running demo chu.

14.13 phs : Effect of the Phase Condition.

This demo illustrates the effect of the phase condition on the computation of periodic solutions. We consider the differential equation

$$\begin{aligned} u_1' &= \lambda u_1 - u_2, \\ u_2' &= u_1(1 - u_1). \end{aligned} \tag{14.17}$$

This equation has a Hopf bifurcation from the trivial solution at $\lambda = 0$. The bifurcating family of periodic solutions is vertical and along it the period increases monotonically. The family terminates in a homoclinic orbit containing the saddle point $(u_1, u_2) = (1, 0)$. Graphical inspection of the computed periodic orbits, for example u_1 versus the scaled time variable t, shows how the phase condition has the effect of keeping the "peak" in the solution in the same location.

AUTO-COMMAND	ACTION

`mkdir phs`
`cd phs`
`demo('phs')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='phs',c='phs.1')`	detect Hopf bifurcation
`r2=run(r1('HB1'),c='phs.2')`	compute periodic solutions. Constants changed : `IRS, IPS, NPR`
`save(r1+r2,'phs')`	save output to b.phs, s.phs, d.phs

Table 14.20: Commands for running demo phs.

14.14 ivp : Time Integration with Euler's Method.

This demo uses Euler's method to locate a stationary solution of the following predator-prey system with harvesting :

$$\begin{aligned} u_1' &= p_2 u_1(1-u_1) - u_1 u_2 - p_1(1-e^{-p_3 u_1}), \\ u_2' &= -u_2 + p_4 u_1 u_2, \end{aligned} \tag{14.18}$$

where all problem parameters have a fixed value. The equations are the same as those in demo pp2. The continuation parameter is the independent time variable, namely `PAR(14)`.

Note that Euler time integration is only first order accurate, so that the time step must be sufficiently small to ensure correct results. Indeed, this option has been added only as a convenience, and should generally be used only to locate stationary states. Note that the AUTO-constants `DS`, `DSMIN`, and `DSMAX` control the step size in the space consisting of time, here `PAR(14)`, and the state vector, here (u_1, u_2).

AUTO-COMMAND	ACTION
`mkdir ivp`	create an empty work directory
`cd ivp`	change directory
`demo('ivp')`	copy the demo files to the work directory
`r1=run(e='ivp',c='ivp')`	time integration
`save(r1,'ivp')`	save output-files as b.ivp, s.ivp, d.ivp

Table 14.21: Commands for running demo ivp.

14.15 r3b : The Circular Restricted 3-Body Problem (CR3BP).

This demo computes periodic solutions and two-dimensional unstable manifolds of those periodic solutions in the restricted three body problem:

$$\begin{aligned}
\dot{x} &= x_p \\
\dot{y} &= y_p \\
\dot{z} &= z_p \\
\dot{x}_p &= 2y_p + x - (1-\mu)\frac{x+\mu}{{d_E}^3} - \mu\frac{x-1+\mu}{{d_M}^3} + lx_p \\
\dot{y}_p &= -2x_p + y - (1-\mu)\frac{y}{{d_E}^3} - \mu\frac{y}{{d_M}^3} + ly_p \\
\dot{z}_p &= -(1-\mu)\frac{z}{{d_E}^3} - \mu\frac{z}{{d_M}^3} + lz_p
\end{aligned}$$

where $d_E = \sqrt{(x+\mu)^2+y^2+z^2}$ and $d_M = \sqrt{(x-1+\mu)^2+y^2+z^2}$. Here $l \neq 0$ breaks the conservativeness of the system. In general, continuations involve l as a parameter, and l will then approximately stay at zero.

14.15.1 Computation of Periodic Solutions of the CR3BP

Running the Python script r3b.auto will generate the families of periodic solutions L1, H1, and V1, for the case of the mass-ratio $\mu = 0.063$:

```
auto r3b.auto | auto('r3b.auto')
```

where, as in the following examples, the left hand side command can be used at the shell prompt, and the right hand side command at the Python CLUI prompt. Note that the commands starting with @ work in both interfaces, but cannot be used in the expert scripts with a .py suffix.

For example, the data generated for the Lyapunov family L1 will consist of

b.L1	the bifurcation diagram data
s.L1	a selection of periodic orbits
d.L1	diagnostic data, including Floquet multipliers

The necessary labeled starting solutions are first computed and stored in the file s.start. Each starting solution is an equilibrium ("libration point"), and its data also contains the period of a bifurcating family of periodic orbits.

The Table below shows the label of each of the starting solutions in s.start, indicating which libration point it corresponds to, and which family of periodic orbits it will generate:

Label	Libration Pt.	Family
1	L1	L1
2	L1	V1
3	L2	L2
4	L2	V2
5	L3	L3
6	L3	V3
7	L4	V4
8	L5	V5

Note (by looking at the constant-files c.r3b.*) that actually only the starting solutions labeled 1 and 2 are used in the current calculations, as executed by the Python script.

Starting solution for other values of μ can be generated using the script compute_lps.py, for instance by running

```
autox compute_lps.py 0.05 | import compute_lps
                          | compute_lps.compute(0.05)
```

After that, it is necessary to run r3b.auto again to regenerate the families.

The demos L1a, H1a, H1b, H1c, V1a, V1b can be run subsequent to the r3b demo to compute 2D unstable manifolds of selected periodic orbits that belong to the L1, V1, and H1 families.

14.15.2 Computing Unstable Manifolds of Periodic Orbits in the CR3BP

Instructions for computing 2-d unstable manifolds of periodic orbits in the Circular Restricted 3-Body Problem (CR3BP) using AUTO-07p.

The instructions below are for the Halo family L1 in AUTO demo L1a.

Instructions for computing 2-d unstable manifolds of other periodic orbits in the CR3BP are similar (Demos H1a, H1b, H1c, V1a, V1b), and are given after these instructions.

Select a labeled solution which has exactly one Floquet multiplier with absolute value greater than 1. (Floquet multipliers can be found in the file d.L1 generated by demo r3b.) Enter the label of the periodic solution in the file L1a.auto at `label=` in L1a.auto. Also enter the size of the initial step into the direction of the unstable manifold there at `step=`. Note that representative values of these three quantities have already been entered there.
Now run the Python script L1a.auto:

```
auto L1a.auto | auto('L1a.auto')
```

This will run r3b.auto as above if this was not already done.

Through various computational steps the execution of the Python script will result in AUTO files b.L1a, s.L1a, and d.L1a, where the orbits in s.L1a constitute the manifold, which can be viewed with the graphics program `plaut04` or `r3bplaut04`:

```
@pl L1a or @r3b L1a | plot3('L1a',r3b=True)
```

The various steps executed by the Python commands in the script file L1a.auto are explained below in Tables 14.22 and 14.23, which also show the equivalent Unix shell versions of these AUTO commands.

The Python script L1aX.auto does the same as L1a.auto, but with additional calculations that generate additional AUTO data files, e.g., to detect heteroclinic connections. Some of these additional runs take quite a bit of CPU time and generate big data files.
Use

```
auto clean.auto | auto('clean.auto')
```

to remove all generated files.
REMARK 1
If the run to compute the Floquet eigenfunction is not successful, i.e., if `PAR(5)` does not become nonzero, then try to compute the Floquet eigenfunction in more stages, as in Table 14.24.
REMARK 2
One can also follow the orbit, its multiplier and eigenfunction, as in Table 14.25.

The instructions below are for the Halo family H1 in AUTO demo H1a.

Follow the instructions for L1a above, where you replace L by H throughout, for instance you can run everything in one go using

```
auto H1a.auto | auto('H1a.auto')
```

or with the extra calculations:

```
auto H1aX.auto | auto('H1aX.auto')
```

The Floquet eigenfunction is now computed from label 7 with step size -10^{-3}. The detailed commands are likewise, except for the manifold calculations in Table 14.23, and those are given in Table 14.26.

The instructions below are for the Halo family H1 in AUTO demo H1b.

Follow the instructions for L1a above, where you replace L by H, and a by b throughout; for instance you can run everything in one go using

```
auto H1b.auto | auto('H1b.auto')
```

or with the extra calculations:

```
auto H1bX.auto | auto('H1bX.auto')
```

The Floquet eigenfunction is now computed from label 3 with step size -10^{-5}. The detailed commands follow the ones for H1a above, except that there is one extra run; see Table 14.27.

`mkdir r3b`
`cd r3b`
`@dm r3b`
	`mkdir r3b`
	`cd r3b`
	`demo('r3b')`

Copy the r3b demo to the local directory r3b.	
`auto r3b.auto`	`auto('r3b.auto')`
Generate the CR3BP AUTO data files.	
`autox ext.py L1 3 -1e-5`	`import ext` `sext=ext.get('L1',3,-1e-5)`
Convert the data for a selected labeled solution from s.L1, adding a zero adjoint variable. The solution label is 3, and the initial step size into the unstable manifold is -10^{-5}. The ext.py script looks for the relevant Floquet multiplier in d.L1. The converted solution will be written in the file s.ext or stored in `sext`.	
`@r flq ext`	`flq=run(sext,c='flq',e='flq')`
Compute the Floquet eigenfunction. Free scalar variables in this run (see c.flq) are: `PAR(1)`=unfolding parameter `PAR(4)`=multiplier `PAR(5)`=norm of eigenfunction If this run is successful then PAR(5) should become nonzero, in fact, PAR(5) should reach the value 1. If the run is not successful then see REMARK 1 below.	
`@sv flq`	`save(flq,'flq')`
Save the results in b.flq, s.flq, and d.flq.	
`autox data.py`	`import data` `startman=data.get(flq('UZ1'))`
Extract data for a selected orbit from s.flq. These data are for both the orbit and its Floquet eigenfunction. It is assumed that s.flq contains only one labeled solution, with label 2. If you did the "optional" computation (see Remark 2) you may need to change the label of the restart solution:	
`autox data.py flq n`	`startman=data.get(flq(n))`
where n is the different label number. The extracted data may be saved in a file called s.startman, which contains a new starting solution that can be used as a base for the manifold computations. The orbit coordinates are at "time zero", and the Floquet eigenfunction are saved at `PAR(25:30)` and `PAR(31:36)`, respectively.	
`@R man L1a.0 startman`	`startL1a = run(startman,e='man',c='man.L1a.0')`
This step does a time integration using continuation in the "period" T, i.e., `PAR(11)`, which here is the "integration time". The labeled solutions from this run all correspond to the same orbit, except that the orbit gets longer and longer. The starting point of the orbit is the point on the periodic orbit at "time zero" plus a small distance (ε) into the direction of the unstable manifold. In AUTO, ε corresponds to `PAR(6)`. This parameter ε is initialized via the script ext.py. (The sign of ε is significant!) The parameters in this run (see c.man.L1a.0) are: `PAR(3)` =energy `PAR(21)`=x-coordinate at end point `PAR(11)`=integration time `PAR(22)`=y-coordinate at end point `PAR(12)`=length of the orbit `PAR(23)`=z-coordinate at end point	
`@sv startL1a`	`save(startL1a,'startL1a')`
Save the results in b.startL1a, s.startL1a, and d.startL1a.	

Table 14.22: Detailed AUTO shell and Python commands for the L1a demo (part 1).

`@R man L1a.1 startL1a`	`L1a=run(startL1a,c='man.L1a.1')`
Look at c.man.L1a.1 to see from which label in s.startL1a this run starts. In this run the y-coordinate of the end point (`PAR(22)`) is kept fixed, while the "period" (`PAR(11)`), i.e., the total integration time, is allowed to vary, as is the value of epsilon, i.e., `PAR(6)`. Note that if `PAR(6)` becomes "large" then the manifold may no longer be accurate. The free parameters in this run are: `PAR(3)` =energy `PAR(12)`=length of the orbit `PAR(6)` ="starting distance" `PAR(21)`=x-coordinate at end point `PAR(11)`=integration time `PAR(23)`=z-coordinate at end point	
`@sv L1a`	`save(L1a,'L1a')`
Save the results in b.L1a, s.L1a, and d.L1a.	
`@R man L1a.2 startL1a`	`L1a2=run(startL1a,c='man.L1a.2')`
Another run, starting from a longer initial orbit, which computes part of the manifold. The free parameters are the same as in the preceding run. This computation results in the orbit winding around the selected periodic L1 orbit.	
`@sv L1a2`	`save(L1a2,'L1a2')`
Save the results in b.L1a2, s.L1a2, and d.L1a2 .	

Table 14.23: Detailed AUTO shell and Python commands for the L1a demo (part 2).

Give the label of the selected solution, and a value that is smaller than the associated Floquet multiplier (magnitude greater than 1) .	
`autox ext.py L1 3 -1e-5 2000`	`import ext` `sext=ext.get('L1',3,-1e-5,2000)`
`@R flq 2 ext`	`flq=run(sext,c='flq.2',e='flq')`
Continue the approximate multiplier; If all goes well then the actual multiplier will be detected as a branch point (BP) with Label 2. Free scalar variables in this run are: `PAR(1)`=unfolding parameter `PAR(4)`=multiplier `PAR(5)`=norm of eigenfunction	
`@sv flq`	`save(flq,'flq')`
`@R flq 3`	`flq=run(flq,e='flq',c='flq.3')`
Switch branches at the BP, thereby generating the nonzero Floquet eigenfunction. The free scalar variables are : `PAR(1)`=unfolding parameter `PAR(4)`=multiplier `PAR(5)`=norm of eigenfunction If all goes well then `PAR(5)` should become nonzero, and the corresponding solution should have Label 4.	
`@sv flq`	`save(flq,'flq')`

Table 14.24: Detailed AUTO shell and Python commands for Remark 1.

`@R flq 4`	`flq=run(flq,e='flq',c='flq.4')`
Free scalar variables in this run are `PAR(1)` =unfolding parameter `PAR(4)` =multiplier `PAR(11)`=period The norm, `PAR(5)`, of the eigenfunction is fixed in this run.	
`@sv flq`	`save(flq,'flq')`

Table 14.25: Detailed AUTO shell and Python commands for Remark 2.

`@R man H1a.1 startH1a`	`H1a=run(startH1a,e='man',c='man.H1a.1')`
Look at c.man.H1a.1 to see from which label in s.startH1a this run starts. In this run the x-coordinate of the end point (`PAR(21)`) is kept fixed, while the "period" (`PAR(11)`), i.e., the total integration time, is allowed to vary, as is the value of ε, i.e., `PAR(6)`. Note that if `PAR(6)` becomes "large" then the manifold may no longer be accurate. The free parameters in this run are: `PAR(3)` =energy `PAR(12)`=length of the orbit `PAR(6)` ="starting distance" `PAR(22)`=y-coordinate at end point `PAR(11)`=integration time `PAR(23)`=z-coordinate at end point	
`@sv H1a`	`save(H1a,'H1a')`
Save the results in b.H1a, s.H1a, and d.H1a.	
`@R man H1a.2 startH1a`	`hetH1a=run(startH1a,e='man',c='man.H1a.2')`
Another run, starting from a longer initial orbit, which computes part of the manifold. The free parameters are the same as in the preceding run. This computation results in the detection of a connecting orbit.	
`@sv hetH1a`	`save(hetH1a,'hetH1a')`
Save the results in b.hetH1a, s.hetH1a, and d.hetH1a.	

Table 14.26: Detailed AUTO shell and Python commands for the H1a demo.

`@R man H1b.3 startH1b`	`het2H1b=run(startH1b,e='man',c='man.H1b.3')`
Another run, starting from a longer initial orbit, which computes part of the manifold. The free parameters are the same as in the preceding run. This computation results in the detection of another connecting orbit.	
`@sv het2H1b`	`save(het2H1b,'het2H1b')`
Save the results in b.het2H1b, s.het2H1b, and d.het2H1b.	

Table 14.27: Detailed AUTO shell and Python commands for the H1b demo.

The instructions below are for the Halo family H1 in AUTO demo H1c.

Follow the instructions for L1a above, where you replace L by H, and a by c throughout; for instance you can run everything in one go using

`auto H1c.auto`	`auto('H1c.auto')`

The Floquet eigenfunction is now computed from label 68 with step size -10^{-2}.

The detailed commands follow the ones for H1a above, except that the last run is left out, and so the H1cX.auto script is not necessary.

The instructions below are for the Halo family V1 in AUTO demo V1a.

Follow the instructions for L1a above, where you replace L by V throughout, for instance you can run everything in one go using

`auto V1a.auto`	`auto('V1a.auto')`

The Floquet eigenfunction is now computed from label 8 with step size -10^{-5}. The detailed commands are likewise, except for the manifold calculations in Table 14.23, and those are given in Table 14.28.

`@R man V1a.1 startV1a`	`V1a=run(startV1a,e='man',c='man.V1a.1')`
Look at c.man.V1a.1 to see from which label in s.startV1a this run starts. In this run the z-coordinate of the end point (`PAR(23)`) is kept fixed, while the "period" (`PAR(11)`), i.e., the total integration time, is allowed to vary, as is the value of ε, i.e., `PAR(6)`. Note that if `PAR(6)` becomes "large" then the manifold may no longer be accurate. The free parameters in this run are: `PAR(3)` =energy `PAR(12)`=length of the orbit `PAR(6)` ="starting distance" `PAR(21)`=x-coordinate at end point `PAR(11)`=integration time `PAR(22)`=y-coordinate at end point	
`@sv V1a`	`save(V1a,'V1a')`
Save the results in b.V1a, s.V1a, and d.V1a.	

Table 14.28: Detailed AUTO shell and Python commands for the V1a demo.

The instructions below are for the Halo family V1 in AUTO demo V1b.

Follow the instructions for L1a above, where you replace L by V, and a by b throughout; for instance you can run everything in one go using

`auto V1b.auto`	`auto('V1b.auto')`

or with the extra calculations:

`auto V1bX.auto`	`auto('V1bX.auto')`

The Floquet eigenfunction is now computed from label 12 with step size 10^{-5}.

The detailed commands are likewise, except for the manifold calculations in Table 14.23, and those are given in Table 14.29.

`@R man V1b.1 startV1b`	`V1b=run(startV1b,e='man',c='man.V1b.1')`
Look at c.man.V1b.1 to see from which label in s.startV1b this run starts. In this run the x-coordinate of the end point (`PAR(21)`) is kept fixed, while the "period" (`PAR(11)`), i.e., the total integration time, is allowed to vary, as is the value of ε, i.e., `PAR(6)`. Note that if `PAR(6)` becomes "large" then the manifold may no longer be accurate. The free parameters in this run are: `PAR(3)` =energy `PAR(12)`=length of the orbit `PAR(6)` ="starting distance" `PAR(22)`=y-coordinate at end point `PAR(11)`=integration time `PAR(23)`=z-coordinate at end point	
`@sv V1b`	`save(V1b,'V1b')`
Save the results in b.V1b, s.V1b, and d.V1b .	
`@R man V1b.2 startV1b`	`hetV1b=run(startV1b,e='man',c='man.V1b.2')`
Another run, starting from a longer initial orbit, which computes part of the manifold. The free parameters are the same as in the preceding run. This computation results in the detection of a connecting orbit.	
`@sv hetV1b`	`save(hetV1b,'hetV1b')`
Save the results in b.hetV1b, s.hetV1b, and d.hetV1b .	

Table 14.29: Detailed AUTO shell and Python commands for the V1b demo.

Chapter 15

AUTO Demos : BVP.

15.1 exp : Bratu's Equation.

This demo illustrates the computation of a solution family to the boundary value problem

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{u_1}, \end{aligned} \tag{15.1}$$

with boundary conditions $u_1(0) = 0, \quad u_1(1) = 0$. This equation is also considered in Doedel, Keller & Kernévez (1991*a*).

AUTO-COMMAND	ACTION

`mkdir exp`
`cd exp`
`demo('exp')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='exp',c='exp')`	1st run; compute solution family containing fold
`r2=run(r1,NTST=20)` `save(r1+r2,'exp')`	2nd run; restart at the last labeled solution, using increased accuracy save output to b.exp, s.exp, d.exp

Table 15.1: Commands for running demo exp.

15.2 int : Boundary and Integral Constraints.

This demo illustrates the computation of a solution family to the equation

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{u_1}, \end{aligned} \tag{15.2}$$

with a non-separated boundary condition and an integral constraint:

$$u_1(0) - u_1(1) - p_2 = 0, \qquad \int_0^1 u(t)dt - p_3 = 0.$$

The solution family contains a fold, which, in the second run, is continued in two equation parameters.

AUTO-COMMAND	ACTION

`mkdir int`
`cd int`
`demo('int')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='int',c='int')` `save(r1,'int')`	1st run; detection of a fold save output-files as b.int, s.int, d.int
`r2=run(r1("LP1"),ICP=[1,2],ISW=2)`	2nd run; generate starting data for a curve of folds.

`r3=run(r2)`
 	
`save(r3,'lp')`
	3rd run; compute a curve of folds; restart
	from the last and only label in `r2`.
	save the output-files as b.lp, s.lp, d.lp

Table 15.2: Commands for running demo int.

15.3 bvp : A Nonlinear ODE Eigenvalue Problem.

This demo illustrates the location of eigenvalues of a nonlinear ODE boundary value problem as bifurcations from the trivial solution family. The families of solutions that bifurcate at all five computed eigenvalues, that is, the eigenfunctions, are computed in both directions. The equations are

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -(p_1\pi)^2 u_1 + u_1^2, \end{aligned} \tag{15.3}$$

with boundary conditions $u_1(0) = 0, \quad u_1(1) = 0.$ We add the integral constraint

$$\int_0^1 u_1(t)dt - p_2 = 0.$$

Then p_2 is simply the average of the first solution component. The integral constaint gives a measure: the exact same continuations could be done without any integral conditions in just the one parameter p_1, however p_2 gives us extra possibilities to plot and stop at desirable solutions. The values that `bvp.auto` sets in `UZR` make sure that solutions are given for $p_2 = \pm 3, \pm 6, \pm 9$, and the continuation stops at $p_2 = \pm 9$, and also makes sure that $0 \le p_1 \le 5.5$.

AUTO-COMMAND	ACTION

`mkdir bvp`
`cd bvp`
`@dm bvp`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`auto bvp.auto` or `auto('bvp.auto')`	Run the script bvp.auto

Table 15.3: Commands for running demo `bvp`.

15.4 lin : A Linear ODE Eigenvalue Problem.

This demo illustrates the location of eigenvalues of a linear ODE boundary value problem as bifurcations from the trivial solution family. By means of branch switching an eigenfunction is computed, as is illustrated for the first eigenvalue. This eigenvalue is then continued in two parameters by fixing the L_2-norm of the first solution component. The eigenvalue problem is given by the equations

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= (p_1\pi)^2 u_1, \end{aligned} \tag{15.4}$$

with boundary conditions $u_1(0) - p_2 = 0$ and $u_1(1) = 0$. We add the integral constraint

$$\int_0^1 u_1(t)^2 dt - p_3 = 0.$$

Then p_3 is simply the L_2-norm of the first solution component. In the first two runs p_2 is fixed, while p_1 and p_3 are free. In the third run p_3 is fixed, while p_1 and p_2 are free.

AUTO-COMMAND	ACTION

`mkdir lin`
`cd lin`
`demo('lin')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='lin',c='lin')`	1st run; compute the trivial solution family and locate eigenvalues.
`r2=run(r1('BP1'),NTST=6,ISW=-1,DSMAX=0.5)`	2nd run; compute a few steps along the bifurcating family.
`save(r1+r2,'lin')`	save all output to b.lin, s.lin, d.lin
`r3=run(r2('UZ1'),ICP=[1,2],NTST=5,ISW=1)`	3rd run; compute a two-parameter curve of eigenvalues.
`save(r3,'2p')`	save the output-files as b.2p, s.2p, d.2p

Table 15.4: Commands for running demo lin.

15.5 non : A Non-Autonomous BVP.

This demo illustrates the continuation of solutions to the non-autonomous boundary value problem

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{x^3 u_1}, \end{aligned} \tag{15.5}$$

with boundary conditions $u_1(0) = 0, \quad u_1(1) = 0$. Here x is the independent variable. This system is first converted to the following equivalent autonomous system :

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{u_3^3 u_1}, \\ u_3' &= 1, \end{aligned} \tag{15.6}$$

with boundary conditions $u_1(0) = 0, \quad u_1(1) = 0, \quad u_3(0) = 0$. (For a periodically forced system see demo frc).

AUTO-COMMAND	ACTION
`mkdir non`	create an empty work directory
`cd non`	change directory
`demo('non')`	copy the demo files to the work directory
`r1=run(e='non',c='non')`	compute the solution family
`save(r1,'non')`	save output-files as b.non, s.non, d.non

Table 15.5: Commands for running demo non.

15.6 kar : The Von Karman Swirling Flows.

The steady axi-symmetric flow of a viscous incompressible fluid above an infinite rotating disk is modeled by the following ODE boundary value problem (Equation (11) in Lentini & Keller (1980) :

$$\begin{aligned} u_1' &= Tu_2, \\ u_2' &= Tu_3, \\ u_3' &= T\big[-2\gamma u_4 + u_2^2 - 2u_1u_3 - u_4^2\big], \\ u_4' &= Tu_5, \\ u_5' &= T\big[2\gamma u_2 + 2u_2u_4 - 2u_1u_5\big], \end{aligned} \tag{15.7}$$

with left boundary conditions

$$u_1(0) = 0, \qquad u_2(0) = 0, \qquad u_4(0) = 1 - \gamma,$$

and (asymptotic) right boundary conditions

$$\begin{aligned} &\big[f_\infty + a(f_\infty,\gamma)\big]\ u_2(1) + u_3(1) - \gamma\ \tfrac{u_4(1)}{a(f_\infty,\gamma)} = 0, \\ &a(f_\infty,\gamma)\ \tfrac{b^2(f_\infty,\gamma)}{\gamma}\ u_2(1) + \big[f_\infty + a(f_\infty,\gamma)\big]\ u_4(1) + u_5(1) = 0, \\ &u_1(1) = f_\infty, \end{aligned} \tag{15.8}$$

where

$$\begin{aligned} a(f_\infty,\gamma) &= \tfrac{1}{\sqrt{2}}\big[(f_\infty^4 + 4\gamma^2)^{1/2} + f_\infty^2\big]^{1/2}, \\ b(f_\infty,\gamma) &= \tfrac{1}{\sqrt{2}}\big[(f_\infty^4 + 4\gamma^2)^{1/2} - f_\infty^2\big]^{1/2}. \end{aligned} \tag{15.9}$$

Note that there are five differential equations and six boundary conditions. Correspondingly, there are two free parameters in the computation of a solution family, namely γ and f_∞. The "period" T is fixed; $T = 500$. The starting solution is $u_i = 0$, $i = 1, \cdots, 5$, at $\gamma = 1$, $f_\infty = 0$.

AUTO-COMMAND	ACTION
`mkdir kar`	create an empty work directory
`cd kar`	change directory
`demo('kar')`	copy the demo files to the work directory
`r1=run(e='kar',c='kar')`	computation of the solution family
`save(r1,'kar')`	save output-files as b.kar, s.kar, d.kar

Table 15.6: Commands for running demo kar.

15.7 spb : A Singularly-Perturbed BVP.

This demo illustrates the use of continuation to compute solutions to the singularly perturbed boundary value problem

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= \tfrac{\lambda}{\epsilon}\big(u_1 u_2(u_1^2 - 1) + u_1\big), \end{aligned} \tag{15.10}$$

with boundary conditions $u_1(0) = 3/2$, $u_1(1) = \gamma$. The parameter λ has been introduced into the equations in order to allow a homotopy from a simple equation with known exact solution to the actual equation. This is done in the first run. In the second run ϵ is decreased by continuation. In the third run ϵ is fixed at $\epsilon = .001$ and the solution is continued in γ. This run takes more than 1500 continuation steps. For a detailed analysis of the solution behavior see Lorenz (1982).

AUTO-COMMAND	ACTION

`mkdir spb`
`cd spb`
`demo('spb')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='spb',c='spb.0')` `save(r1,'0')`	1st run; homotopy from $\lambda = 0$ to $\lambda = 1$ save output-files as b.0, s.0, d.0

`r2=run(r1,c='spb.1')`
 	
`save(r2,'1')`
	2nd run; let ϵ tend to zero; restart from the last label of `r0`. constants changed : `IRS, ICP(1), NTST, DS, UZR, STOP`
	save the output-files as b.1, s.1, d.1

`r3=run(r2('UZ2'),c='spb.3')`
 	
`save(r3,'2')`
	3rd run; continuation in γ; $\epsilon = 0.001$; restart from 2nd UZ label of `r2`. Constants changed : `IRS, ICP(1), ITNW, EPSL, EPSU, UZR`
	save the output-files as b.2, s.2, d.2

Table 15.7: Commands for running demo spb.

15.8 ezp : Complex Bifurcation in a BVP.

This demo illustrates the computation of a solution family to the complex boundary value problem

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{u_1}, \end{aligned} \tag{15.11}$$

with boundary conditions $u_1(0) = 0,\ u_1(1) = 0$. Here u_1 and u_2 are allowed to be complex, while the parameter p_1 can only take real values. In the real case, this is Bratu's equation, whose solution family contains a fold; see the demo exp. It is known (Henderson & Keller (1990)) that a simple quadratic fold gives rise to a pitch fork bifurcation in the complex equation. This bifurcation is located in the first computation below. In the second and third run, both legs of the bifurcating solution family are computed. On it, both solution components u_1 and u_2 have nontrivial imaginary part.

AUTO-COMMAND	ACTION

`mkdir ezp`
`cd ezp`
`demo('ezp')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`ezp=run(e='ezp',c='ezp')`	1st run; compute solution family containing fold

`ezp=ezp+run(ezp('BP1'),ISW=-1)`
 	
`ap('ezp')`
	2nd run; compute bifurcating complex solution family.
	append output-files to p.ezp, s.ezp, d.ezp

`ezp=ezp+run(ezp('BP1'),ISW=-1,DS='-')`
 	
`save(ezp,'ezp')`
	3rd run; compute 2nd leg of bifurcating family.
	save combined output to b.ezp, s.ezp, d.ezp

Table 15.8: Commands for running demo ezp.

15.9 um2 : Basic computation of a 2D unstable manifold.

This demo shows how one can compute a 2D unstable manifold of an equilibrium using *orbit continuation*. The model equations are given by

$$\begin{aligned} x' &= \epsilon x - y^3 \ , \\ y' &= y + x^3 \ . \end{aligned} \tag{15.12}$$

The origin has eigenvalues ϵ and 1, where $\epsilon > 0$, so that its unstable manifold is indeed 2-dimensional. Since the phase space itself is 2-dimensional, one can also consider this demo as showing how to generate part of a 2-dimensional phase portrait. However, the basic steps in this demo also apply to the computation of 2-dimensional unstable manifolds of equilibria in higher-dimensional phase space.

In the computations the independent time variable t is scaled to vary in the unit interval, so that the actual integration time T becomes an explicit parameter in the equations, namely,

$$\begin{aligned} x' &= T\ (\epsilon x - y^3) \ , \\ y' &= T\ (\ y + x^3) \ . \end{aligned} \tag{15.13}$$

To carry out the calculations run the Python script um2.auto included in the demo. In order to better appreciate the power of orbit continuation for computing such manifolds, one can also run the script for a smaller value of ϵ, *e.g.*, $\epsilon = 10^{-2}, \cdots, 10^{-6}$, by changing the value entered on the last line of the constants-file c.umn.2. One can view the phase portrait by plotting the solutions in the solutions-file s.3 in the x-y plane.

In the first run an orbit is "grown" by continuation in the integration time T, starting from a very small value of T, so that a solution that is constant in time is an accurate initial approximation. The starting solution is in fact a point on a circle of small radius r_0 around the stationary point, *i.e.*, around the origin. For illustrative purpose the value of r_0 is 0.1 in this demo, but could be smaller if more accuracy is needed. The precise starting point is in the strongly unstable direction, namely, in the y-direction, which is the direction of the eigenvector associated with the "strongly unstable eigenvalue", which here has value 1. The growing of the initial orbit is terminated when the norm of its endpoint reaches the value 0.6, *i.e.*, when $\sqrt{x(1)^2 + y(1)^2} = 0.6$.

The value of ϵ is 0.5 in the first run. In the second run continuation is used to decrease the value of ϵ to 0.1 (or, if desired, to a smaller value, as already mentioned above). The norm of the endpoint $(x(1), y(1))$ is fixed in the second run, while T is variable.

In the third run the norm of the endpoint remains fixed. However, the initial point $(x(0), y(0))$ is allowed to move around the small circle of radius r_0 around the orgin. The endpoint thereby moves around the "large" circle of radius 0.6. The integration time T remains variable. The orbits computed in this run generate the local manifold.

When viewing the orbits computed in the third run, as written in the file s.3, notice that orbits near the "weakly unstable direction", which here corresponds to the x direction, have been well-computed. Such orbits are sensitively dependent on the initial condition $(x(0), y(0))$ when the problem is considered as an initial value problem. It is in fact the *continuation* of the *entire orbits* using a *boundary value approach* which enables their determination. As already mentioned, this feature is even more visible when running this demo with a smaller value of ϵ.

15.10 um3 : A 2D unstable manifold in 3D.

This demo uses *orbit continuation* to compute part of the 2D unstable manifold of the origin of the equations

$$\begin{aligned} x' &= \epsilon x - y^3 + z^3 \ , \\ y' &= y + x^3 \ , \\ z' &= -z + x^2 \ . \end{aligned} \tag{15.14}$$

The origin has unstable eigenvalues ϵ and 1, where $\epsilon > 0$, so that its unstable manifold is 2-dimensional. In the computations the independent time variable t is scaled to vary in the unit interval, so that the actual integration time T becomes an explicit parameter in the equations; see, for example, demo um2. The calculations can be done by running the Python script um3.auto included in the demo.

In the first run an orbit is "grown" by continuation in the integration time T, starting from a very small value of T, so that a solution that is constant in time is an accurate initial approximation. The starting solution is in fact a point on a circle of small radius $r_0 = 0.03$ in the unstable eigenspace of the origin. The precise starting point is in the strongly unstable direction, namely, the y-direction. The initial orbit is grown until the L_2-norm of its endpoint $(x(1), y(1), z(1))$ reaches the value 1. The value of ϵ is fixed at 0.5 in the first run. In the second run continuation is used to decrease ϵ to 0.01. The norm of the endpoint $(x(1), y(1), z(1))$ is fixed in this run, while T is variable.

In the third run the norm of the endpoint remains fixed, but the initial point $(x(0), y(0), z(0))$ is allowed to move around the small circle of radius r_0 in the unstable eigenspace of the origin. The endpoint thereby moves on the surface of the "large" sphere of radius 1. The integration time T remains variable. The orbits computed in this run generate the local manifold.

When viewing the orbits computed in the third run, as written in the file s.3, notice that there appears to be second equilibrium with a 2D stable manifold which intersects the 2D unstable manifold of the origin. The intersection curve, which corresponds to a heteroclinic orbit, is visible in the graphical representation of the manifold.

15.11 p2c : Point to cycle connections.

In this demo a point to cycle heteroclinic connection is computed via homotopy, and then continued in two system parameters, in the Lorenz equations

$$\begin{aligned} u_1' &= p_3(u_2 - u_1), \\ u_2' &= p_1 u_1 - u_2 - u_1 u_3, \\ u_3' &= u_1 u_2 - p_2 u_3. \end{aligned}$$

Type `auto p2c.auto` to run the demo and `auto clean.auto` to remove generated files.

Refer to Doedel, Kooi, van Voorn & Kuznetsov (2008*a*) and `http://www.bio.vu.nl/thb/research/project/globif/index_main.html` for background information.

15.12 c2c : Cycle to cycle connections.

In this demo a cycle to cycle heteroclinic connection is computed via homotopy, and then continued in one system parameter, in a food chain model:

$$\begin{aligned} x' &= x(1-x) - \frac{5xy}{1+3x}, \\ y' &= \frac{5xy}{1+3x} - p_1 y - \frac{0.1yz}{1+2y}, \\ z' &= \frac{0.1yz}{1+2y} - p_2 z. \end{aligned}$$

Type `auto c2c.auto` to run the demo. After that it is possible to compute two-parameter continuations of folds of the cycle-to-cycle connection, but since this is computationally intensive it is put in a seperate file: type `auto c2cfolds.auto`. Type `auto clean.auto` to remove all generated files.

Refer to Doedel, Kooi, van Voorn & Kuznetsov (2008*b*) and `http://www.bio.vu.nl/thb/research/project/globif/index_main.html` for background information.

15.13 pcl : Lorenz: Point-to-cycle connections with Lin's method.

This demo computes a point-to-cycle connection (or EtoP connection; for equilibrium to periodic orbit) in the Lorenz equations

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= \rho x - y - xz, \\ z' &= xy - \beta z, \end{aligned} \tag{15.15}$$

using Lin's method, as described in Krauskopf & Rieß (2008). Initially, we fix $\beta = 8/3$, $\sigma = 10$, and let ρ vary, starting from 0. Here we have a transition from simple to chaotic dynamics. For approximate values of ρ in the interval $[13.9265, 24.0579]$ one finds preturbulence, organized by a pair of symmetrically related periodic orbits that emanate from a homoclinic bifurcation at $\rho \approx 13.9265$. For $\rho \approx 24.0579$ there exist two symmetric point-to-cycle connections that mark the appearance of a chaotic attractor for higher values of ρ.

The computation to find one of these two connections uses the following steps:

1. Find the secondary equilibria emanating from the pitchfork bifurcation of the equilibrium at 0, and their Hopf bifurcations, similarly to the demo lrz.

2. Follow the periodic orbit emanating from the Hopf bifurcation in ρ and its period T, until $\rho = 24.0579$, a value close to where a point-to-cycle connection is known to exist.

3. Extend the system, putting the variational equation (the eigenfunction) into solution coordinates 4, 5, and 6. The trivial (0) eigenvector is continued until we hit a branch point, corresponding to where $\mu =$ `PAR(12)` equals the natural logarithm of a Floquet multiplier.

4. Switch branches and continue the non-trivial eigenvector until its norm h equals 1.

5. Extend the system again from 6 to 9 dimensions to calculate a connection from the cross section Σ, given by $x = 10$, to the periodic orbit. The connection starts at the non-trivial eigenvector with respect to the periodic orbit with a distance of $\delta = 10^{-7}$, and grows backwards in time until it hits the section Σ at time T^+.

6. Extend the system one last time from 9 to 12 dimensions. We calculate a connection from the equilibrium at 0, starting at its eigenvector over a distance of $\varepsilon = 10^{-7}$, to the cross section Σ, at time T^-. The second intersection is the one that is closest to the intersection computed in step 5, and it is the one we want.

7. Firstly, in the routine **PVLS** in the equations file pcl.f90, the Lin vector, a normalized vector between the points of intersection computed in steps 5 and 6 is put in Z_x, Z_y and Z_z (`PAR(24)-PAR(26)`). Starting data for the Lin gap, which measures the distance between these two points of intersection, is put in $\eta =$ `PAR(23)`. Subsequently, close this gap by continuing in η, ρ, δ, ε, T^-, T^+, μ, and T. This process is illustrated in Figure 15.1.

8. Continue the point to cycle connection obtained in step 7 in the system parameters ρ and β, together with δ, ε, T^-, T^+, μ, and T. Connections for various values of (ρ, β) are shown in Figure 15.2.

The above sequence of calculations can be carried out by running the Python script pcl.auto (without constants files) or pclc.auto (with constants files) included in the demo. See the script and the Fortran file pcl.f90 for details on how all parameters are mapped and which precise AUTO constants are changed at every step.

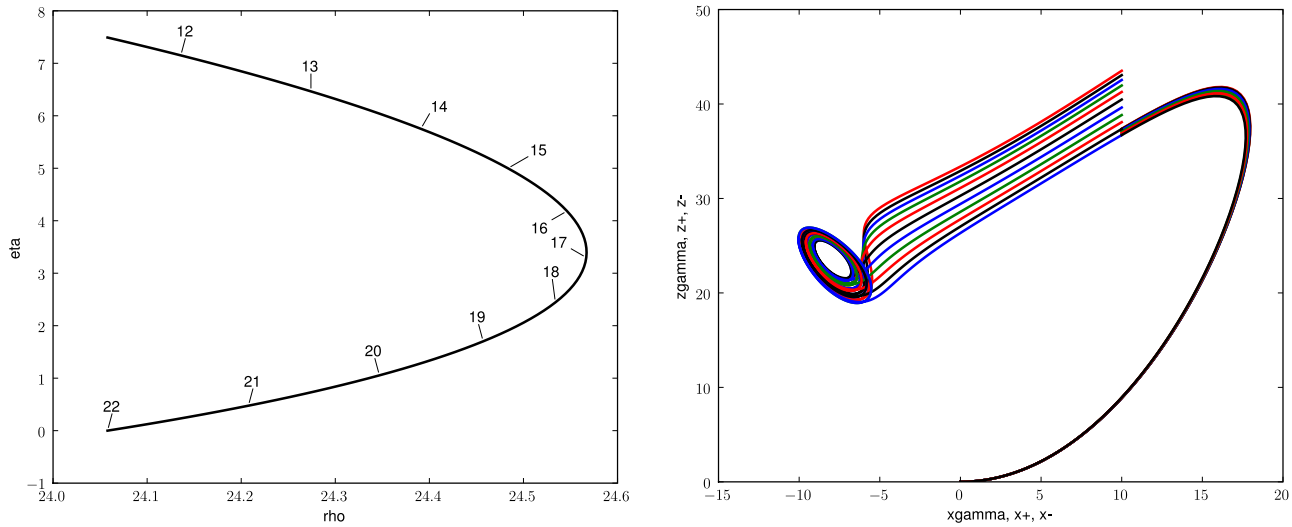


Figure 15.1: Closing the Lin-gap to obtain the point-to-cycle connection. The left panel is a plot of ρ versus the gap size η, and the right panel shows the corresponding orbit segments, projected onto the (x, z)-plane. To obtain these figures run `plot('closegap')` or `@pp closegap`.

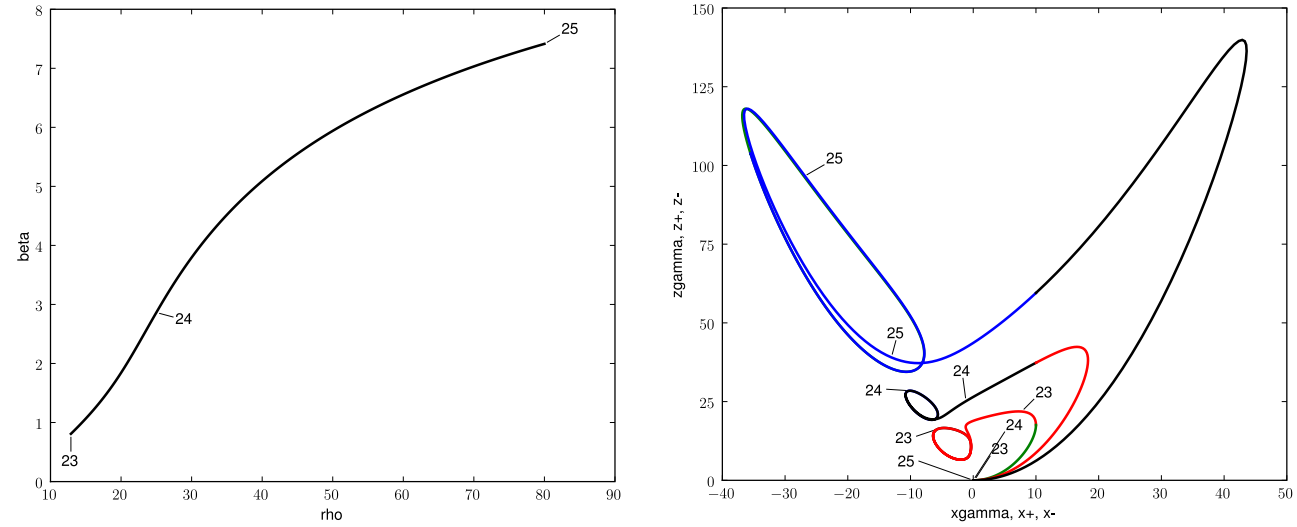


Figure 15.2: Parameter space diagram (left) and corresponding orbit segments in phase space (right), where the connection is continued in ρ and β. To obtain these figures run `plot('cont')` or `@pp cont`.

15.14 snh : SNH with Global reinjection: Point-to-cycle connections with Lin's method.

This demo computes point-to-cycle (or EtoP connection; for equilibrium to periodic orbit) and homoclinic point-to-point connections in the model vector field

$$\begin{aligned} x' &= \nu_1 x - \omega y - (\alpha x - \beta y)\sin\varphi - (x^2+y^2)x + d(2\cos\varphi + \nu_2)^2, \\ y' &= \nu_1 y + \omega x - (\alpha y + \beta x)\sin\varphi - (x^2+y^2)y + f(2\cos\varphi + \nu_2)^2, \\ \varphi' &= \nu_2 + s(x^2+y^2) + 2\cos\varphi + c(x^2+y^2)^2, \end{aligned} \tag{15.16}$$

using Lin's method, as described in Krauskopf and Rieß(2008). This system describes the dynamics near a saddle-node Hopf bifurcation with global reinjection, as discussed in Krauskopf and Oldeman(2006). We keep the following parameters fixed throughout: $\omega = 1$, $\alpha = -1.0$, $\beta = 0$, $s = -1$, $c = 0$, $d = 0.01$, and $f = \pi d$.

Initially, we fix $\nu_2 = -1.46$, since for that value of ν_2, close to $\nu_1 = 0.74$, there exists a codimension-one connection from a periodic orbit (Γ) to an equilibrium (b). Together with a codimension-zero connection back to Γ, it forms a heteroclinic cycle. In this example, the flow is such that the codimension-one connecting orbit is an EtoP orbit where the flow is from the periodic orbit to the equilibrium. A homoclinic orbit to b also approaches this cycle. Below, we compute all three connecting orbits.

The below sequences of calculations can be carried out by running the Python script snh.auto included in the demo. The individual connections can be computed by running the scripts h1b.auto (homoclinic orbit), cb.auto (codimension-one EtoP), and tb.auto (codimension-zero EtoP). See the scripts and the Fortran file snh.f90 for details on how all parameters are mapped and which precise AUTO constants are changed at every step.

15.14.1 The homoclinic point-to-point connection.

The starting point for this investigation is a homoclinic orbit connecting the point b to itself, where the phase of φ is shifted by 2π (in other words, the homoclinic orbit reinjects once and it is a heteroclinic orbit in the covering space). We can most easily compute the homoclinic orbit using a homotopy method (see the HomCont section 20.7 for details):

1. We locate the homoclinic orbit, or here, the heteroclinic orbit in the covering space by continuing the one-dimensional stable manifold in negative time. This way, HomCont views the stable manifold as a one-dimension unstable manifold to which its standard homotopy method can be applied and which makes the method much more straightfoward than starting with a two-dimensional unstable manifold. We reach the unstable eigenspace of $E^u(b)$ as soon as the artificial dummy parameter ω_1, measuring a distance to $E^u(b)$, vanishes.

2. We can now improve this connection by continuing in decreasing negative time, keeping ω_1 fixed, and freeing up the system parameter ν_1.

3. The resulting orbit can be continued forwards and backwards in the system parameter ν_1 and ν_2, using standard HomCont settings. Note the setting of `IEQUIB=1`: HomCont

auto-detects the phase shift and only continues one equilibrium instead of treating the orbit as a general heteroclinic orbit.

The resulting homoclinic orbit snakes in parameter space between the two tangencies of the codimension-zero connection and terminates at a segment of the codimension-one EtoP connection. We show this in Figure 15.5.

15.14.2 The codimension-one point-to-cycle connection.

To compute the codimension-one point-to-cycle connection, the following steps are used (very similar to those in the demo `pcl`):

1. Continue the first equilibrium a at $x = y = 0$, $\varphi = -\arccos(-\nu_2/2)$, which undergoes a Hopf bifurcation at $\nu_1 \approx 0.683447$.

2. Follow the periodic orbit, emanating from the Hopf bifurcation, in ν_1 and its period T, until $\nu_1 = 0.74$.

3. Extend the system, putting the variational equation (the eigenfunction) into solution coordinates 4, 5, and 6. The trivial (0) eigenvector is continued until we hit a branch point, corresponding to where $\mu =$ `PAR(12)` equals the natural logarithm of a Floquet multiplier.

4. Switch branches and continue the non-trivial eigenvector until its norm h equals 1.

5. Extend the system again from 6 to 9 dimensions to calculate a connection from the periodic orbit to the cross section Σ, given by $\varphi = \pi$. The connection starts at the non-trivial eigenvector with respect to the periodic orbit with a distance of $\delta = -10^{-5}$, and grows forwards in time until it hits the section Σ at time T^-.

6. Extend the system one last time from 9 to 12 dimensions. We calculate a connection backwards in time from the second equilibrium b at $x = y = 0$, $\varphi = \arccos(-\nu_2/2)$, starting at its eigenvector over a distance of $\varepsilon = 10^{-6}$, to the cross section Σ, at time T^+. The Lin vector, a normalized vector between the points of intersection computed in steps 5 and 6, is put in Z_x, Z_y and Z_z (`PAR(24)-PAR(26)`). Starting data for the Lin gap, which measures the distance between these two points of intersection is put in $\eta =$ `PAR(23)`.

7. Close the gap computed in step 6 by continuing in η, ν_1, δ, ε, T^+, T^-, μ, and T. The connection is found at $\nu_1 = 7.41189$. This process is illustrated in Figure 15.3.

8. Continue the point to cycle connection obtained in step 7 in the system parameters ν_1 and ν_2, together with δ, ε, T^+, T^-, μ, and T. Connections for various values of (ν_1, ν_2) are shown in Figure 15.4.

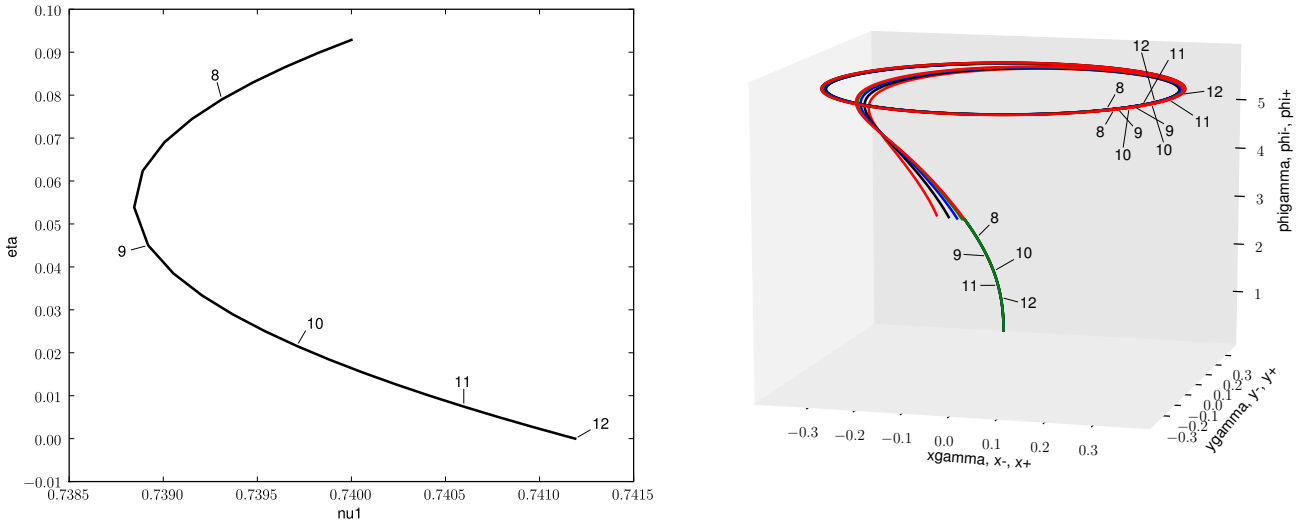


Figure 15.3: Closing the Lin-gap to obtain the point-to-cycle connection. The left panel is a plot of ν_1 versus the gap size η, and the right panel shows the corresponding orbit segments. To obtain these figures run `plot('closegap')` or `@pp closegap`, and `plot3('closegap')` or `@pl closegap`.

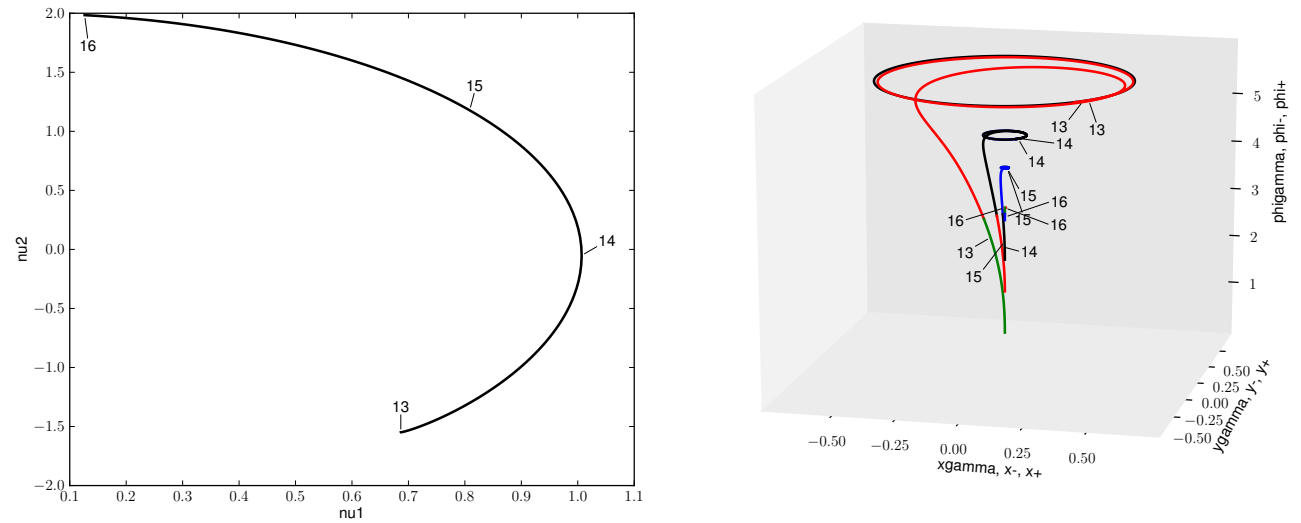


Figure 15.4: Parameter space diagram (left) and corresponding orbit segments in phase space (right), where the connection is continued in ν_1 and ν_2. To obtain these figures run `plot('cb')` or `@pp cb`, and `plot3('cb')` or `@pl cb`. Label 13 denotes the largest connection, at a saddle-node bifurcation of limit cycles, and label 16 the smallest one, where the periodic orbit disappears in a Hopf bifurcation.

15.14.3 The codimension-zero point-to-cycle connection.

Next we compute the codimension-zero connection back to the cycle which must also exist near the accumulation of the parameter space curve h_1^b. This computation starts in the same way as

the computation of the codimension-one connection: steps 1 to 4 are the same except that in step 3 we now find the negative Floquet exponent μ, instead of the positive exponent. Steps 5 to 8 proceed as follows:

5. Similarly to step 5 before, we compute an orbit in the stable manifold of the periodic orbit. However, Lin's method is not used, because it is easier to use a homotopy method to connect directly to the unstable eigenspace $E^u(b)$, which is given by the section where $\varphi = \arccos(-\nu_2/2)$. Because we compute an approximation to the stable rather than the unstable manifold using almost the same boundary value problem, we let T^- be negative. Also the distance from the periodic orbit to the connection is now given by $\delta = 10^{-4}$, flipping its sign.

6. Improve the connection computed in step 5, by decreasing the negative value of T^-, fixing the starting point in $E^u(b)$ and freeing δ.

7. Follow the codimension-zero connection in the system parameter ν_1, together with μ, T, δ, and T^-, also adding an integral condition for the connection. AUTO detects two fold points (LP), corresponding to tangencies of $W^u(b)$ and $W^s(\Gamma)$.

8. Continue the two folds forwards and backwards in two parameters, by adding the system parameter ν_2 and setting the AUTO constant `ISW=2`. The folds terminate where Γ disappears in a Hopf bifurcations (small ν_1) and disappears in a saddle-node bifurcation of limit cycles (large ν_1). In both cases the continued connection and orbit stop converging, so AUTO reports MX.

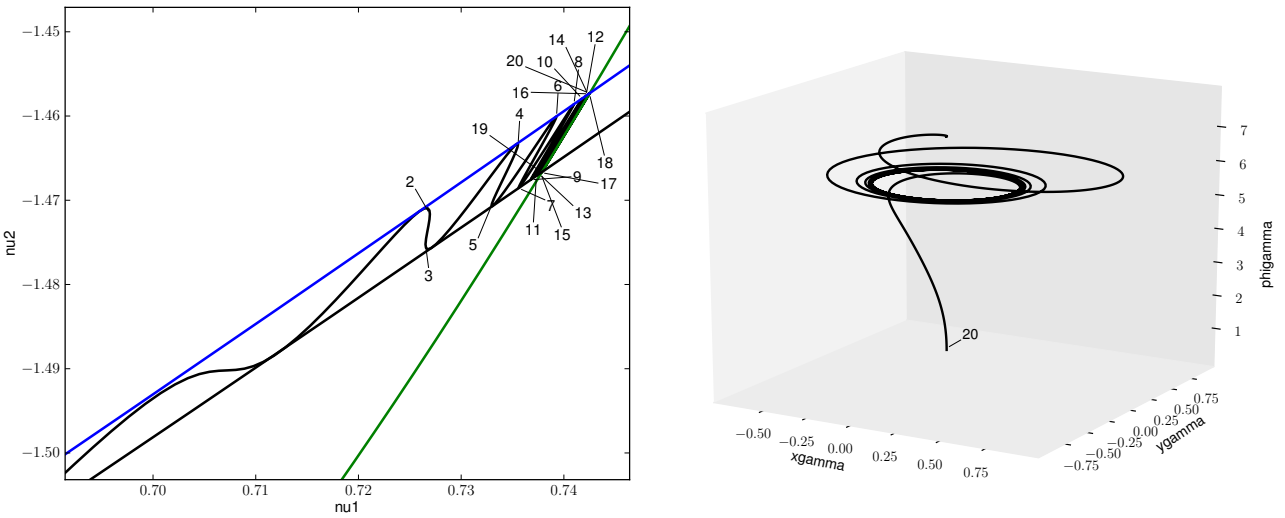


Figure 15.5: Parameter space diagram (left) and the corresponding homoclinic orbit on the snaking curve h_1^b for label 20 (right). The snaking curve is in between the two tangencies for the codimension-zero EtoP connection t_b and terminates at a segment of the codimension-one EtoP connection c_b. To obtain these figures run `plot('all')` or `@pp all`, and `plot3('all')` or `@pl all`.

15.15 fnc : Canards in the FitzHugh-Nagumo system.

This demo computes attracting and repelling slow manifolds in the self-coupled FitzHugh-Nagumo system:

$$\begin{aligned} v' &= h - (v^3 - v + 1)/2 - \gamma s v, \\ h' &= -\varepsilon(2h + 2.6v), \\ s' &= -\varepsilon\delta s. \end{aligned} \tag{15.17}$$

Furthermore, this demo continues canard orbits in parameter space. Typically, trajectories in slow-fast systems such as this one consist of a slow part that follows the attracting slow manifold, followed by a fast part when the trajectory hits a fold with respect to the fast direction. After this jump, the trajectory follows a slow segment again. A canard orbit, on the other hand, does not jump at the fold but follows the repelling slow manifold. A central role is played by the two-dimensional critical manifold S which is given by the nullcline of the fast variable in the limit for $\varepsilon = 0$. It consists of attracting and repelling sheets S^a and S^r, which generically meet at fold curves F with respect to the fast flow direction. For details, see Desroches, Krauskopf & Osinga (2008, **?**).

For system (15.17), where we fix $\gamma = 0.5$ and $\delta = 0.565$, the critical manifold is given by

$$S = \{(v, h, s) \in \mathbb{R}^3 | 2h - v^3 + v - 1 - vs = 0\}, \tag{15.18}$$

which is folded with respect to the fast variable v along the folded node curve

$$F = \{(v, h, s) \in S | 1 - 3v^2 = 0\}. \tag{15.19}$$

The computation of the slow manifolds is performed in three steps. They start with a constant-in-time solution at the folded node singularity $(v, h, s) = (-0.49, 0.6176, 0.2797)$ for $\varepsilon = 0.015$ on the scaled time interval $[0, 1]$ with time lengths $T^a = 0$ and $T^r = 0$, respectively. Then, for the computation of the attracting manifold (v^a, h^a, s^a), via boundary conditions we always keep the starting point $(v^a(0), h^a(0), s^a(0))$ on S and $s^a(1)$ fixed to 0.2797. The steps are as follows:

1. Homotopy step 1: grow the orbit segment in T^a, continuing also in $v^a(0)$, $h^a(0)$, and $s^a(0)$, where the starting point is kept on the folded node F until $s^a(0) = 0.6$.

2. Homotopy step 2: the extra boundary condition for F is dropped, and we now instead fix $s^a(0) = 0.6$. We continue in $v^a(0)$, $h^a(0)$, and T^a until $h^a(0) = -6.0$.

3. Actual computation: we continue in $v^a(0)$, $s^a(0)$, and T^a, fixing $h^a(0) = -6.0$. The end-point coordinates $v^a(1)$ and $h^a(1)$ are monitored so they can be matched with starting points of the repelling manifold. These matches were found manually and are now indicated at specific values of T^a as UZ labels.

For the repelling manifold (v^r, h^r, s^r), we always keep the end point $(v^r(1), h^r(1), s^r(1))$ on S and $s^r(0)$ fixed to 0.2797. The steps are now as follows:

1. Homotopy step 1: grow the orbit segment in T^r until $s^r(1) = 0.05$.

2. Homotopy step 2: the extra boundary condition for F is dropped, and we now instead fix $s^r(1) = 0.05$. We continue until $v^r(1) = 0$.

3. Actual computation: we continue in $s^r(0)$ and T^r, fixing $v^r(1) = 0$. The starting point coordinates $v^r(0)$ and $h^r(0)$ are now monitored for matches with the attracting manifold.

The demo contains one folder attr for the "attracting" slow manifold and one folder rep for the "repelling" slow manifold. In each of these two folders, the manifolds can be computed by running the commands auto attr.auto and auto rep.auto, respectively. The alternative script files attrc.auto and repc.auto use constants files for every step instead of specifying the constants in the script.

The main folder contains AUTO files to continue in parameter space six of the secondary canards of this systems. The script fnc.auto first computes the attracting and repelling slow manifolds in their respective folders and next concatenates matching manifolds in Python. Six of these canard orbits are then continued in ε in both decreasing and increasing direction.

If T_j^a is the integration time for the canard segment ξ_j^a on the attracting slow manifold and T_j^r is the integration time for the orbit segment ξ_j^r on the repelling slow manifold corresponding to the same canard solution, then the concatenated orbit segment ξ_j has $T_j^c = T_j^a + T_j^r$ as integration time. The orbit ξ_j is obtained from ξ_j^a and ξ_j^r by, concatenating ξ_j^a and ξ_j^r, and rescaling the time so that it runs monotonically from 0 to 1.

Parameters are then also copied, where only those relating to the start of the attracting manifold and the end of the repelling manifold are kept, for these parameters are used for the boundary conditions of the canard orbits. The new integration time T_j^c is stored in PAR(11).

Two constant files are provided. They correspond to the continuation in epsilon in both decreasing (c.fnc) and and increasing (c.fnc.epsplus) direction. Continuation in any other system parameter is easily obtained by editing the ICP entry.

The script plot.auto produces the plots we show here, in Figures 15.6, 15.7, and 15.8.

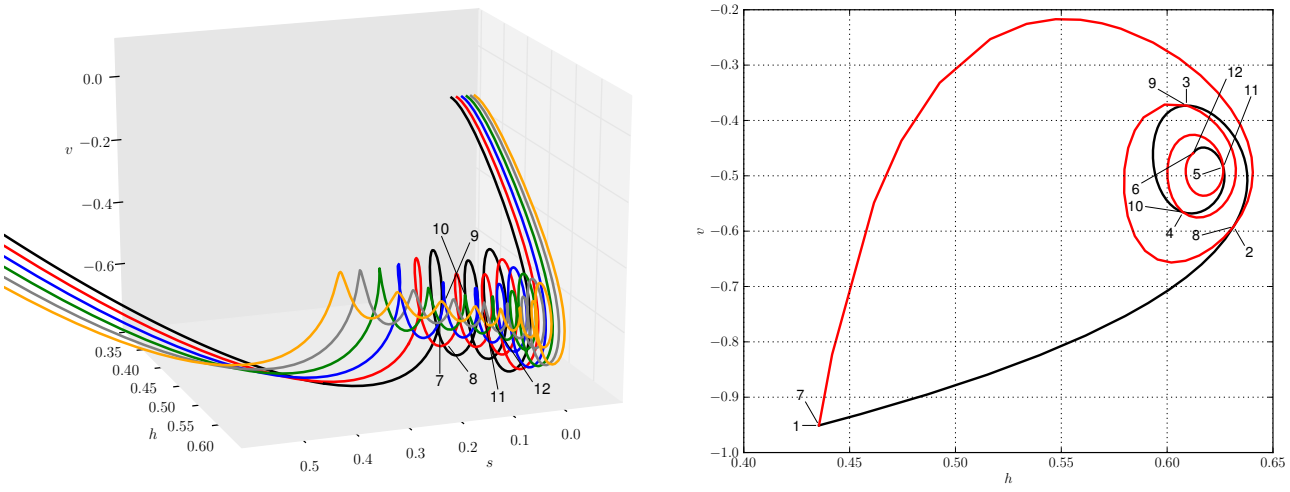


Figure 15.6: Repelling and attracting slow manifold curves where they match up (left) and their intersection curves at $s = 0.2797$ in the (h, v) plane (right).

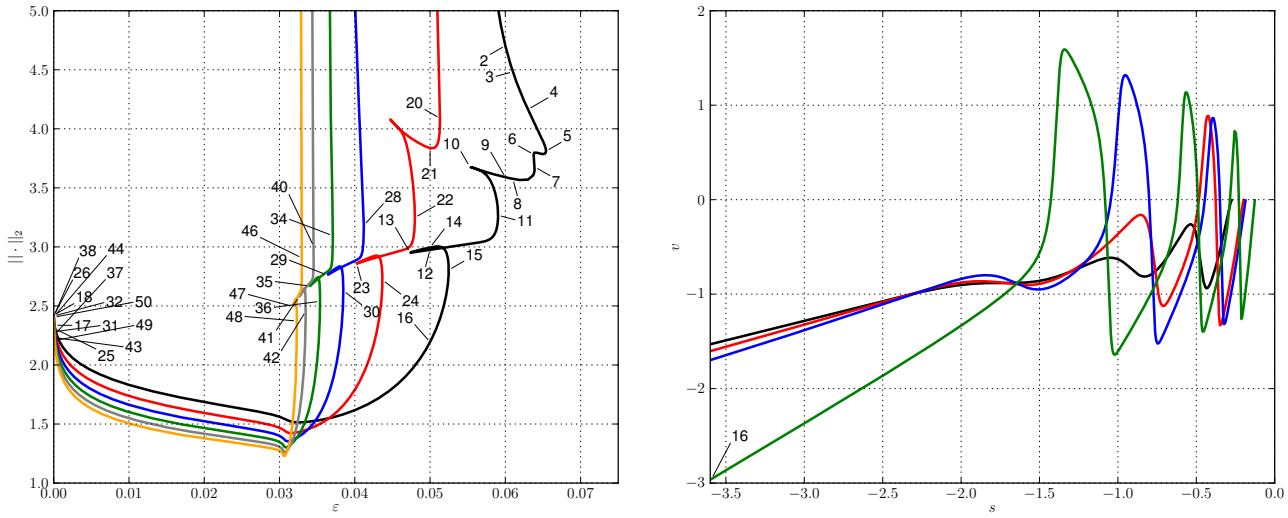


Figure 15.7: Continuation of canard orbits: the AUTO L_2 norm as a function of ε for all six orbits (left), and a projection of labels 4, 8, 11, and 16 (black, red, blue, green) of the orbit ξ_3 on the (v, s) plane (right).

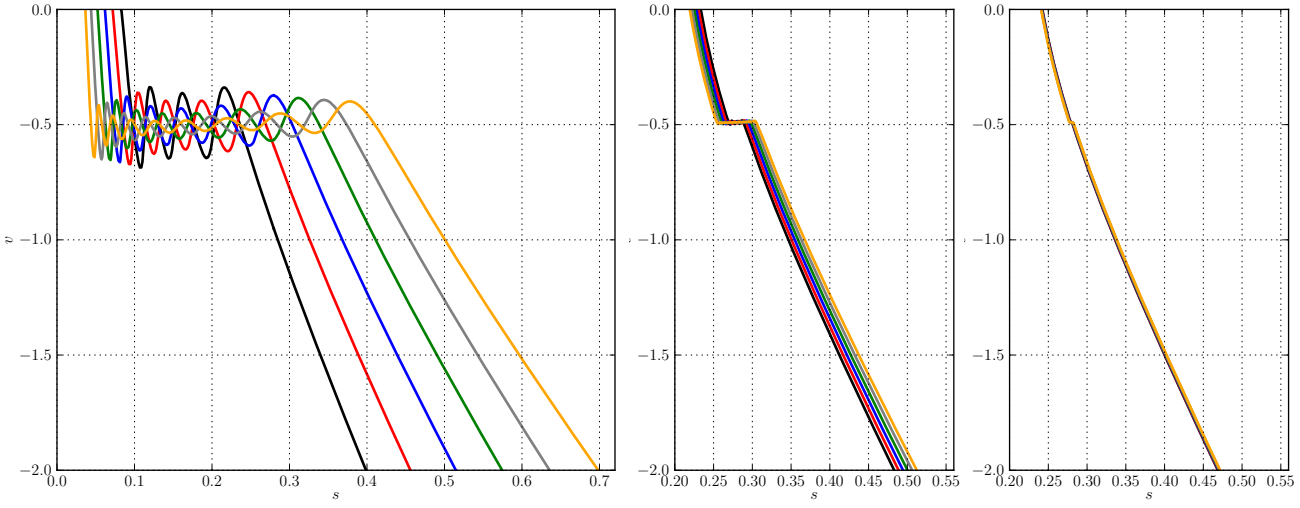


Figure 15.8: Continuation of canard orbits: projection of all canard orbits on the (v, s) plane for $\varepsilon = 0.015$ (left), $\varepsilon = 10^{-4}$ (middle), and $\varepsilon = 10^{-6}$ (right).

Chapter 16

AUTO Demos : Parabolic PDEs.

16.1 pd1 : Stationary States (1D Problem).

This demo uses Euler's method to locate a stationary solution of a nonlinear parabolic PDE, followed by continuation of this stationary state in a free problem parameter. The equation is

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} \;+\; p_1\; u\;(1-u),$$

on the space interval $[0, L]$, where $L =$ `PAR(11)` $= 10$ is fixed throughout, as is the diffusion constant $D =$ `PAR(15)` $= 0.1$. The boundary conditions are $u(0) = u(L) = 0$ for all time.

In the first run the continuation parameter is the independent time variable, namely `PAR(14)`, while $p_1 = 1$ is fixed. The AUTO-constants `DS`, `DSMIN`, and `DSMAX` then control the step size in space-time, here consisting of `PAR(14)` and $u(x)$. Initial data are $u(x) = \sin(\pi x/L)$ at time zero. Note that in the subroutine **STPNT** the initial data must be scaled to the unit interval, and that the scaled derivative must also be provided; see the equations-file pv1.f90. In the second run the continuation parameter is p_1.

Euler time integration is only first order accurate, so that the time step must be sufficiently small to ensure correct results. Indeed, this option has been added only as a convenience, and should generally be used only to locate stationary states.

AUTO-COMMAND	ACTION

`mkdir pd1`
`cd pd1`
`demo('pd1')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='pd1',c='pd1')` `save(r1,'1')`	time integration towards stationary state save output-files as b.1, s.1, d.1

`r2=run(r1,IPS=17,ICP=[1],NTST=20,`
`NMX=100,RL1=50,NPR=25,DS=0.1,DSMAX=0.5)`
`save(r2,'2')`
	continuation of stationary states; read
	restart data from the last label of `r1`
	save output-files as b.2, s.2, d.2

Table 16.1: Commands for running demo pd1.

16.2 pd2 : Stationary States (2D Problem).

This demo uses Euler's method to locate a stationary solution of a nonlinear parabolic PDE, followed by continuation of this stationary state in a free problem parameter. The equations are

$$\begin{aligned} \partial u_1/\partial t &= D_1\ \partial^2 u_1/\partial x^2\ +\ p_1\ u\ (1-u)\ -\ u_1 u_2, \\ \partial u_2/\partial t &= D_2\ \partial^2 u_2/\partial x^2\ -\ u_2\ +\ u_1 u_2, \end{aligned} \tag{16.1}$$

on the space interval $[0, L]$, where $L =$ `PAR(11)` $= 1$ is fixed throughout, as are the diffusion constants $D_1 =$ `PAR(15)` $= 1$ and $D_2 =$ `PAR(16)` $= 1$. The boundary conditions are $u_1(0) = u_1(L) = 0$ and $u_2(0) = u_2(L) = 1$, for all time.

In the first run the continuation parameter is the independent time variable, namely `PAR(14)`, while $p_1 = 12$ is fixed. The AUTO-constants `DS`, `DSMIN`, and `DSMAX` then control the step size in space-time, here consisting of `PAR(14)` and $(u_1(x), u_2(x))$. Initial data at time zero are $u_1(x) = \sin(\pi x/L)$ and $u_2(x) = 1$. Note that in the subroutine **STPNT** the initial data must be scaled to the unit interval, and that the scaled derivatives must also be provided; see the equations-file pv2.f90. In the second run the continuation parameter is p_1. A branch point is located during this run.

Euler time integration is only first order accurate, so that the time step must be sufficiently small to ensure correct results. Indeed, this option has been added only as a convenience, and should generally be used only to locate stationary states.

AUTO-COMMAND	ACTION

`mkdir pd2`
`cd pd2`
`demo('pd2')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='pd2',c='pd2')` `save(r1,'1')`	time integration towards stationary state save output-files as b.1, s.1, d.1

`r2 = run(r1,IPS=17,ICP=[1],ISP=2,NMX=15,`
`NPR=50,DS=-0.1,DSMAX=1.0,UZR={-1:0.0})`
`save(r2,'2')`
	continuation of stationary states; read restart data from s.1
	save output-files as b.2, s.2, d.2

Table 16.2: Commands for running demo pd2.

16.3 wav : Periodic Waves.

This demo illustrates the computation of various periodic wave solutions to a system of coupled parabolic partial differential equations on the spatial interval $[0, 1]$. The equations, that model an enzyme catalyzed reaction (Doedel & Kernévez (1986*b*)) are :

$$\begin{aligned} \partial u_1/\partial t &= \partial^2 u_1/\partial x^2 - p_1\big[p_4 R(u_1, u_2) - (p_2 - u_1)\big], \\ \partial u_2/\partial t &= \beta\partial^2 u_2/\partial x^2 - p_1\big[p_4 R(u_1, u_2) - p_7(p_3 - u_2)\big]. \end{aligned} \tag{16.2}$$

All equation parameters, except p_3, are fixed throughout.

AUTO-COMMAND	ACTION

`mkdir wav`
`cd wav`
`demo('wav')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='wav',c='wav')`
 	
`save(r1,"ode")`
	1st run; stationary solutions of the system without diffusion
	save output-files as b.ode, s.ode, d.ode

`r2=run(e='wav',c='wav',IPS=11)`	2nd run; detect bifurcations to wave train solutions

`r3=run(r2("HB1"),IPS=12,ICP=[3,11],ILP=0,`
`ISP=0,RL1=700,DS=0.1,DSMAX=1.0,`
`UZR={3:[610.0,638.0], -11:500.0})`
`save(r2+r3,'wav')`
	3rd run; wave train solutions of fixed wave speed
	
	save output to b.wav, s.wav, d.wav

`uz3=load(r3("UZ3"),RL1=1000,`
`DS=0.5,DSMAX=2.0,UZR={})`
`r4=run(uz3,ICP=[3,10],NPR=50)`
 	
`save(r4,'rng')`
	load restart label
	
	4th run; wave train solutions of fixed wave length
	save output-files as b.rng, s.rng, d.rng

`r5=run(uz3,IPS=14,ICP=[14],NMX=230,NPR=5)` `save(r5,'tim')`	5th run; time evolution computation save output-files as b.tim, s.tim, d.tim

Table 16.3: Commands for running demo wav.

16.4 brc : Chebyshev Collocation in Space.

This demo illustrates the computation of stationary solutions and periodic solutions to systems of parabolic PDEs in one space variable, using Chebyshev collocation in space. More precisely, the approximate solution is assumed of the form $u(x,t) = \sum_{k=0}^{n+1} u_k(t)\ell_k(x)$. Here $u_k(t)$ corresponds to $u(x_k,t)$ at the Chebyshev points $\left\{x_k\right\}_{k=1}^{n}$ with respect to the interval $[0,1]$. The polynomials $\left\{\ell_k(x)\right\}_{k=0}^{n+1}$ are the Lagrange interpolating coefficients with respect to points $\left\{x_k\right\}_{k=0}^{n+1}$, where $x_0 = 0$ and $x_{n+1} = 1$. The number of Chebyshev points in $[0,1]$, as well as the number of equations in the PDE system, can be set by the user in the file brc.f90.

As an illustrative application we consider the Brusselator (Holodniok, Knedlik & Kubíček (1987))

$$\begin{aligned} u_t &= D_x/L^2 u_{xx} + u^2 v - (B+1)u + A, \\ v_t &= D_y/L^2 v_{xx} - u^2 v + Bu, \end{aligned} \tag{16.3}$$

with boundary conditions $u(0,t) = u(1,t) = A$ and $v(0,t) = v(1,t) = B/A$.

Note that, given the non-adaptive spatial discretization, the computational procedure here is not appropriate for PDEs with solutions that rapidly vary in space, and care must be taken to recognize spurious solutions and bifurcations.

AUTO-COMMAND	ACTION

`mkdir brc`
`cd brc`
`demo('brc')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='brc',c='brc')`	compute the stationary solution family with Hopf bifurcations.
`r2=run(r1("HB1"),IPS=2,ICP=[5,11])`	compute a family of periodic solutions from the first Hopf point.
`r3=run(r2("BP1"),ISW=-1)`	compute a solution family from a secondary periodic bifurcation.
`save(r1+r2+r3,'brc')`	save all output to b.brc, s.brc, d.brc

Table 16.4: Commands for running demo brc.

16.5 brf : Finite Differences in Space.

This demo illustrates the computation of stationary solutions and periodic solutions to systems of parabolic PDEs in one space variable. A fourth order accurate finite difference approximation is used to approximate the second order space derivatives. This reduces the PDE to an autonomous ODE of fixed dimension which AUTO is capable of treating. The spatial mesh is uniform; the number of mesh intervals, as well as the number of equations in the PDE system, can be set by the user in the file brf.f90.

As an illustrative application we consider the Brusselator (Holodniok, Knedlik & Kubíček (1987))

$$\begin{aligned} u_t &= D_x/L^2 u_{xx} + u^2 v - (B+1)u + A, \\ v_t &= D_y/L^2 v_{xx} - u^2 v + Bu, \end{aligned} \tag{16.4}$$

with boundary conditions $u(0,t) = u(1,t) = A$ and $v(0,t) = v(1,t) = B/A$.

Note that, given the non-adaptive spatial discretization, the computational procedure here is not appropriate for PDEs with solutions that rapidly vary in space, and care must be taken to recognize spurious solutions and bifurcations.

AUTO-COMMAND	ACTION

`mkdir brf`
`cd brf`
`demo('brf')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='brf',c='brf')`	compute the stationary solution family with Hopf bifurcations
`r2=run(r1("HB1"),IPS=2,ICP=[5,11],` `NMX=120,EPSL=1e-8)`	compute a family of periodic solutions from the first Hopf point.
`r3=run(r2("BP1"),ISW=-1,` `NMX=100,EPSL=1e-7)`	compute a solution family from a secondary periodic bifurcation.
`save(r1+r2+r3,'brf')`	save all output to b.brf, s.brf, d.brf

Table 16.5: Commands for running demo brf.

16.6 bru : Euler Time Integration (the Brusselator).

This demo illustrates the use of Euler's method for time integration of a nonlinear parabolic PDE. The example is the Brusselator (Holodniok, Knedlik & Kubíček (1987)), given by

$$\begin{aligned} u_t &= D_x/L^2 u_{xx} + u^2 v - (B+1)u + A, \\ v_t &= D_y/L^2 v_{xx} - u^2 v + Bu, \end{aligned} \tag{16.5}$$

with boundary conditions $u(0,t) = u(1,t) = A$ and $v(0,t) = v(1,t) = B/A$. All parameters are given fixed values for which a stable periodic solution is known to exist.

The continuation parameter is the independent time variable, namely `PAR(14)`. The AUTO-constants `DS`, `DSMIN`, and `DSMAX` then control the step size in space-time, here consisting of `PAR(14)` and $(u(x), v(x))$. Initial data at time zero are $u(x) = A - 0.5\sin(\pi x)$ and $v(x) = B/A + 0.7\sin(\pi x)$. Note that in the subroutine **STPNT** the space derivatives of u and v must also be provided; see the equations-file bru.f90.

Euler time integration is only first order accurate, so that the time step must be sufficiently small to ensure correct results. This option has been added only as a convenience, and should generally be used only to locate stationary states. Indeed, in the case of the asymptotic periodic state of this demo, the number of required steps is very large and use of a better time integrator is advisable.

AUTO-COMMAND	ACTION
`mkdir bru`	create an empty work directory
`cd bru`	change directory
`demo('bru')`	copy the demo files to the work directory
`r1=run(e='bru',c='bru')`	time integration
`save(r1,'bru')`	save output-files as b.bru, s.bru, d.bru

Table 16.6: Commands for running demo bru.

Chapter 17

AUTO Demos : Optimization.

17.1 opt : A Model Algebraic Optimization Problem.

This demo illustrates the method of successive continuation for constrained optimization problems by applying it to the following simple problem : Find the maximum sum of coordinates on the unit sphere in R^5. Coordinate 1 is treated as the state variable. Coordinates 2-5 are treated as control parameters. For details on the successive continuation procedure see Doedel, Keller & Kernévez (1991*a*), Doedel, Keller & Kernévez (1991*b*).

AUTO-COMMAND	ACTION
`mkdir opt`	create an empty work directory
`cd opt`	change directory
`demo('opt')`	copy the demo files to the work directory
`r1=run(e='opt',c='opt')`	one free equation parameter
`save(r1,'1')`	save output-files as `b.1`, `s.1`, `d.1`
`r2=run(r1("LP1"))`	two free equation parameters; read restart data from `r1`
`save(r2,'2')`	save output-files as `b.2`, `s.2`, `d.2`
`r3=run(r2("LP1"))`	three free equation parameters; read restart data from `r2`
`save(r3,'3')`	save output-files as `b.3`, `s.3`, `d.3`
`run(r3("LP1"))`	four free equation parameters; read restart data from `r3`
`save(r4,'4')`	save output-files as `b.4`, `s.4`, `d.4`

Table 17.1: Commands for running demo `opt`.

17.2 ops : Optimization of Periodic Solutions.

This demo illustrates the method of successive continuation for the optimization of periodic solutions. For a detailed description of the basic method see Doedel, Keller & Kernévez (1991*b*). The illustrative system of autonomous ODEs, taken from Rodríguez-Luis (1991), is

$$\begin{aligned} x'(t) &= [-\lambda_4(x^3/3 - x) + (z - x)/\lambda_2 - y]/\lambda_1, \\ y'(t) &= x - \lambda_3, \\ z'(t) &= -(z - x)/\lambda_2, \end{aligned} \tag{17.1}$$

with objective functional

$$\omega = \int_0^1 g(x, y, z; \lambda_1, \lambda_2, \lambda_3, \lambda_4)\ dt,$$

where $g(x, y, z; \lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv \lambda_3$. Thus, in this application, a one-parameter extremum of g corresponds to a fold with respect to the problem parameter λ_3, and multi-parameter extrema correspond to generalized folds. Note that, in general, the objective functional is an integral along the periodic orbit, so that a variety of optimization problems can be addressed.

For the case of periodic solutions, the extended optimality system can be generated automatically, i.e., one need only define the vector field and the objective functional, as in done in the file ops.f90. For reference purpose it is convenient here to write down the full extended system in its general form :

$$\begin{aligned} & u'(t) = Tf\big(u(t), \lambda\big), \qquad T \in \mathrm{R}\ \text{(period)},\ u(\cdot), f(\cdot,\cdot) \in \mathrm{R}^n,\ \lambda \in \mathrm{R}^{n_\lambda}, \\ & w'(t) = -Tf_u\big(u(t), \lambda\big)^* w(t) + \kappa u_0'(t) + \gamma g_u\big(u(t), \lambda\big)^*, \qquad w(\cdot) \in \mathrm{R}^n,\ \kappa, \gamma \in \mathrm{R}, \\ & u(1) - u(0) = 0, \qquad w(1) - w(0) = 0, \\ & \textstyle\int_0^1 u(t)^* u_0'(t)\ dt = 0, \\ & \textstyle\int_0^1 \omega - g\big(u(t), \lambda\big)\ dt = 0, \\ & \textstyle\int_0^1 w(t)^* w(t) + \kappa^2 + \gamma^2 - \alpha\ dt = 0, \qquad \alpha \in \mathrm{R}, \\ & \textstyle\int_0^1 f\big(u(t), \lambda\big)^* w(t) - \gamma g_T\big(u(t), \lambda\big) - \tau_0\ dt = 0, \qquad \tau_0 \in \mathrm{R}, \\ & \textstyle\int_0^1 Tf_{\lambda_i}\big(u(t), \lambda\big)^* w(t) - \gamma g_{\lambda_i}\big(u(t), \lambda\big) - \tau_i\ dt = 0, \qquad \tau_i \in \mathrm{R}, \quad i = 1, \cdots, n_\lambda. \end{aligned} \tag{17.2}$$

Above u_0 is a reference solution, namely, the previous solution along a solution family.

In the computations below, the two preliminary runs, with `IPS`=1 and `IPS`=2, respectively, locate periodic solutions. The subsequent runs are with `IPS`=15 and hence use the automatically generated extended system.

- `Run 1.` Locate a Hopf bifurcation. The free system parameter is λ_3.

- `Run 2.` Compute a family of periodic solutions from the Hopf bifurcation.

- `Run 3.` This run retraces part of the periodic solution family, using the full optimality system, but with all adjoint variables, $w(\cdot), \kappa, \gamma$, and hence α, equal to zero. The optimality parameters τ_0 and τ_3 are zero throughout. An extremum of the objective functional with respect to λ_3 is located. Such a point corresponds to a branch point of the extended system. Given the choice of objective functional in this demo, this extremum is also a fold with respect to λ_3.

- `Run 4.` Branch switching at the above-found branch point yields nonzero values of the adjoint variables. Any point on the bifurcating family away from the branch point can serve as starting solution for the next run. In fact, the branch-switching can be viewed as generating a nonzero eigenvector in an eigenvalue-eigenvector relation. Apart from the adjoint variables, all other variables remain unchanged along the bifurcating family.

- `Run 5.` The above-found starting solution is continued in two system parameters, here λ_3 and λ_2; i.e., a two-parameter family of extrema with respect to λ_3 is computed. Along this family the value of the optimality parameter τ_2 is monitored, i.e., the value of the functional that vanishes at an extremum with respect to the system parameter λ_2. Such a zero of τ_2 is, in fact, located, and hence an extremum of the objective functional with respect to both λ_2 and λ_3 has been found. Note that, in general, τ_i is the value of the functional that vanishes at an extremum with respect to the system parameter λ_i.

- `Run 6.` In the final run, the above-found two-parameter extremum is continued in three system parameters, here λ_1, λ_2, and λ_3, toward $\lambda_1 = 0$. Again, given the particular choice of objective functional, this final continuation has an alternate significance here : it also represents a three-parameter family of transcritical secondary periodic bifurcations points.

Although not illustrated here, one can restart an ordinary continuation of periodic solutions, using `IPS`=2 or `IPS`=3, from a labeled solution point on a family computed with `IPS`=15.

The free scalar variables specified in the AUTO constants-files for Run 3 and Run 4 are shown in Table 17.2.

Index	3	11	12	22	-22	-23	-31
Variable	λ_3	T	α	τ_2	$[\lambda_2]$	$[\lambda_3]$	$[T]$

Table 17.2: `Runs 3 and 4` (files c.ops.3 and c.ops.4).

The parameter α, which is the norm of the adjoint variables, becomes nonzero after branch switching in Run 4. The negative indices (-22, -23, and -31) set the active optimality functionals, namely for λ_2, λ_3, and T, respectively, with corresponding variables τ_2, τ_3, and τ_0, respectively. These should be set in the first run with `IPS`=15 and remain unchanged in all subsequent runs.

Index	3	2	11	22	-22	-23	-31
Variable	λ_3	λ_2	T	τ_2	$[\lambda_2]$	$[\lambda_3]$	$[T]$

Table 17.3: `Run 5` (file c.ops.5).

In Run 5 the parameter α, which has been replaced by λ_2, remains fixed and nonzero. The variable τ_2 monitors the value of the optimality functional associated with λ_2. The zero of τ_2 located in this run signals an extremum with respect to λ_2.

Index	3	2	1	11	-22	-23	-31
Variable	λ_3	λ_2	λ_1	T	$[\lambda_2]$	$[\lambda_3]$	$[T]$

Table 17.4: `Run 6` (file c.ops.6).

In Run 6 τ_2, which has been replaced by λ_1, remains zero.

Note that τ_0 and τ_3 are not used as variables in any of the runs; in fact, their values remain zero throughout. Also note that the optimality functionals corresponding to τ_0 and τ_3 (or, equivalently, to T and λ_3) *are* active in all runs. This set-up allows the detection of the extremum of the objective functional, with T and λ_3 as scalar equation parameters, as a bifurcation in the third run.

The parameter λ_4, and its corresponding optimality variable τ_4, are not used in this demo. Also, λ_1 is used in the last run only, and its corresponding optimality variable τ_1 is never used.

AUTO-COMMAND	ACTION

`mkdir ops`
`cd ops`
`demo('ops')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='ops',c='ops')`	locate a Hopf bifurcation

`uzr={3:[0.92,0.93]}`
`r2=run(r1("HB1"),IPS=2,ICP=[3,11],`
`NMX=150,RL0=0.9,UZR=uzr)`
`save(r1+r2,'0')`
	set variable for `UZR`
	compute a family of periodic solutions; restart from `r1`
	save output to b.0, s.0, d.0

`icp=[3,11,12,22,-22,-23,-31]`
`r3=run(r2("UZ1"),IPS=15,ILP=0,`
`ICP=icp,ISP=2,NMX=25,ITNW=7,DS=-0.05)`
	set variable for `ICP`
	locate a 1-parameter extremum as a bifurcation; restart from `r2`

`r4=run(r3("BP1"),ISW=-1,ISP=0,NMX=5)`
 	
`save(r3+r4,'1')`
	switch branches to generate optimality starting data; restart from `r3`
	save output to b.1, s.1, d.1

`icp[1:3]=[2,11]`
`uzr[22]=0.0`
`r5=run(r4,ICP=icp,ISW=1,NMX=150,`
`RL0=0.8,RL1=1.9,DS='-',UZR=uzr)`
 	
`save(r5,'2')`
	set variable for `ICP`
	set variable for `UZR`
	compute 2-parameter family of 1-parameter extrema; restart from `r4`
	
	save the output-files as b.2, s.2, d.2

`icp[2:4]=[1,11]`
`r6=run(r5("UZ4"),IRS=15,ICP=icp,NTST=50,`
`UZR={1:[0.1,0.05,0.01,0.005,0.001]})`
 	
`save(r6,'3')`
	set variable for `ICP`
	compute 3-parameter family of 2-parameter extrema; restart from `r5`
	
	save the output-files as b.3, s.3, d.3

Table 17.5: Commands for running demo ops.

17.3 obv : Optimization for a BVP.

This demo illustrates use of the method of successive continuation for a boundary value optimization problem. A detailed description of the basic method, as well as a discussion of the specific application considered here, is given in Doedel, Keller & Kernévez (1991*b*). The required extended system is fully programmed here in the user-supplied routines in obv.f90. For the case of periodic solutions the optimality system can be generated automatically; see the demo ops.

Consider the system

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= -\lambda_1 e^{p(u_1,\lambda_2,\lambda_3)}, \end{aligned} \tag{17.3}$$

where $p(u_1, \lambda_2, \lambda_3) \equiv u_1 + \lambda_2 u_1^2 + \lambda_3 u_1^4$, with boundary conditions

$$\begin{aligned} u_1(0) &= 0, \\ u_1(1) &= 0. \end{aligned} \tag{17.4}$$

The objective functional is

$$\omega = \int_0^1 (u_1(t) - 1)^2 \, dt + \frac{1}{10} \sum_{k=1}^{3} \lambda_k^2.$$

The successive continuation equations are given by

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= -\lambda_1 e^{p(u_1,\lambda_2,\lambda_3)}, \\ w_1'(t) &= \lambda_1 e^{p(u_1,\lambda_2,\lambda_3)} p_{u_1} w_2(t) + 2\gamma(u_1(t) - 1), \\ w_2'(t) &= -w_1(t), \end{aligned} \tag{17.5}$$

where

$$p_{u_1} \equiv \frac{\partial p}{\partial u_1} = 1 + 2\lambda_2 u_1 + 4\lambda_3 u_1^3,$$

with

$$\begin{aligned} &u_1(0) = 0, \qquad w_1(0) - \beta_1 = 0, \qquad w_2(0) = 0, \\ &u_1(1) = 0, \qquad w_1(1) + \beta_2 = 0, \qquad w_2(1) = 0, \end{aligned} \tag{17.6}$$

$$\int_0^1 \left[\omega - (u_1(t) - 1)^2 - \frac{1}{10} \sum_{k=1}^{3} \lambda_k^2\right] dt = 0,$$

$$\int_0^1 \left[w_1^2(t) - \alpha_0\right] dt = 0,$$

$$\begin{aligned} &\textstyle\int_0^1 \left[-e^{p(u_1,\lambda_2,\lambda_3)} w_2(t) - \frac{1}{5}\gamma\lambda_1\right] dt = 0, \\ &\textstyle\int_0^1 \left[-\lambda_1 e^{p(u_1,\lambda_2,\lambda_3)} u_1(t)^2 w_2(t) - \frac{1}{5}\gamma\lambda_2 - \tau_2\right] dt = 0, \\ &\textstyle\int_0^1 \left[-\lambda_1 e^{p(u_1,\lambda_2,\lambda_3)} u_1(t)^4 w_2(t) - \frac{1}{5}\gamma\lambda_3 - \tau_3\right] dt = 0. \end{aligned} \tag{17.7}$$

In the first run the free equation parameter is λ_1. All adjoint variables are zero. Three extrema of the objective function are located. These correspond to branch points and, in the second run, branch switching is done at one of these. Along the bifurcating family the adjoint

variables become nonzero, while state variables and λ_1 remain constant. Any such non-trivial solution point can be used for continuation in two equation parameters, after fixing the L_2-norm of one of the adjoint variables. This is done in the third run. Along the resulting family several two-parameter extrema are located by monotoring certain inner products. One of these is further continued in three equation parameters in the final run, where a three-parameter extremum is located.

AUTO-COMMAND	ACTION

`mkdir obv`
`cd obv`
`demo('obv')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='obv',c='obv')` `save(r1,'obv')`	locate 1-parameter extrema as branch points save output-files as b.obv, s.obv, d.obv

`r2=run(r1("BP1"),ISW=-1,NMX=5)`
 	
`save(r2,'1')`
	compute a few step on the first bifurcating family
	save the output-files as b.1, s.1, d.1

`r3=run(r2,ICP=[10,1,2,17,18,13,14,15],`
`ISW=1,NMX=100)`
`save(r3,'2')`
	locate 2-parameter extremum; restart from r2
	save the output-files as b.2, s.2, d.2

`r4=run(r3("UZ2"),`
`ICP=[10,1,2,3,18,13,14,15],NMX=25)`
`save(r4,'3')`
	locate 3-parameter extremum; restart from r3
	save the output-files as b.3, s.3, d.3

Table 17.6: Commands for running demo obv.

Chapter 18

AUTO Demos : Connecting orbits.

18.1 fsh : A Saddle-Node Connection.

This demo illustrates the computation of travelling wave front solutions to the Fisher equation,

$$\begin{aligned} &w_t = w_{xx} + f(w), \qquad -\infty < x < \infty, \quad t > 0, \\ &f(w) \equiv w(1-w). \end{aligned} \tag{18.1}$$

We look for solutions of the form $w(x,t) = u(x+ct)$, where c is the wave speed. This gives the first order system

$$\begin{aligned} u_1'(z) &= u_2(z), \\ u_2'(z) &= cu_2(z) - f\big(u_1(z)\big). \end{aligned} \tag{18.2}$$

Its fixed point $(0,0)$ has two positive eigenvalues when $c > 2$. The other fixed point, $(1,0)$, is a saddle point. A family of orbits connecting the two fixed points requires one free parameter; see Friedman & Doedel (1991). Here we take this parameter to be the wave speed c.

In the first run a starting connecting orbit is computed by continuation in the period T. This procedure can be used generally for time integration of an ODE with AUTO. Starting data in **STPNT** correspond to a point on the approximate stable manifold of $(1,0)$, with T small. In this demo the "free" end point of the orbit necessary approaches the unstable fixed point $(0,0)$. A computed orbit with sufficiently large T is then chosen as restart orbit in the second run, where, typically, one replaces T by c as continuation parameter. However, in the second run below, we also add a phase condition, and both c and T remain free.

AUTO-COMMAND	ACTION

`mkdir fsh`
`cd fsh`
`demo('fsh')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='fsh',c='fsh')` `save(r1,'0')`	continuation in the period T, with c fixed; no phase condition save output-files as b.0, s.0, d.0

`r2 = run(r1("EP2"),`
`ICP=[2,11,12,13,14],`
`NINT=1, DS='-',`
`UZR={2:[1,2,3,5,10]})`
`save(r2,'fsh')`
	continuation in c and T, with active phase condition.
	
	save output-files as b.fsh, s.fsh, d.fsh

Table 18.1: Commands for running demo fsh.

18.2 nag : A Saddle-Saddle Connection.

This demo illustrates the computation of traveling wave front solutions to Nagumo's equation,

$$\begin{aligned} &w_t = w_{xx} + f(w,a), \qquad -\infty < x < \infty, \quad t > 0, \\ &f(w,a) \equiv w(1-w)(w-a), \qquad 0 < a < 1. \end{aligned} \tag{18.3}$$

We look for solutions of the form $w(x,t) = u(x+ct)$, where c is the wave speed. This gives the first order system

$$\begin{aligned} u_1'(z) &= u_2(z), \\ u_2'(z) &= cu_2(z) - f\big(u_1(z), a\big), \end{aligned} \tag{18.4}$$

where $z = x + ct$, and $' = d/dz$. If $a = 1/2$ and $c = 0$ then there are two analytically known heteroclinic connections, one of which is given by

$$u_1(z) = \frac{e^{\frac{1}{2}\sqrt{2}z}}{1 + e^{\frac{1}{2}\sqrt{2}z}}, \qquad u_2(z) = u_1'(z), \qquad -\infty < z < \infty.$$

The second heteroclinic connection is obtained by reflecting the phase plane representation of the first with respect to the u_1-axis. In fact, the two connections together constitute a heteroclinic cycle. One of the exact solutions is used below as starting orbit. To start from the second exact solution, change SIGN=-1 in the routine **STPNT** in nag.f90 and repeat the computations below; see also Friedman & Doedel (1991).

AUTO-COMMAND	ACTION
`mkdir nag`	create an empty work directory
`cd nag`	change directory
`demo('nag')`	copy the demo files to the work directory
`r1=run(e='nag',c='nag')`	compute part of first family of heteroclinic orbits
`r2=run(e='nag',c='nag',DS='-')`	compute first family in opposite direction
`save(r1+r2,'nag')`	save all output to b.nag, s.nag, d.nag

Table 18.2: Commands for running demo nag.

18.3 stw : Continuation of Sharp Traveling Waves.

This demo illustrates the computation of sharp traveling wave front solutions to nonlinear diffusion problems of the form

$$w_t = A(w)w_{xx} + B(w)w_x^2 + C(w),$$

with $A(w) = a_1w+a_2w^2$, $B(w) = b_0+b_1w+b_2w^2$, and $C(w) = c_0+c_1w+c_2w^2$. Such equations can have `sharp traveling wave fronts` as solutions, i.e., solutions of the form $w(x,t) = u(x+ct)$ for which there is a z_0 such that $u(z) = 0$ for $z \geq z_0$, $u(z) \neq 0$ for $z < z_0$, and $u(z) \to constant$ as $z \to -\infty$. These solutions are actually generalized solutions, since they need not be differentiable at z_0.

Specifically, in this demo a homotopy path will be computed from an analytically known exact sharp traveling wave solution of

$$w_t = 2ww_{xx} + 2w_x^2 + w(1-w), \tag{1}$$

to a corresponding sharp traveling wave of

$$w_t = (2w + w^2)w_{xx} + ww_x^2 + w(1-w). \tag{2}$$

This problem is also considered in Doedel, Keller & Kernévez (1991*b*). For these two special cases the functions A, B, C are defined by the coefficients in Table 18.3.

	a_1	a_2	b_0	b_1	b_2	c_0	c_1	c_2
Case (1)	2	0	2	0	0	0	1	-1
Case (2)	2	1	0	1	0	0	1	-1

Table 18.3: Problem coefficients in demo stw.

With $w(x,t) = u(x+ct)$, $z = x+ct$, one obtains the reduced system

$$\begin{aligned} u_1'(z) &= u_2, \\ u_2'(z) &= \big[cu_2 - B(u_1)u_2^2 - C(u_1)\big]/A(u_1). \end{aligned} \tag{18.5}$$

To remove the singularity when $u_1 = 0$, we apply a nonlinear transformation of the independent variable (see Aronson (1980)), viz., $d/d\tilde{z} = A(u_1)d/dz$, which changes the above equation into

$$\begin{aligned} u_1'(\tilde{z}) &= A(u_1)u_2, \\ u_2'(\tilde{z}) &= cu_2 - B(u_1)u_2^2 - C(u_1). \end{aligned} \tag{18.6}$$

Sharp traveling waves then correspond to heteroclinic connections in this transformed system.

Finally, we map $[0,T] \to [0,1]$ by the transformation $\xi = \tilde{z}/T$. With this scaling of the independent variable, the reduced system becomes

$$\begin{aligned} u_1'(\xi) &= TA(u_1)u_2, \\ u_2'(\xi) &= T\big[cu_2 - B(u_1)u_2^2 - C(u_1)\big]. \end{aligned} \tag{18.7}$$

For Case 1 this equation has a known exact solution, namely,

$$u(\xi) = \frac{1}{1+\exp(T\xi)}, \qquad v(\xi) = \frac{-\frac{1}{2}}{1+\exp(-T\xi)}.$$

This solution has wave speed $c = 1$. In the limit as $T \to \infty$ its phase plane trajectory connects the stationary points $(1,0)$ and $(0,-\frac{1}{2})$.

The sharp traveling wave in Case 2 can now be obtained using the following homotopy. Let $(a_1, a_2, b_0, b_1, b_2) = (1-\lambda)(2,0,2,0,0) + \lambda(2,1,0,1,0)$. Then as λ varies continuously from 0 to 1, the parameters $(a_1, a_2, b_0, b_1, b_2)$ vary continously from the values for Case 1 to the values for Case 2.

AUTO-COMMAND	ACTION
`mkdir stw`	create an empty work directory
`cd stw`	change directory
`demo('stw')`	copy the demo files to the work directory
`r1=run(e='stw',c='stw')`	continuation of the sharp traveling wave
`save(r1,'stw')`	save output-files as b.stw, s.stw, d.stw

Table 18.4: Commands for running demo stw.

Chapter 19

AUTO Demos : Miscellaneous.

19.1 pvl : Use of the Routine PVLS.

Consider Bratu's equation

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -p_1 e^{u_1}, \end{aligned} \tag{19.1}$$

with boundary conditions $u_1(0) = 0$, $u_1(1) = 0$. As in demo exp, a solution curve requires one free parameter; here p_1.

Note that additional parameters are specified in the user-supplied subroutine **PVLS** in file pvl.f90, namely, p_2 (the L_2-norm of u_1), p_3 (the minimum of u_2 on the space-interval $[0, 1]$), p_4 (the boundary value $u_2(0)$), p_5 (the same boundary value $u_2(0)$). These additional parameters should be considered as "solution measures" for output purposes; they should not be treated as true continuation parameters.

Note also that four free parameters are specified in the AUTO-constants file c.pvl, namely, p_1, p_2, p_3, p_4, and p_5. The first one in this list, p_1, is the true continuation parameter. The parameters p_2, p_3, p_4, and p_5 are *overspecified* so that their values will appear in the output. However, *it is essential that the true continuation parameter appear first.* For example, it would be an error to specify the parameters in the following order : p_2, p_1, p_3, p_4, p_5.

In general, true continuation parameters must appear first in the parameter-specification in the AUTO constants-file. Overspecified parameters will be printed, and can be defined in **PVLS**, but they are not part of the intrinsic continuation procedure.

As this demo also illustrates (see the UZR values in c.pvl), labeled solutions can also be output at selected values of the overspecified parameters.

AUTO-COMMAND	ACTION
`mkdir pvl`	create an empty work directory
`cd pvl`	change directory
`demo('pvl')`	copy the demo files to the work directory
`r1=run(e='pvl',c='pvl')`	compute a solution family
`save(r1,'pvl')`	save output-files as b.pvl, s.pvl, d.pvl

Table 19.1: Commands for running demo pvl.

19.2 ext : Spurious Solutions to BVP.

This demo illustrates the computation of spurious solutions to the boundary value problem

$$\begin{aligned} &u_1' - u_2 = 0, \\ &u_2' + \lambda^2\pi^2 \sin(u_1 + u_1^2 + u_1^3) = 0, \qquad t \in [0,1], \\ &u_1(0) = 0, \quad u_1(1) = 0. \end{aligned} \tag{19.2}$$

Here the differential equation is discretized using a fixed uniform mesh. This results in spurious solutions that disappear when an adaptive mesh is used. See the AUTO-constant `IAD` in Section 10.3. This example is also considered in Beyn & Doedel (1981) and Doedel, Keller & Kernévez (1991*b*).

AUTO-COMMAND	ACTION

`mkdir ext`
`cd ext`
`demo('ext')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run(e='ext',c='ext')`	detect bifurcations from the trivial solution family
`r2=run(r1("BP3"),ISW=-1,NCOL=3)`	compute a bifurcating family containing spurious bifurcations.
`save(r1+r2,'ext')`	save all output to b.ext, s.ext, d.ext

Table 19.2: Commands for running demo ext.

19.3 tim : A Test Problem for Timing AUTO.

This demo is a boundary value problem with variable dimension NDIM. It can be used to time the performance of AUTO for various choices of NDIM (which must be even), NTST, and NCOL. The equations are

$$\begin{aligned} u_i' &= u_i, \\ v_i' &= -p_1\ e(u_i), \end{aligned} \tag{19.3}$$

$i = 1, \cdots,$NDIM$/2$, with boundary conditions $u_i(0) = 0$, $u_i(1) = 0$. Here

$$e(u) = \sum_{k=0}^{n} \frac{u^k}{k!}\ ,$$

with $n = 25$. The computation requires 10 full LU-decompositions of the linearized system that arises from Newton's method for solving the collocation equations. The commands for running the timing problem for a particular choice of NDIM, NTST, and NCOL are given below. (Note that if NDIM is changed then NBC must be changed accordingly.)

AUTO-COMMAND	ACTION
`mkdir tim`	create an empty work directory
`cd tim`	change directory
`demo('tim')`	copy the demo files to the work directory
`r1=run(e='tim',c='tim')`	Timing run
`save(r1,'tim')`	save output-files as b.tim, s.tim, d.tim

Table 19.3: Commands for running demo tim.

Chapter 20

HomCont.

20.1 Introduction.

HomCont is a collection of routines for the continuation of homoclinic solutions to ODEs in two or more parameters. The accurate detection and multi-parameter continuation of certain codimension-two singularities is allowed for, including all known cases that involve a unique homoclinic orbit at the singular point. Homoclinic connections to hyperbolic and non-hyperbolic equilibria are allowed as are certain heteroclinic orbits. Homoclinic orbits in reversible systems can also be computed. The theory behind the methods used is explained in Champneys & Kuznetsov (1994), Bai & Champneys (1996), Sandstede (1995*b*, 1995*c*), Champneys, Kuznetsov & Sandstede (1996) and references therein. The final cited paper contains a concise description of the present version.

The current implementation of HomCont must be considered as experimental, and updates are anticipated. The HomCont routines are in the file `auto/07p/src/autlib5.f`. Expert users wishing to modify the routines may look there. Note also that at present, HomCont can be run only in AUTO Command Mode and not with the GUI.

20.2 HomCont Files and Routines.

In order to run HomCont one must prepare an equations file xxx.f90, where xxx is the name of the example, and constants-file c.xxx. These files are in the standard AUTO format, but the c.xxx file almost always also must contain constants that are specific to homoclinic continuation. The choice `IPS`=9 in c.xxx specifies the problem as being homoclinic continuation.

The equation-file kpr.f90 serves as a sample for new equation files. It contains the Fortran routines **FUNC**, **STPNT**, **PVLS**, **BCND**, **ICND** and **FOPT**. The final three are dummy routines which are never needed for homoclinic continuation. Note a minor difference in **STPNT** and **PVLS** with other AUTO equation-files, in that the common block `/BLHOM/` is required.

The constants-file c.xxx is identical in format to other AUTO constants-files, except for the added constants. Note that the values of the constants `NBC` and `NINT` are irrelevant, as these are set automatically by the choice `IPS`=9. Also, the choice `JAC=1` is strongly recommended, because the Jacobian is used extensively for calculating the linearization at the equilibria and

hence for evaluating boundary conditions and certain test functions. However, note that JAC=1 does not necessarily mean that auto will use the analytically specified Jacobian for continuation.

The earlier HomCont files h.xxx are obsolete but still supported. You can convert from the old to the new format by running the command @cnvc xxx yyy where xxx refers to either or both of the files c.xxx and h.xxx. A new-style constants file, c.yyy is written. The command @cnvc xxx overwrites c.xxx and deletes h.xxx.

20.3 HomCont-Constants.

An example for the HomCont-specific constants in c.xxx is listed below:

```
NUNSTAB=1, NSTAB=2, IEQUIB=1, ITWIST=1, ISTART=5
IREV = []
IFIXED = [13]
IPSI = [9,10]
```

These constants have the following meaning.

20.3.1 NUNSTAB

Number of unstable eigenvalues of the left-hand equilibrium (the equilibrium approached by the orbit as $t \to -\infty$). The default value is -1, which means autodetection. In almost all cases autodetection is possible. The exception is when starting from a homoclinic to a saddle-node equilibrium when IEQUIB$\neq$1. Examples for this exception are in runs 9, 12, and 13 of the demo kpr (see Chapter 23). Even in this case only one of NUNSTAB and NSTAB needs to be specified, the other one being computed as NDIM minus the specified constant.

20.3.2 NSTAB

Number of stable eigenvalues of the right-hand equilibrium (the equilibrium approached by the orbit as $t \to +\infty$). The same default as for NUNSTAB applies here.

20.3.3 IEQUIB

- IEQUIB=0 : Homoclinic orbits to hyperbolic equilibria; the equilibrium is specified explicitly in **PVLS** and stored in PAR(11+I), I=1,NDIM.
- IEQUIB=1 (default): Homoclinic orbits to hyperbolic equilibria; the equilibrium is solved for during continuation. Initial values for the equilibrium are stored in PAR(11+I), I=1,NDIM in **STPNT**.
- IEQUIB=2 : Homoclinic orbits to a saddle-node; initial values for the equilibrium are stored in PAR(11+I), I=1,NDIM in **STPNT**.
- IEQUIB=-1 : Heteroclinic orbits to hyperbolic equilibria; the equilibria are specified explicitly in **PVLS** and stored in PAR(11+I), I=1,NDIM (left-hand equilibrium) and PAR(11+I), I=NDIM+1,2*NDIM (right-hand equilibrium).

- IEQUIB=-2 : Heteroclinic orbits to hyperbolic equilibria; the equilibria are solved for during continuation. Initial values are specified in **STPNT** and stored in PAR(11+I), I=1,NDIM (left-hand equilibrium), PAR(11+I), I=NDIM+1,2*NDIM (right-hand equilibrium).

20.3.4 ITWIST

- ITWIST=0 (default): the orientation of the homoclinic orbit is not computed.

- ITWIST=1 : the orientation of the homoclinic orbit is computed. For this purpose, the adjoint variational equation is solved for the unique bounded solution. If IRS = 0, an initial solution to the adjoint equation must be specified as well. However, if IRS>0 and ITWIST has just been increased from zero, then AUTO will automatically generate the initial solution to the adjoint. In this case, a dummy Newton-step should be performed, see Section 20.7 for more details.

20.3.5 ISTART

- ISTART=1 : No special action is taken.

- ISTART=2 : If IRS=0, an explicit solution must be specified in the subroutine **STPNT** in the usual format.

- ISTART=3 : The "homotopy" approach is used for starting, see Section 20.7 for more details. Note that this is not available with the choice IEQUIB=2.

- ISTART=4 : A phase-shift is performed for homoclinic orbits to let the equilibrium (either fixed or non-fixed, depending on IEQUIB) correspond to $t = 0$ and $t = 1$. This is necessary if a periodic orbit that is close to a homoclinic orbit is continued into a homoclinic orbit.

- ISTART=5 (default): If IRS=0, the restart solution comes from a data file, or the restart solution is a homoclinic orbit with problem type IPS=9, no special action is taken, as for ISTART=1. For other problem types, use a phase-shift, as for ISTART=4.

- ISTART=-N, $N = 1, 2, 3, \ldots$: Homoclinic branch switching: this description is for reference only. We refer to the demo in Chapter 27 to see how this can be used in actual practice and to Oldeman, Champneys & Krauskopf (2003) for theory and background.

  The orbit is split into $N + 1$ parts and AUTO sees it as an $(N + 1)\times$NDIM-dimensional object. The first part u_0 goes from the equilibrium to the point x_0 that is furthest from the equilibrium. Then follow $N - 1$ shifted copies of the orbit, which travel from the point x_0 back to the point x_0. The last part U_N goes from the point x_0 back to the equilibrium. The derivatives $\dot{x}_0$ with respect to time of the point that is furthest from the equilibrium are stored at the parameters PAR(NP-NDIM+1...NP), where NP=max(NPARX,NPAR).

  If ITWIST=1, and this was also the case in the preceding run, then a copy of the adjoint vector Ψ at x_0 is stored at the parameters PAR(NP-NDIM*2+1...NP-NDIM) and Lin's method can be used to do homoclinic branch switching. To be more precise, the individual parts u_i and u_{i+1} are at distances ε_i away from each other, along the Lin vector Psi, at the

left- and right-hand end points. These gaps ε_i are at parameters PAR(20+2*i). Moreover, each part (except u_{N+1}) ends at at a Poincaré section which goes through x_0 and is perpendicular to $\dot{x}_0$.

The times T_i that each part u_i takes are stored as follows: T_0 =PAR(10), T_N =PAR(11) and T_i =PAR(19+2*i) for $i = 1 \ldots N-1$. Through a continuation in problem parameters, gaps ε_i, and times T_i it is possible to switch from a 1-homoclinic to an N-homoclinic orbit.

If ITWIST=0, the adjoint vector is not computed and Lin's method is not used. Instead, AUTO produces a gap ε=PAR(22) at the right-hand end point p of u_{N+1}, measuring the distance between the stable manifold of the equilibrium and p. This technique can also be used to find 2-homoclinic orbits, by varying in ε and T_1, similar to the method described before, but only if the unstable manifold in one-dimensional. Because this method is more limited than the method using Lin vectors, we do not recommend it for normal usage.

To switch back to a normal homoclinic orbit, set ISTART back to a positive value such as 1. Now HomCont has lost all the information about the adjoint, so if ITWIST is set to 0, HomCont does a normal continuation without the adjoint, and if ITWIST is set to 1, one needs to do a Newton dummy step first to recalculate the adhoint.

20.3.6 IREV

If IREV$\neq$[] then it is assumed that the system is reversible under the transformation $t \to -t$ and $U(i) \to -U(i)$ for all i with IREV(i)>0. Then only half the homoclinic solution is solved for with right-hand boundary conditions specifying that the solution is symmetric under the reversibility (see Champneys & Spence (1993)). The number of free parameters is then reduced by one. Otherwise the default applies, IREV=[].

20.3.7 IFIXED

Labels of test functions that are held fixed. E.g., with IFIXED=[n] one can compute a locus in one extra parameter of a singularity defined by test function PSI(n)=0. The default is IFIXED=[].

20.3.8 IPSI

Labels of activated test functions for detecting homoclinic bifurcations, see Section 20.6 for a list. If a test function is activated then the corresponding parameter (IPSI(I)+20) must be added to the list of continuation parameters ICP and zero of this parameter added to the list of user-defined output points UZR, in c.xxx. The default is IPSI=[].

20.4 Restrictions on HomCont Constants.

Note that certain combinations of these constants are not allowed in the present implementation. In particular,

- The computation of orientation ITWIST=1 is not implemented for IEQUIB<0 (heteroclinic orbits), IEQUIB=2 (saddle-node homoclinics), IREV≠[] (reversible systems), ISTART=3 (homotopy method for starting), or if the equilibrium contains complex eigenvalues in its linearization.

- The homotopy method ISTART=3 is not fully implemented for heteroclinic connections IEQUIB<0, saddle-node homoclinic orbits IEQUIB=2 or reversible systems IREV≠[].

- Certain test functions are not valid for certain forms of continuation (see Section 20.6 below); for example PSI(13) and PSI(14) only make sense if ITWIST=1 and PSI(15) and PSI(16) only apply to IEQUIB=2.

20.5 Restrictions on the Use of PAR.

The parameters PAR(1) – PAR(9) can be used freely by the user. The other parameters are used as follows :

- PAR(11) : The value of PAR(11) equals the length of the time interval over which a homoclinic solution is computed. Also referred to as "period". This must be specified in **STPNT**.

- PAR(10) : If ITWIST=1 then PAR(10) is used internally as a dummy parameter so that the adjoint equation is well-posed.

- PAR(12)-PAR(20) : These are used for specifying the equilibria and (if ISTART=3) the artificial parameters of the homotopy method (see Section 20.7 below).

- PAR(21)-PAR(36) : These parameters are used for storing the test functions (see Section 20.6).

The output is in an identical format to AUTO except that additional information at each computed point is written in fort.9. This information comprises the eigenvalues of the (left-hand) equilibrium, the values of each activated test function and, if ITWIST=1, whether the saddle homoclinic loop is orientable or not. Note that the statement about orientability is only meaningful if the leading eigenvalues are not complex and the homoclinic solution is not in a flip configuration, that is, none of the test functions ψ_i for $i = 11, 12, 13, 14$ is zero (or close to zero), see Section 20.6. Finally, the values of the IPSI activated test functions are written.

20.6 Test Functions.

Codimension-two homoclinic orbits are detected along branches of codim 1 homoclinics by locating zeroes of certain test functions ψ_i. The test functions that are "switched on" during any continuation are given by the choice of the labels i, and are specified by the parameters IPSI) in c.xxx. Here IPSI=[IPSI(1),...,IPSI(NPSI)] gives the labels of the test functions (numbers between 1 and 16). A zero of each labeled test function defines a certain codimension-two homoclinic singularity, specified as follows. The notation used for eigenvalues is the same as that in Champneys & Kuznetsov (1994) or Champneys et al. (1996).

- $i = 1$: Resonant eigenvalues (neutral saddle); $\mu_1 = -\lambda_1$.

- $i = 2$: Double real leading stable eigenvalues (saddle to saddle-focus transition); $\mu_1 = \mu_2$.

- $i = 3$: Double real leading unstable eigenvalues (saddle to saddle-focus transition); $\lambda_1 = \lambda_2$.

- $i = 4$: Neutral saddle, saddle-focus or bi-focus (includes $i = 1$); $\mathrm{Re}(\mu_1) = -\mathrm{Re}(\lambda_1)$.

- $i = 5$: Neutrally-divergent saddle-focus (stable eigenvalues complex); $\mathrm{Re}(\lambda_1) = -\mathrm{Re}(\mu_1) - \mathrm{Re}(\mu_2)$.

- $i = 6$: Neutrally-divergent saddle-focus (unstable eigenvalues complex); $\mathrm{Re}(\mu_1) = -\mathrm{Re}(\lambda_1) - \mathrm{Re}(\lambda_2)$.

- $i = 7$: Three leading eigenvalues (stable); $\mathrm{Re}(\lambda_1) = -\mathrm{Re}(\mu_1) - \mathrm{Re}(\mu_2)$.

- $i = 8$: Three leading eigenvalues (unstable); $\mathrm{Re}(\mu_1) = -\mathrm{Re}(\lambda_1) - \mathrm{Re}(\lambda_2)$.

- $i = 9$: Local bifurcation (zero eigenvalue or Hopf): number of stable eigenvalues decreases; $\mathrm{Re}(\mu_1) = 0$.

- $i = 10$: Local bifurcation (zero eigenvalue or Hopf): number of unstable eigenvalues decreases; $\mathrm{Re}(\lambda_1) = 0$.

- $i = 11$: Orbit flip with respect to leading stable direction (e.g., 1D unstable manifold).

- $i = 12$: Orbit flip with respect to leading unstable direction, (e.g., 1D stable manifold).

- $i = 13$: Inclination flip with respect to stable manifold (e.g., 1D unstable manifold).

- $i = 14$: Inclination flip with respect to unstable manifold (e.g., 1D stable manifold).

- $i = 15$: Non-central homoclinic to saddle-node (in stable manifold).

- $i = 16$: Non-central homoclinic to saddle-node (in unstable manifold).

Expert users may wish to add their own test functions by editing the function **PSIHO** in autlib5.f.

It is important to remember that, in order to specify activated test functions, it is required to also add the corresponding label +20 to the list of continuation parameters and a zero of this parameter to the list of user-defined output points. Having done this, the corresponding parameters are output to the screen and zeros are accurately located.

20.7 Starting Strategies.

There are four possible starting procedures for continuation.

(i) Data can be read from a previously-obtained output point from AUTO (e.g., from continuation of a periodic orbit up to large period; note that if the end-point of the data stored is not close to the equilibrium, a phase shift must be performed by setting `ISTART=4`). These data can be read from fort.8 (saved to `s.xxx`) by making `IRS` correspond to the label of the data point in question.

(ii) Data from numerical integration (e.g., computation of a stable periodic orbit, or an approximate homoclinic obtained by shooting) can be read in from a data file using the AUTO constant `dat` (see Section 10.8.7). The numerical data should be stored in a file `xxx.dat`, in multi-column format according to the read statement

```
READ(...,*) T(J),(U(I,J),I=1,NDIM)
```

where T runs in the interval `[0,1]`. When starting from this solution `IRS` should be set to 0 and the value of `ISTART` is irrelevant.

(iii) By setting `ISTART=2`, an explicit homoclinic solution can be specified in the routine **STPNT** in the usual AUTO format, that is $U = ...(T)$ where T is scaled to lie in the interval $[0, 1]$.

(iv) The choice `ISTART=3`, allows for a homotopy method to be used to approach a homoclinic orbit starting from a small approximation to a solution to the linear problem in the unstable manifold (Doedel, Friedman & Monteiro 1993). For details of implementation, the reader is referred to Section 5.1.2. of Champneys & Kuznetsov (1994), under the simplification that we do not solve for the adjoint $u(t)$ here. The basic idea is to start with a small solution in the unstable manifold, and perform continuation in `PAR(11)`$=2T$ and dummy initial-condition parameters ξ_i in order to satisfy the correct right-hand boundary conditions, which are defined by zeros of other dummy parameters ω_i. More precisely, the left-hand end point is placed in the tangent space to the unstable manifold of the saddle and is characterized by `NUNSTAB` coordinates ξ_i satisfying the condition

$$\xi_1^2 + \xi_2^2 + \ldots + \xi_{\texttt{NUNSTAB}}^2 = \epsilon_0^2,$$

where ϵ_0 is a user-defined small number. At the right-hand end point, `NUNSTUB` values ω_i measure the deviation of this point from the tangent space to the stable manifold of the saddle.

Suppose that `IEQUIB=0,1` and set `IP=12+IEQUIB*NDIM`. Then

`PAR(IP)`	:	ϵ_0
`PAR(IP+i)`	:	$\xi_{\texttt{i}}$, `i=1,2,...,NUNSTAB`
`PAR(IP+NUNSTAB+i)`	:	$\omega_{\texttt{i}}$, `i=1,2,...,NUNSTAB`

Note that to avoid interference with the test functions (i.e. `PAR(21)-PAR(36)`*), one must have* `IP+2*NUNSTAB < 21`*.*

If an ω_i is vanished, it can be frozen while another dummy or system parameter is allowed to vary in order to make consequently all $\omega_i = 0$. The resulting final solution gives the initial homoclinic orbit provided the right-hand end point is sufficiently close to the saddle. See Chapter 23 for an example, however, we recommend the homotopy method only for "expert users".

To compute the orientation of a homoclinic orbit (in order to detect inclination-flip bifurcations) it is necessary to compute, in tandem, a solution to the modified adjoint variational equation, by setting `ITWIST=1`. In order to obtain starting data for such a computation when restarting from a point where just the homoclinic is computed, upon increasing `ITWIST` to 1, AUTO generates trivial data for the adjoint. Because the adjoint equations are linear, only a single step of Newton's method is required to enable these trivial data to converge to the correct unique bounded solution. This can be achieved by making a single continuation step in a trivial parameter (i.e. a parameter that does not appear in the problem).

Decreasing `ITWIST` to 0 automatically deletes the data for the adjoint from the continuation problem.

20.8 Notes on Running HomCont Demos.

HomCont demos are given in the following chapters. To copy all files of a demo `xxx` (for example, `san`), move to a clean directory and type `demo('xxx')`. Simply typing `auto xxx.auto` will then automatically execute all runs of the demo. At each step, the user is encouraged to plot the data saved by using the command `plot` (e.g., `plot(r)` plots the data saved in the Python variable `r`).

Of course, in a real application, the runs will not have been prepared in advance, and AUTO-commands must be used. Such commands can be found in a table at the end of each chapter. A sequence of detailed AUTO-commands will be given in these tables as illustrated in Table 20.1 and Table 20.2 for two representative runs of HomCont demo `san`.

The user is encouraged to copy the format of one of these demos when constructing new examples.

The output of the HomCont demos reproduced in the following chapters is somewhat machine dependent, as already noted in Section 12.4. In exceptional circumstances, AUTO may reach its maximum number of steps `NMX` before a certain output point, or the label of an output point may change. In such case the user may have to make appropriate changes in the AUTO constants-files.

COMMAND	ACTION

`san=load('san',IPS=9,NDIM=3,ISP=0,ILP=0,`
` ITNW=7,JAC=1,NTST=35,IEQUIB=0,DS=0.05)`
`r1=run(san,ICP=[1,8],UZR={-1:0.25})`
`save(r1,'6')`
	set common AUTO constants
	run AUTO using the specified constants
	save output-files as b.6, s.6, d.6

`@R san 1` `@sv 6`	use the constants file c.san.1

Table 20.1: These two sets of AUTO-Commands are equivalent.

COMMAND	ACTION
`r9=run(r8,ICP=[2,8],UZR={-2:3.0})`	get the AUTO constants-file and run AUTO/HomCont; restart from the last label in `r8`
`r6=r6+r9`	append output to the Python variable `r6`
`save(r6,'6')`	save output to the files b.6, s.6, d.6
`@R san 9 6`	use the constants file c.san.9, start from the file s.6
`@ap 6`	append output to the files b.6, s.6, d.6

Table 20.2: These two sets of AUTO-Commands behave similarly.

Chapter 21

HomCont Demo : san.

21.1 Sandstede's Model.

Consider the system (Sandstede 1995*a*)

$$\begin{aligned} \dot{x} &= a\,x + b\,y - a\,x^2 + (\tilde{\mu} - \alpha\,z)\,x\,(2 - 3x) \\ \dot{y} &= b\,x + a\,y - \tfrac{3}{2}\,b\,x^2 - \tfrac{3}{2}\,a\,x\,y - (\tilde{\mu} - \alpha\,z)\,2\,y \\ \dot{z} &= c\,z + \mu\,x + \gamma\,x\,y + \alpha\,\beta\,(x^2\,(1-x) - y^2) \end{aligned} \tag{21.1}$$

as given in the file san.f90. Choosing the constants appearing in (21.1) appropriately allows for computing inclination and orbit flips as well as non-orientable resonant bifurcations, see (Sandstede 1995*a*) for details and proofs. The starting point for all calculations is $a = 0$, $b = 1$ where there exists an explicit solution given by

$$(x(t), y(t), z(t)) = \left(1 - \left(\frac{1 - e^t}{1 + e^t}\right)^2, 4\,e^t\,\frac{1 - e^t}{(1 + e^t)^3}, 0\right).$$

This solution is specified in the routine **STPNT**.

21.2 Inclination Flip.

We start by copying the demo to the current work directory and running the first step

```
demo('san')
san=load('san',IPS=9,NDIM=3,ISP=0,ILP=0,ITNW=7,JAC=1,NTST=35,IEQUIB=0,DS=0.05)
r1=run(san,ICP=[1,8],UZR={-1:0.25})
```

This computation starts from the analytic solution above with $a = 0$, $b = 1$, $c = -2$, $\alpha = 0$, $\beta = 1$ and $\gamma = \mu = \tilde{\mu} = 0$. The homoclinic solution is followed in the parameters $(a, \tilde{\mu})$ `=(PAR(1), PAR(8))` up to $a = 0.25$. The output is summarised on the screen as

```
  BR    PT  TY  LAB    PAR(1)        L2-NORM            PAR(8)
   1     1  EP    1   0.00000E+00   4.00000E-01 ...   0.00000E+00
   1     5  UZ    2   2.50000E-01   4.03054E-01 ...  -1.85981E-11
```

and saved in more detail in the Python variable `r1`.

Next we want to add a solution to the adjoint equation to the solution obtained at $a = 0.25$. This is achieved by starting from the last label, and making the changes `ITWIST = 1`, `NMX = 2` and `ICP(1) = 9`. We also disable any user-defined functions `UZR={}`. The computation so-defined is a single step in a trivial parameter `PAR(9)` (namely a parameter that does not appear in the problem). The effect is to perform a Newton step to enable AUTO to converge to a solution of the adjoint equation.

```
r2=run(r1,ICP=[9,8],ITWIST=1,NMX=2,UZR={})
```

The output is stored in the Python variable `r2`.

We can now continue the homoclinic plus adjoint in $(\alpha, \tilde{\mu})$ `=(PAR(4), PAR(8))` by changing the constants to read `NMX = 50` and `ICP(1) = 4`. We also add `PAR(10)` to the list of continuation parameters `ICP`. Here `PAR(10)` is a dummy parameter used in order to make the continuation of the adjoint well posed. Theoretically, it should be zero if the computation of the adjoint is successful (Sandstede 1995*a*). The test functions for detecting resonant bifurcations (`ISPI(1)=1`) and inclination flips (`ISPI(1)=13`) are also activated. Recall that this should be specified in three ways. First we add `PAR(21)` and `PAR(33)` to the list of continuation parameters, second we set up user defined output at zeros of these parameters, and finally we set `IPSI=[1,13]` We also add another user zero for detecting when `PAR(4)=1.0`. Running

```
r3=run(r2,ICP=[4,8,10,21,33],IPSI=[1,13],NMX=50,NPR=20,UZR={4:1.0,21:0,33:0})
                                        save(r3,'r3')
```

starts from the last, and in this case, only, labelled solution in `r2` and outputs to the screen

```
 BR  PT  TY LAB    PAR(4)      ...     PAR(8)        PAR(10)    ...     PAR(33)
  1  20       4  7.84722E-01  ... -2.72146E-11 -4.21812E-09 ...  1.44112E+01
  1  27  UZ   5  1.00000E+00  ... -3.91152E-11 -4.38659E-09 ...  5.70167E+00
  1  35  UZ   6  1.23086E+00  ... -6.18304E-11 -4.62672E-09 ... -9.48584E-06
  1  40       7  1.38397E+00  ... -8.41993E-11 -4.63701E-09 ... -1.34882E+00
  1  50  EP   8  1.69521E+00  ... -1.36449E-10 -5.35972E-09 ... -5.31105E-01
```

Full output is stored in `b.3`, `s.3` and `d.3`. Note that the artificial parameter ϵ =`PAR(10)` is zero within the allowed tolerance. At label `6`, a zero of test function ψ_{13} has been detected which corresponds to an inclination flip with respect to the stable manifold. That the orientation of the homoclinic loop changes as the family passes through this point can be read from the information in `d.3`. However in `d.3`, the line

```
ORIENTABLE (    0.2982090775E-03)
```

at `PT=35` would seems to contradict the detection of the inclination flip at this point. Nonetheless, the important fact is the zero of the test function; and note that the value of the variable indicating the orientation is small compared to its value at the other regular points. Data for the adjoint equation at `LAB= 4, 6` and `8` at and on either side of the inclination flip are presented in Fig. 21.1. The switching of the solution between components of the leading unstable left eigenvector is apparent. Finally, we remark that the Newton step in the dummy parameter `PAR(20)` performed above is crucial to obtain convergence. Indeed, if instead we try to continue the homoclinic orbit and the solution of the adjoint equation directly by setting

```
ITWIST = 1   IRS = 2   NMX = 50   ICP(1) = 4
```

and running

```
r4=run(r1,ICP=[4,8,10,21,33],ITWIST=1,IPSI=[1,13],NMX=50,UZR={33:0})
```

we obtain a no convergence error.

21.3 Non-orientable Resonant Eigenvalues.

Inspecting the output saved in the third run, we observe the existence of a non-orientable homoclinic orbit at the second `UZ` label `6`. We restart at this label, with the first continuation parameter being once again $a =$`PAR(1)`, by changing constants according to

```
DS = -0.05    NMX = 20    ICP(1) = 1
```

Running,

```
r5=run(r3('UZ2'),ICP=[1,8,10,21,33],NMX=20,DS='-',sv='5')
```

the output at label `9`

```
BR    PT  TY LAB     PAR(1)           PAR(8)        PAR(10)       PAR(21)
 1     8  UZ   9  -1.30447E-07 ...  3.41490E-12  -1.63406E-09  -2.60894E-07
```

indicates that AUTO has detected a zero of `PAR(21)`, implying that a non-orientable resonant bifurcation occurred at that point.

21.4 Orbit Flip.

In this section we compute an orbit flip. To this end we restart from the original explicit solution, without computing the orientation. We begin by separately performing continuation in $(\alpha, \tilde{\mu})$, $(\beta, \tilde{\mu})$, $(a, \tilde{\mu})$, $(b, \tilde{\mu})$ and $(\mu, \tilde{\mu})$ in order to reach the parameter values $(a, b, \alpha, \beta, \mu) = (0.5, 3, 1, 0, 0.25)$. The sequence of continuations up to the desired parameter values are run via

```
r6=run(san,ICP=[4,8],UZR={-4:1})
r7=run(r6,ICP=[5,8],UZR={-5:0},DS='-')
r8=run(r7,ICP=[1,8],UZR={-1:0.5},DS='-')
r9=run(r8,ICP=[2,8],UZR={-2:3.0})
r10=run(r9,ICP=[7,8],UZR={-7:0.25})
```

with appropriate continuation parameters and user output values set. The desired output is stored in `r10`.

The final saved point `LAB=6` contains a homoclinic solution at the desired parameter values. From here we perform continuation in the negative direction of $(\mu, \tilde{\mu}) =$ (`PAR(7),PAR(8)`) with the test function ψ_{11} for orbit flips with respect to the stable manifold activated.

```
r11=run(r10,ICP=[7,8,31],IPSI=[11],UZR={31:0.0,-7:-0.5},DS='-')
                              save(r11,'11')
```

The output detects an inclination flip (by a zero of `PAR(31)`) at `PAR(7)=0`

```
 BR    PT  TY LAB    PAR(7)       ...    PAR(8)        PAR(31)
  1     5  UZ   7  6.33545E-06    ...  1.70968E-06  -8.70508E-05
```

at which parameter value the homoclinic orbit is contained in the (x, y)-plane (see Fig. 21.2).

Finally, we demonstrate that the orbit flip can be continued as three parameters (`PAR(6), PAR(7), PAR(8)`) are varied.

```
                              of = r11('UZ1')
   r12=run(of,ICP=[7,8,6],IPSI=[],NPR=5,NMX=20,IFIXED=[11],UZR={},DS='-')
                              save(r12,'12')
```

```
BR    PT  TY LAB    PAR(7)        ...    PAR(8)        PAR(6)
 1     5       9  -7.38787E-19    ...  -2.91178E-10  -3.25000E-01
 1    10      10  -5.27166E-19    ...  -2.23972E-10  -8.25000E-01
 1    15      11  -6.15227E-19    ...  -2.91908E-10  -1.32500E+00
 1    20  EP  12  -5.96426E-19    ...  -3.20088E-10  -1.82500E+00
```

The orbit flip continues to be defined by a planar homoclinic orbit at `PAR(7)=PAR(8)=0`.

21.5 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION
`mkdir san`	create an empty work directory
`cd san`	change directory
`demo('san')`	copy the demo files to the work directory
`san=load('san',IPS=9,NDIM=3,ISP=0,ILP=0,` ` ITNW=7,JAC=1,NTST=35,IEQUIB=0,DS=0.05)`	configure common constants
`r1=run(san,ICP=[1,8],UZR={-1:0.25})`	continuation in `PAR(1)`
`r2=run(r1,ICP=[9,8],ITWIST=1,NMX=2,UZR={})`	generate adjoint variables

`r3=run(r2,ICP=[4,8,10,21,33],IPSI=[1,13],`
` NMX=50,NPR=20,UZR={4:1.0,21:0,33:0})`
`save(r3,'3')`
	continue homoclinic orbit and adjoint
	save output-files as b.3, s.3, d.3

`r4=run(r1,ICP=[4,8,10,21,33],ITWIST=1,`
` IPSI=[1,13],NMX=50,UZR={33:0})`
`sv('4')`
	no convergence without dummy step
	save output-files as b.4, s.4, d.4

`r5=run(r3('UZ2'),ICP=[1,8,10,21,33],NMX=20,` ` DS='-',sv='5')`	continue non-orientable orbit save output-files as b.5, s.5, d.5

Table 21.1: Detailed AUTO-Commands for running demo san.

AUTO-COMMAND	ACTION
`r6=run(san,ICP=[4,8],UZR={-4:1})`	restart and homotopy to `PAR(4)`=1.0
`r7=run(r6,ICP=[5,8],UZR={-5:0},DS='-')`	homotopy to `PAR(5)`=0.0
`r8=run(r7,ICP=[1,8],UZR={-1:0.5},DS='-')`	homotopy to `PAR(1)`=0.5
`r9=run(r8,ICP=[2,8],UZR={-2:3.0})`	homotopy to `PAR(2)`=3.0
`r10=run(r9,ICP=[7,8],UZR={-7:0.25})`	homotopy to `PAR(7)`=0.25

`r11=run(r10,ICP=[7,8,31],IPSI=[11],`
` UZR={31:0.0,-7:-0.5},DS='-')`
`save(r11,'11')`
	
	continue in `PAR(7)` to detect orbit flip
	save output-files as b.11, s.11, d.11

`of=r11('UZ1')`
`r12=run(of,ICP=[7,8,6],IPSI=[],NPR=5,`
` NMX=20,IFIXED=[11],UZR={},DS='-')`
`save(r12,'12')`
	select first UZ-labelled point of `r11` to start from
	
	three-parameter continuation of orbit flip
	save output-files as b.12, s.12, d.12

Table 21.2: Detailed AUTO-Commands for running demo san.

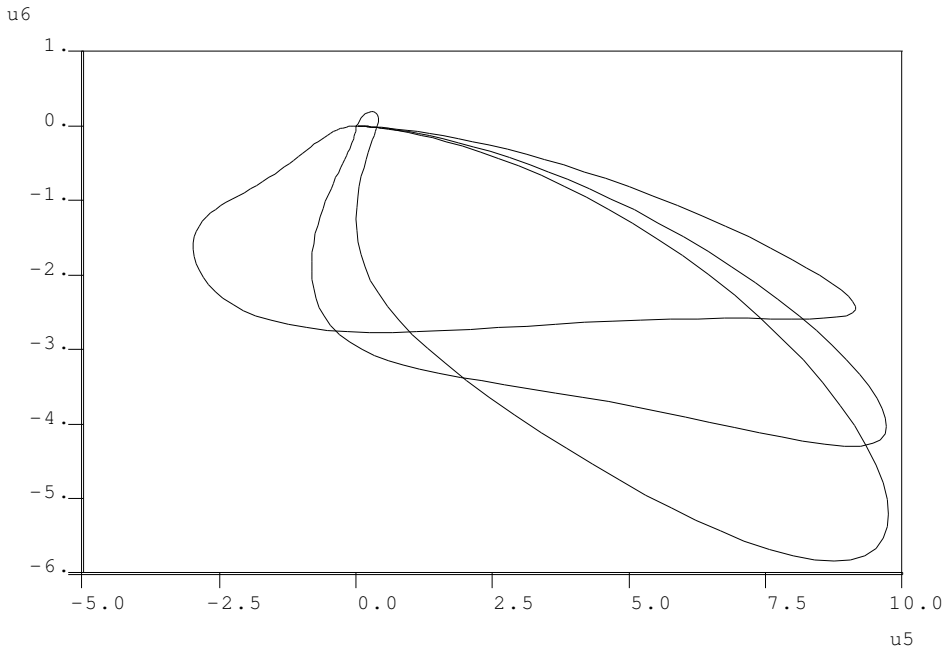


Figure 21.1: Second versus third component of the solution to the adjoint equation at labels 4, 6 and 8

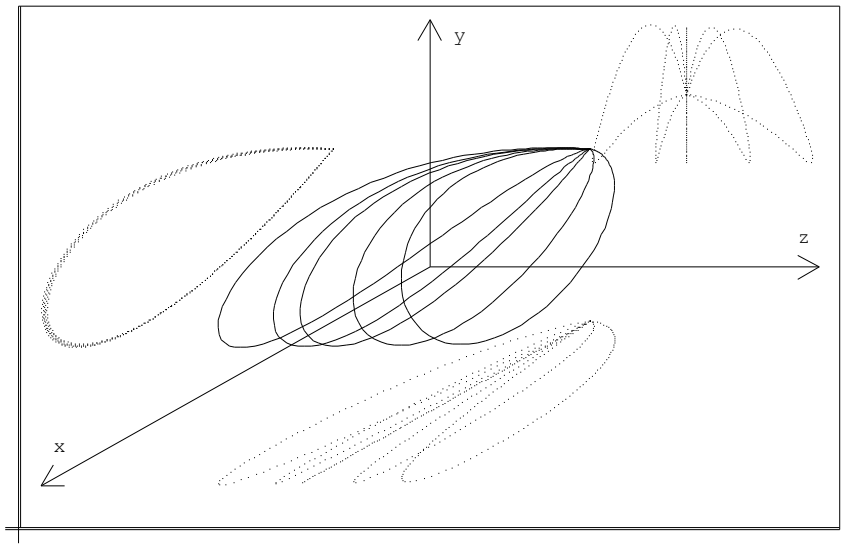


Figure 21.2: Orbits on either side of the orbit flip bifurcation. The critical orbit is contained in the (x, y)-plane

Chapter 22

HomCont Demo : mtn.

22.1 A Predator-Prey Model with Immigration.

Consider the following system of two equations (Scheffer 1995)

$$\begin{aligned} \dot{X} &= RX\left(1-\frac{X}{K}\right)-\frac{A_1XY}{B_1+X}+D_0K \\ \dot{Y} &= E_1\frac{A_1XY}{B_1+X}-D_1Y-\frac{A_2ZY^2}{B_2^2+Y^2}. \end{aligned} \tag{22.1}$$

The values of all parameters except (K, Z) are set as follows :

$$R=0.5,\ A_1=0.4,\ B_1=0.6,\ D_0=0.01,\ E_1=0.6,\ A_2=1.0,\ B_2=0.5,\ D_1=0.15.$$

The parametric portrait of the system (22.1) on the (Z, K)-plane is presented in Figure 22.1. It contains fold ($t_{1,2}$) and Hopf (H) bifurcation curves, as well as a homoclinic bifurcation curve P. The fold curves meet at a cusp singular point C, while the Hopf and the homoclinic curves originate at a Bogdanov-Takens point BT. Only the homoclinic curve P will be considered here, the other bifurcation curves can be computed using AUTO or, for example, locbif (Khibnik, Kuznetsov, Levitin & Nikolaev 1993).

22.2 Continuation of Central Saddle-Node Homoclinics.

Local bifurcation analysis shows that at $K=6.0,\ Z=0.06729762\ldots$, the system has a saddle-node equilibrium

$$(X^0, Y^0)=(5.738626\ldots, 0.5108401\ldots),$$

with one zero and one negative eigenvalue. Direct simulations reveal a homoclinic orbit to this saddle-node, departing and returning along its central direction (i.e., tangent to the null-vector).

Starting from this solution, stored in the file mtn.dat, we continue the saddle-node central homoclinic orbit with respect to the parameters K and Z by copying the demo and running it

```
dm('mtn')
r1=run('mtn',c='mtn.1',sv='1')
```

The file mtn.f90 contains approximate parameter values

$$K = \texttt{PAR(1)} = 6.0,\ Z = \texttt{PAR(2)} = 0.06729762,$$

as well as the coordinates of the saddle-node

$$X^0 = \texttt{PAR(12)} = 5.738626,\ Y^0 = \texttt{PAR(13)} = 0.5108401,$$

and the length of the truncated time-interval

$$T_0 = \texttt{PAR(11)} = 1046.178\,.$$

Since a homoclinic orbit to a saddle-node is being followed, we have also made the choice

```
IEQUIB = 2
```

in c.mtn.1. The two test-functions, ψ_{15} and ψ_{16}, to detect non-central saddle-node homoclinic orbits are also activated, which must be specified in three ways. Firstly, in c.mtn.1, IPSI is set to [15,16] so the active test functions are chosen as 15 and 16. This sets up the monitoring of these test functions. Secondly, in c.mtn.1 user-defined functions (UZR) are set up to look for zeros of the parameters corresponding to these test functions. Recall that the parameters to be zeroed are always the test functions plus 20. Finally, these parameters are included in the list of continuation parameters (ICP).

Among the output there is a line

```
 BR    PT  TY  LAB    PAR(1)      ...    PAR(2)         PAR(35)        PAR(36)
  1    26  UZ    4   6.61046E+00 ...   6.93248E-02   5.23950E-09  -6.42344E-02
```

indicating that a zero of the test function IPSI(1)=15 This means that at

$$D_1 = (K^1, Z^1) = (6.6105\ldots, 0.069325\ldots)$$

the homoclinic orbit to the saddle-node becomes *non-central*, namely, it returns to the equilibrium along the stable eigenvector, forming a non-smooth loop. The output is saved in b.1, s.1 and d.1. Repeating computations in the opposite direction along the curve, IRS=1, DS=-0.01 in c.mtn.2,

```
r1=r1+run(r1(1),c='mtn.2',ap='1')
```

one obtains

```
 BR    PT  TY  LAB    PAR(1)      ...    PAR(2)         PAR(35)        PAR(36)
  1    34  UZ   10   5.18039E+00 ...   6.38554E-02   8.86654E-10  -7.28132E-02
```

which means another non-central saddle-node homoclinic bifurcation occurs at

$$D_2 = (K^2, Z^2) = (5.1803\ldots, 0.063855\ldots).$$

Note that these data were obtained using a smaller value of NTST than the original computation (compare c.mtn.1 with c.mtn.2). The high original value of NTST was only necessary for the first few steps because the original solution is specified on a uniform mesh.

22.3 Switching between Saddle-Node and Saddle Homoclinic Orbits.

Now we can switch to continuation of saddle homoclinic orbits at the located codim 2 points D_1 and D_2.

```
r1=r1+run(r1('UZ1'),c='mtn.3',ap='1')
```

starts from D_1. Note that now

```
IEQUIB = 1
```

has been specified in c.mtn.3. Also, test functions ψ_9 and ψ_{10} have been activated in order to monitor for non-hyperbolic equilibria along the homoclinic locus. We get the following output

```
 BR    PT  TY  LAB    PAR(1)      ...    PAR(2)        PAR(29)       PAR(30)
  1    10       12   7.11454E+00 ...   7.08176E-02  -4.64986E-01   3.18355E-03
  1    20       13   9.17683E+00 ...   7.67874E-02  -4.68491E-01   1.60931E-02
  1    30       14   1.21084E+01 ...   8.54348E-02  -4.71887E-01   3.06966E-02
  1    40  EP   15   1.50379E+01 ...   9.42805E-02  -4.74379E-01   4.14457E-02
```

The fact that PAR(29) and PAR(30) do not change sign indicates that there are no further non-hyperbolic equilibria along this family. Note that restarting in the opposite direction with IRS=15,DS=-0.02

```
r4=run(r1,c='mtn.4',sv='4')
```

will detect the same codim 2 point D_1 but now as a zero of the test-function ψ_{10}

```
 BR    PT  TY  LAB    PAR(1)      ...    PAR(2)        PAR(29)       PAR(30)
  1    38  UZ   11   6.61046E+00 ...   6.93248E-02  -4.63660E-01   3.13439E-08
```

Note that the values of PAR(1) and PAR(2) are equal to those at label 4 up to at least six significant figures.

Actually, the program runs further and eventually computes the point D_2 and the whole lower family of P emanating from it, however, the solutions between D_1 and D_2 should be considered as spurious[1], therefore we do not save these data. The reliable way to compute the lower family of P is to restart computation of saddle homoclinic orbits in the other direction from the point D_2

```
r1=r1+run(r1('UZ3'),c='mtn.5',ap='1')
```

This gives the lower family of P approaching the BT point (see Figure 22.1)

[1] The program actually computes the saddle-saddle heteroclinic orbit bifurcating from the non-central saddle-node homoclinic at the point D_1, see Champneys et al. (1996, Fig. 2), and continues it to the one emanating from D_2.

```
 BR    PT  TY  LAB    PAR(1)      ...   PAR(2)        PAR(29)       PAR(30)
  1    10       16   4.96649E+00 ...  6.29843E-02  -4.38247E-01   4.94481E-03
  1    20       17   4.92531E+00 ...  7.96087E-02  -3.39922E-01   3.28829E-02
  1    30       18   7.09217E+00 ...  1.58708E-01  -1.69289E-01   3.87631E-02
  1    40  EP   19   1.10181E+01 ...  2.80980E-01  -3.48294E-02   2.10449E-02
```

The data are appended to the stored results in b.1, s.1 and d.1. One could now display all data using the AUTO command `@pp 1` to reproduce the curve P shown in Figure 22.1.

It is worthwhile to compare the homoclinic curves computed above with a curve $T_0 = const$ along which the system has a limit cycle of constant large period $T_0 = 1046.178$, which can easily be computed using AUTO or locbif. Such a curve is plotted in Figure 22.2. It obviously approximates well the saddle homoclinic loci of P, but demonstrates much bigger deviation from the saddle-node homoclinic segment D_1D_2. This happens because the period of the limit cycle grows to infinity while approaching both types of homoclinic orbit, but with *different asymptotics*: as $-\ln\|\alpha - \alpha^*\|$, in the saddle homoclinic case, and as $\|\alpha - \alpha^*\|^{-1}$ in the saddle-node case.

22.4 Three-Parameter Continuation.

Finally, we can follow the curve of non-central saddle-node homoclinic orbits in three parameters. The extra continuation parameter is D_0=`PAR(3)`. To achieve this we restart at label `4`, corresponding to the codim 2 point D_1. We return to continuation of saddle-node homoclinics, `NUNSTAB=0`,`IEQUIB=2`, but append the defining equation $\psi_{15} = 0$ to the continuation problem (via `IFIXED=[15]`). The new continuation problem is specified in c.mtn.6.

```
r6=run(r1('UZ1'),c='mtn.6',sv='6')
```

Notice that we set `ILP=1` and choose `PAR(3)` as the first continuation parameter so that AUTO can detect limit points with respect to this parameter. We also make a user-defined function (`UZR`) to detect intersections with the plane $D_0 = 0.01$. We get among other output

```
 BR    PT  TY  LAB    PAR(3)        L2-NORM     ...    PAR(1)        PAR(2)
  1    22  LP   20   1.08120E-02   5.32589E+00 ...   5.67363E+00   6.60818E-02
  1    31  UZ   21   1.00000E-02   4.81969E+00 ...   5.18032E+00   6.38551E-02
```

the first line of which represents the D_0 value at which the homoclinic curve P has a tangency with the family t_2 of fold bifurcations. Beyond this value of D_0, P consists entirely of saddle homoclinic orbits. The data at label `20` reproduce the coordinates of the point D_2. The results of this computation and a similar one starting from D_1 in the opposite direction (with `DS=-0.01`) are displayed in Figure 22.3.

22.5 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION

`mkdir mtn`
`cd mtn`
`demo('mtn')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run('mtn',c='mtn.1',sv='1')`	continue saddle-node homoclinic orbit from mnt.dat save output-files as b.1, s.1, d.1
`r1=r1+run(r1(1),c='mtn.2',ap='1')`	continue in opposite direction; restart from label 1 append output-files to b.1, s.1, d.1
`r1=r1+run(r1('UZ1'),c='mtn.3',ap='1')`	switch to saddle homoclinic orbit ; restart: 1st UZ append output-files to b.1, s.1, d.1
`r4=run(r1,c='mtn.4',sv='4')`	continue in reverse direction; restart from last label save output-files as b.4, s.4, d.4
`r1=r1+run(r1('UZ3'),c='mtn.5',ap='1')`	other saddle homoclinic orbit family; restart: 3rd UZ append output-files to b., s.1, d.1
`r6=run(r1('UZ1'),c='mtn.6',sv='6')`	3-parameter non-central saddle-node homoclinic. save output-files as b.6, s.6, d.6

Table 22.1: Detailed AUTO-Commands for running demo mtn.

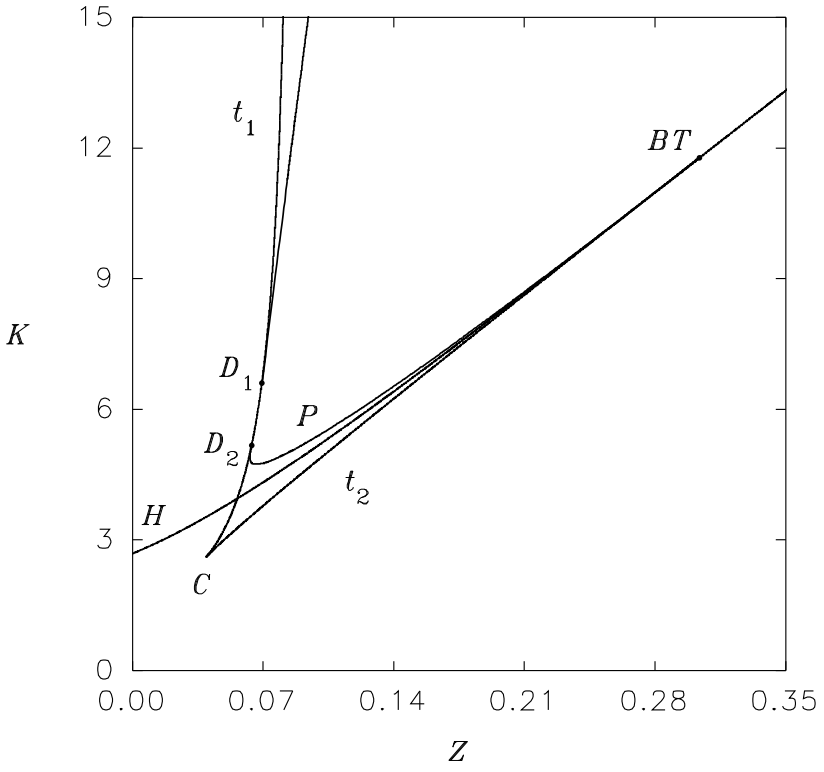


Figure 22.1: Parametric portrait of the predator-prey system

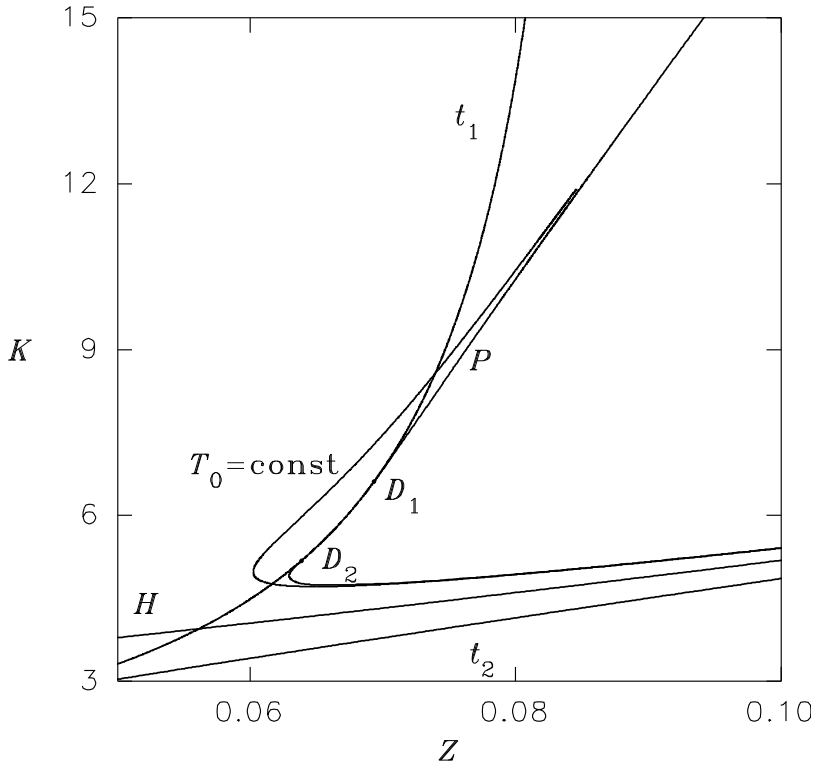


Figure 22.2: Approximation by a large-period cycle

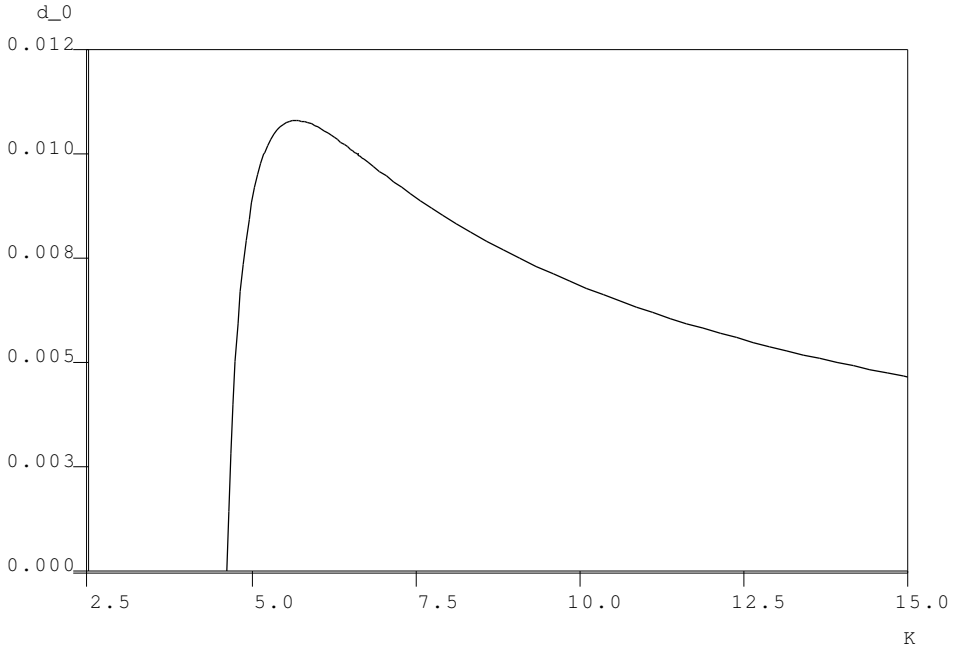


Figure 22.3: Projection onto the (K, D_0)-plane of the three-parameter curve of non-central saddle-node homoclinic orbit

Chapter 23

HomCont Demo : kpr.

23.1 Koper's Extended Van der Pol Model.

The equation-file kpr.f90 contains the equations

$$\begin{aligned} \dot{x} &= \epsilon_1^{-1}\,(k\,y - x^3 + 3\,x - \lambda) \\ \dot{y} &= x - 2\,y + z \\ \dot{z} &= \epsilon_2(y - z), \end{aligned} \tag{23.1}$$

with $\epsilon_1 = 0.1$ and $\epsilon_2 = 1$ (Koper 1995).

To copy across the demo kpr and compile we type

```
demo('kpr')
```

23.2 The Primary Branch of Homoclinics.

First, we locate a homoclinic orbit using the homotopy method. The file kpr.f90 already contains approximate parameter values for a homoclinic orbit, namely λ =PAR(1)=-1.851185, k =PAR(2)=-0.15. The file c.kpr.1 specifies the appropriate constants for continuation in $2T$=PAR(11) (also referred to as PERIOD) and the dummy parameter ω_1=PAR(17) starting from a small solution in the local unstable manifold;

```
r1=run('kpr',c='kpr.1',sv='1')
```

Among the output there is the line

```
  BR    PT  TY LAB    PERIOD        L2-NORM      ...     PAR(17)     ...
   1    29  UZ   2  1.90018E+01   1.69382E+00    ...  4.46147E-09   ...
```

which indicates that a zero of the artificial parameter ω_1 has been located. This means that the right-hand end point of the solution belongs to the plane that is tangent to the stable manifold at the saddle. The output is stored in files b.1, s.1, d.1. Upon plotting the data at label 2 (see Figure 23.1) it can be noted that although the right-hand projection boundary condition is satisfied, the solution is still quite away from the equilibrium.

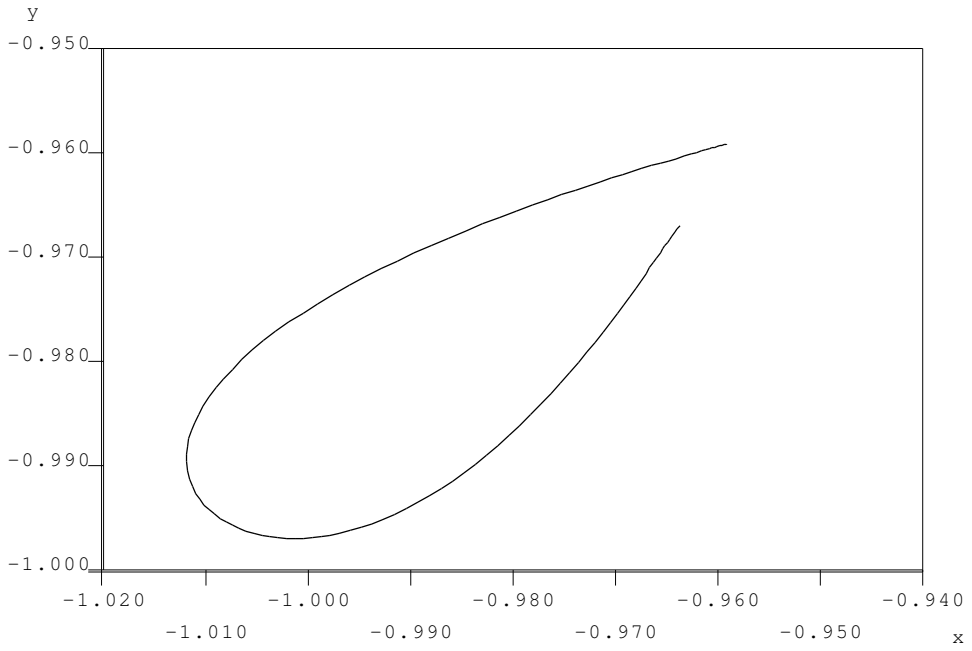


Figure 23.1: Projection on the (x, y)-plane of solutions of the boundary value problem with $2T = 19.08778$.

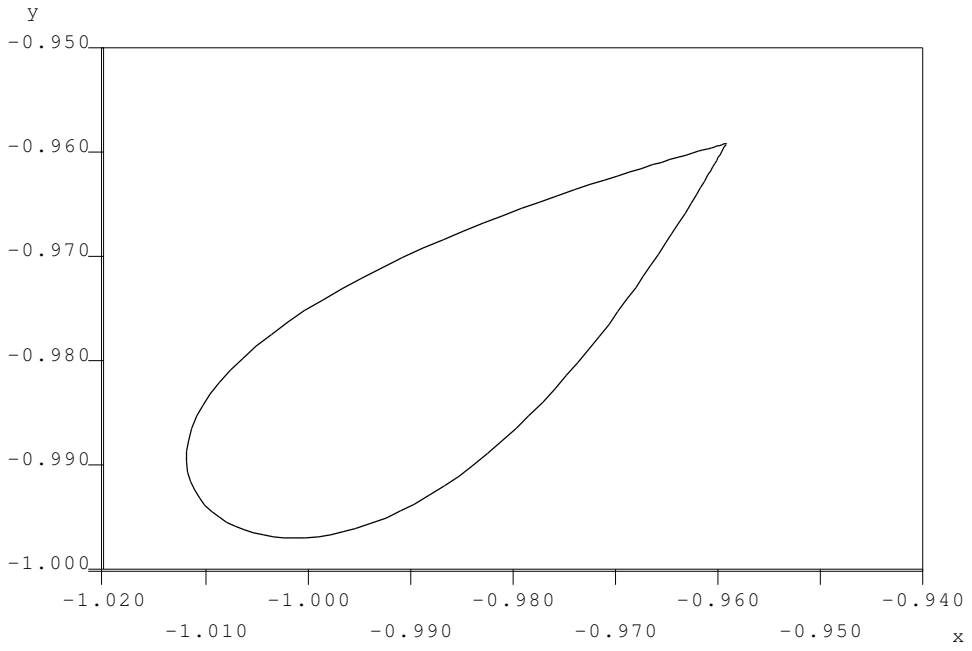


Figure 23.2: Projection on the (x, y)-plane of solutions of the boundary value problem with $2T = 60.0$.

The right-hand endpoint can be made to approach the equilibrium by performing a further continuation in T with the right-hand projection condition satisfied (PAR(17) fixed) but with λ allowed to vary.

```
r2=run(r1('UZ1'),c='kpr.2',sv='2')
```

the output at label 4, stored in s.2,

```
 BR    PT TY  LAB    PERIOD        L2-NORM      ...     PAR(1)      ...
  1    35 UZ    4   6.00000E+01  1.67281E+00   ...  -1.85119E+00 ...
```

provides a good approximation to a homoclinic solution (see Figure 23.2).

The second stage to obtain a starting solution is to add a solution to the modified adjoint variational equation. This is achieved by setting both ITWIST and ISTART to 1 (in c.kpr.3), which generates a trivial guess for the adjoint equations. Because the adjoint equations are linear, only a single Newton step (by continuation in a trivial parameter) is required to provide a solution. Rather than choose a parameter that might be used internally by AUTO, in c.kpr.3 we take the continuation parameter to be PAR(11), which is not quite a trivial parameter but whose affect upon the solution is mild.

```
r3=run(r2('UZ1'),c='kpr.3',sv='3')
```

The output at the second point (label 6) contains the converged homoclinic solution (variables (U(1), U(2), U(3)) and the adjoint (U(4), U(5), U(6))). We now have a starting solution and are ready to perform two-parameter continuation.

The fourth run

```
r3=r3+run(r3,c='kpr.4',ap='3')
```

continues the homoclinic orbit in PAR(1) and PAR(2). Note that several other parameters appear in the output. PAR(10) is a dummy parameter that should be zero when the adjoint is being computed correctly; PAR(29), PAR(30), PAR(33) correspond to the test functions ψ_9,ψ_{10} and ψ_{13}. That these test functions were activated is specified in three places in c.kpr.4 as described in Section 20.6.

Note that at the end-point of the family (reached when after NMX=50 steps) PAR(29) is approximately zero which corresponds to a zero of ψ_9, a non-central saddle-node homoclinic orbit. We shall return to the computation of this codimension-two point later. Before reaching this point, among the output we find two zeroes of PAR(33) (test function ψ_{13}) which gives the accurate location of two inclination-flip bifurcations,

```
 BR  PT  TY LAB    PAR(1)      ...     PAR(2)        PAR(10)     ...   PAR(33)
  1   6  UZ   7  -1.80166E+00 ...  -2.00266E-01  -7.25140E-07 ...  1.14077E-04
  1  12  UZ   8  -1.56876E+00 ...  -4.39547E-01  -2.15617E-07 ... -1.48740E-07
```

That the test function really does have a regular zero at this point can be checked from the data saved in b.3, plotting PAR(33) as a function of PAR(1) or PAR(2). Figure 23.3 presents solutions $\phi(t)$ of the modified adjoint variational equation (for details see Champneys et al. (1996)) at parameter values on the homoclinic family before and after the first detected inclination flip.

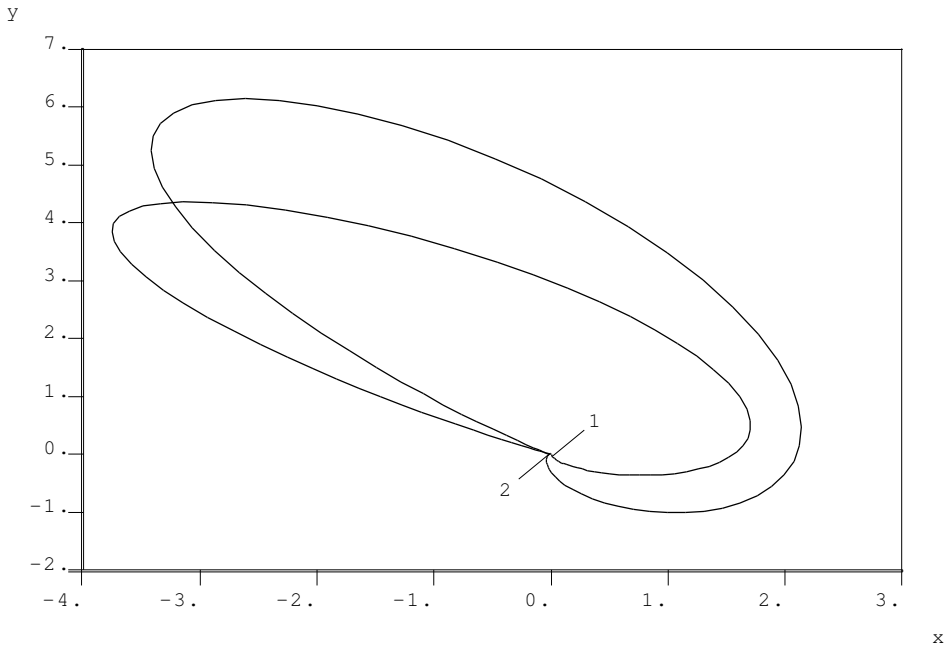


Figure 23.3: Projection on the (x, y)-plane of solutions $\phi(t)$ at `1` ($\lambda = -1.825470, k = -0.1760749$) and `2` ($\lambda = -1.686154, k = -0.3183548$).

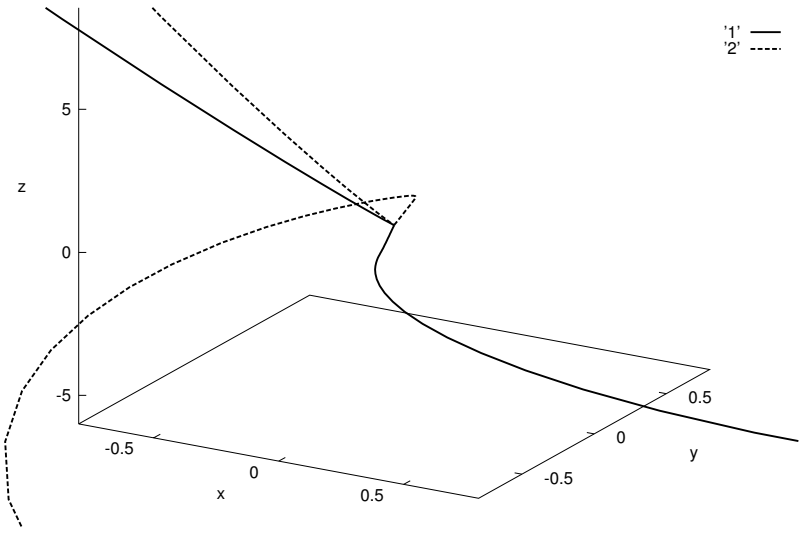


Figure 23.4: Three-dimensional blow-up of the solution curves $\phi(t)$ at labels `1` (dotted) and `2` (solid line) from Figure 3.8.

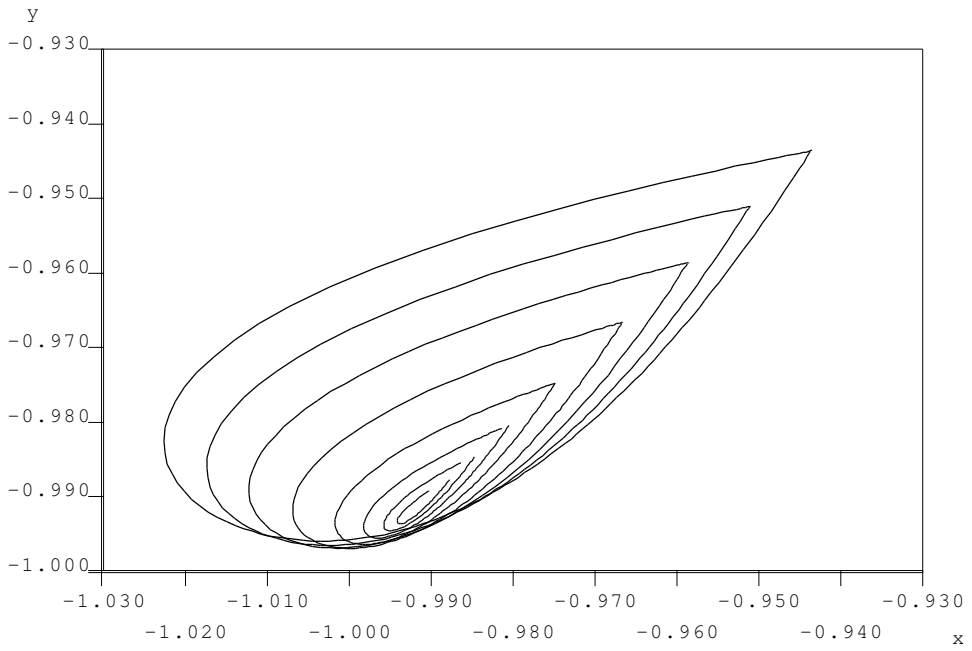


Figure 23.5: Computed homoclinic orbits approaching the BT point

Note that these solutions were obtained by choosing a smaller step `DS` and more output (smaller `NPR`) in `c.kpr.4`. A blow-up of the region close to the origin of this figure is shown in Figure 23.4. It illustrates the flip of the solutions of the adjoint equation while moving through the bifurcation point. Note that the data in this figure were plotted after first performing an additional continuation of the solutions with respect to `PAR(11)`.

Continuing in the other direction

```
r3=r3+run(r3()[0],c='kpr.5',ap='3')
```

we approach a Bogdanov-Takens point

```
 BR    PT  TY LAB    PAR(1)      ...     PAR(10)     ...     PAR(33)
  1    50  EP  10 -1.93828E+00  ... -7.52334E+00  ... -3.19793E+01
```

Note that the numerical approximation has ceased to become reliable, since `PAR(10)` has now become large. Phase portraits of homoclinic orbits between the BT point and the first inclination flip are depicted in Figure 23.5. Note how the computed homoclinic orbits approaching the BT point have their endpoints well away from the equilibrium. To follow the homoclinic orbit to the BT point with more precision, we would need to first perform continuation in T (`PAR(11)`) to obtain a more accurate homoclinic solution.

23.3 More Accuracy and Saddle-Node Homoclinic Orbits.

Continuation in T in order to obtain an approximation of the homoclinic orbit over a longer interval is necessary for parameter values near a non-hyperbolic equilibrium (either a saddle-

node or BT) where the convergence to the equilibrium is slower. First, we start from the original homoclinic orbit computed via the homotopy method, label 4, which is well away from the non-hyperbolic equilibrium. Also, we shall no longer be interested in in inclination flips so we set ITWIST=0 in c.kpr.6, and in order to compute up to PAR(11)=1000, we set up a user-defined function for this. Running AUTO with PAR(11) and PAR(2) as free parameters

```
r6=run(r2('EP1'),c='kpr.6',sv='6')
```

we obtain among the output

```
 BR    PT  TY LAB     PERIOD        L2-NORM     ...     PAR(2)
  1    35  UZ   6   1.00000E+03  1.66191E+00  ... -1.50000E-01
```

We can now repeat the computation of the family of saddle homoclinic orbits in PAR(1) and PAR(2) from this point with the test functions ψ_9 and ψ_{10} for non-central saddle-node homoclinic orbits activated

```
r7=run(r6('UZ1'),c='kpr.7',sv='7')
```

The saddle-node point is now detected at

```
 BR    PT  TY LAB    PAR(1)      ...     PAR(2)         PAR(29)        PAR(30)
  1    29  UZ   8  1.76505E-01   ... -2.40533E+00  -1.74004E-06    2.30933E+01
```

which is stored in s.7. That PAR(29) (ψ_9) is zeroed shows that this is a non-central saddle-node connecting the centre manifold to the strong stable manifold. Note that all output beyond this point, although a well-posed solution to the boundary-value problem, is spurious in that it no longer represents a homoclinic orbit to a saddle equilibrium (see Champneys et al. (1996)). If we had chosen to, we could continue in the other direction in order to approach the BT point more accurately by reversing the sign of DS in c.kpr.7.

The file c.kpr.8 contains the constants necessary for switching to continuation of the central saddle-node homoclinic curve in two parameters starting from the non-central saddle-node homoclinic orbit stored as label 8 in s.7.

```
r8=run(r7('UZ1'),c='kpr.8',sv='8')
```

In this run we have activated the test functions for saddle to saddle-node transition points along curves of saddle homoclinic orbits (ψ_{15} and ψ_{16}). Among the output we find

```
 BR    PT  TY LAB    PAR(1)      ...     PAR(2)         PAR(35)        PAR(36)
  1    38  UZ  11   1.76509E-01 ...  -2.40533E+00   6.89014E-03    3.09956E-05
```

which corresponds to the family of homoclinic orbits leaving the locus of saddle-nodes in a second non-central saddle-node homoclinic bifurcation (a zero of ψ_{16}).

Note that the parameter values do not vary much between the two codimension-two non-central saddle-node points (labels 8 and 11). However, Figure 23.6 shows clearly that between the two codimension-two points the homoclinic orbit rotates between the two components of the 1D stable manifold, i.e. between the two boundaries of the center-stable manifold of the saddle node. The overall effect of this process is the transformation of a nearby "small" saddle

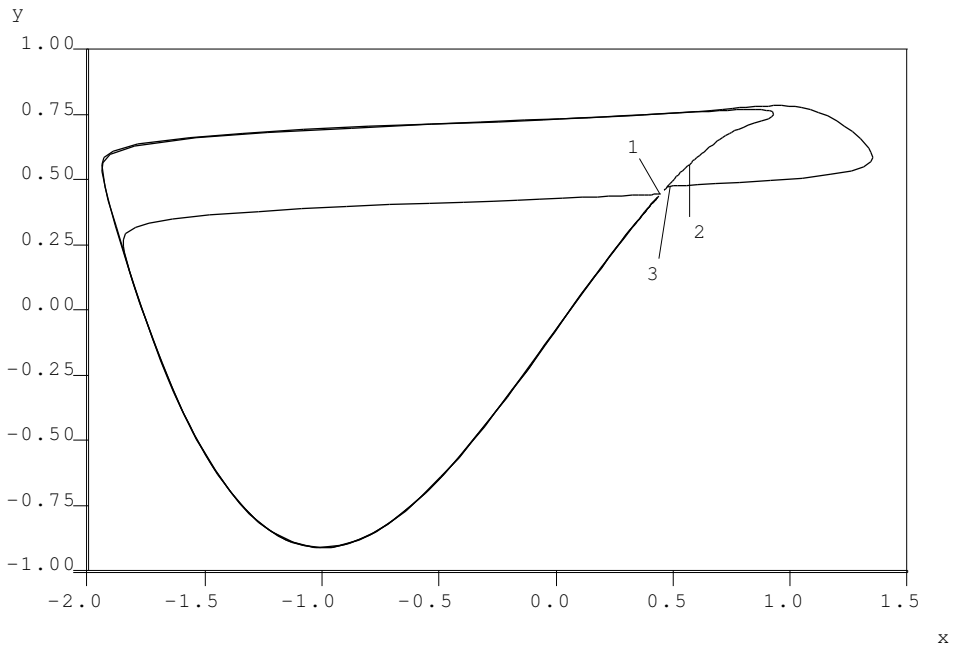


Figure 23.6: Two non-central saddle-node homoclinic orbits, 1 and 3; and, 2, a central saddle-node homoclinic orbit between these two points

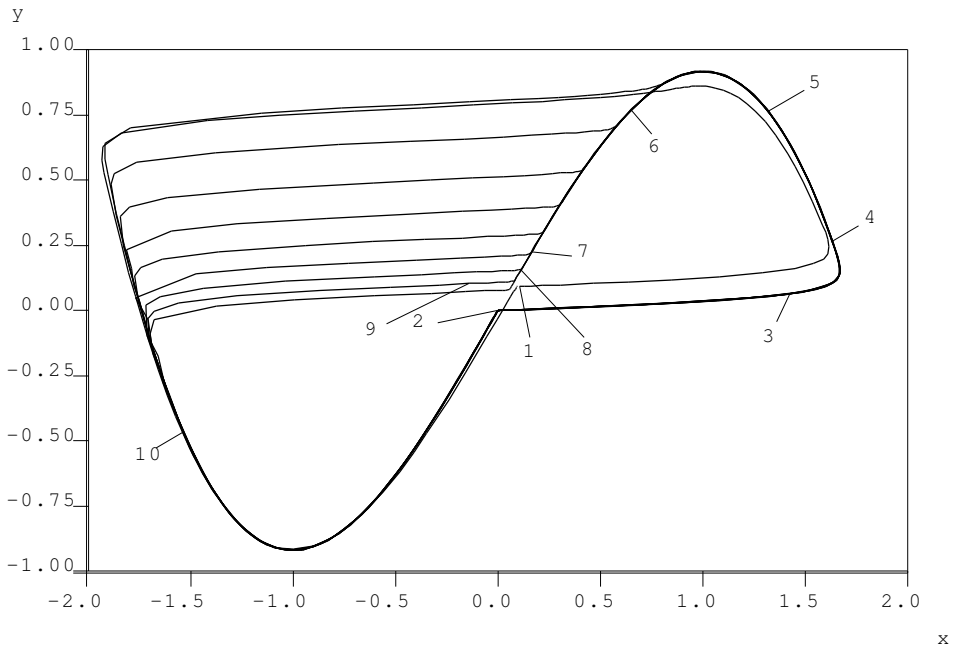


Figure 23.7: The big homoclinic orbit approaching a figure-of-eight

homoclinic orbit to a "big" saddle homoclinic orbit (i.e. with two extra turning points in phase space).

Finally, we can switch to continuation of the big saddle homoclinic orbit from the new codim 2 point at label 11.

```
r9=run(r8('UZ1'),c='kpr.9',sv='9')
```

Note that AUTO takes a large number of steps near the line PAR(1)=0, while PAR(2) approaches $-2.189\ldots$ (which is why we chose such a large value NMX=500 in c.kpr.9). This particular computation ends at

```
 BR    PT  TY LAB    PAR(1)        L2-NORM     ...    PAR(2)
  1   500  EP  24  2.04263E-05   2.18126E-01  ... -2.18951E+00
```

By plotting phase portraits of orbits approaching this end point (see Figure 23.7) we see a "canard-like" like transformation of the big homoclinic orbit to a pair of homoclinic orbits in a figure-of-eight configuration. That we get a figure-of-eight is not a surprise because PAR(1)=0 corresponds to a symmetry in the differential equations (Koper 1994); note also that the equilibrium, stored as (PAR(12), PAR(13), PAR(14)) in d.9, approaches the origin as we approach the figure-of-eight homoclinic.

23.4 Three-Parameter Continuation.

We now consider curves in three parameters of each of the codimension-two points encountered in this model, by freeing the parameter ϵ = PAR(3). First we continue the first inclination flip stored at label 7 in s.3

```
r10=run(r3('UZ1'),c='kpr.10',sv='10')
```

Note that ITWIST=1 in c.kpr.10, so that the adjoint is also continued, and there is one fixed condition IFIXED(1)=13 so that test function ψ_{13} has been frozen. Among the output there is a codimension-three point (zero of ψ_9) where the neutrally twisted homoclinic orbit collides with the saddle-node curve

```
 BR  PT  TY LAB    PAR(1)      ...   PAR(2)         PAR(3)         PAR(29)     ...
  1  18  UZ  11   1.28270E-01 ... -2.51932E+00   5.74477E-01  -2.59151E-06 ...
```

The other detected inclination flip (at label 8 in s.3) is continued similarly

```
r11=run(r3('UZ2'),c='kpr.11',sv='11')
```

giving among its output another codim 3 saddle-node inclination-flip point

```
 BR  PT  TY LAB    PAR(1)      ...   PAR(2)         PAR(3)         PAR(29)     ...
  1  27  UZ  11   1.53542E-01 ... -2.45810E+00   1.17171E+00  -1.13312E-06 ...
```

Output beyond both of these codim 3 points is spurious and both computations end in an MX point (no convergence).

To continue the non-central saddle-node homoclinic orbits it is necessary to work on the data without the solution $\phi(t)$. We restart from the data saved at LAB=8 and LAB=11 in s.7 and s.8 respectively. We could continue these codim 2 points in two ways, either by appending the defining condition $\psi_{16} = 0$ to the continuation of saddle-node homoclinic orbits (with IEQUIB=2, etc.), or by appending $\psi_9 = 0$ to the continuation of a saddle homoclinic orbit (with IEQUIB=1). The first approach is used in the example mtn, for contrast we shall adopt the second approach here.

```
r12=run(r7('UZ1'),c='kpr.12',sv='12')
r12=r12+run(r8('UZ1'),c='kpr.13',ap='12')
```

The projection onto the (ϵ, k)-plane of all four of these codimension-two curves is given in Figure 23.8. The intersection of the inclination-flip lines with one of the non-central saddle-node homoclinic lines is apparent. Note that the two non-central saddle-node homoclinic orbit curves are almost overlaid, but that as in Figure 23.6 the orbits look quite distinct in phase space.

23.5 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION

`mkdir kpr`
`cd kpr`
`demo('kpr')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run('kpr',c='kpr.1',sv='1')`	continuation in the time-length parameter PAR(11) save output-files as b.1, s.1, d.1
`r2=run(r1('UZ1'),c='kpr.2',sv='2')`	locate the homoclinic orbit save output-files as b.2, s.2, d.2
`r3=run(r2('UZ1'),c='kpr.3',sv='3')`	generate adjoint variables save output-files as b.3, s.3, d.3
`r3=r3+run(r3,c='kpr.4',ap='3')`	continue the homoclinic orbit append output-files to b.3, s.3, d.3
`r3=r3+run(r3()[0],c='kpr.5',ap='3')`	continue in reverse direction append output-files to b.3, s.3, d.3
`r6=run(r2('EP1'),c='kpr.6',sv='6')`	increase the period save output-files as b.6, s.6, d.6

Table 23.1: Detailed AUTO-Commands for running demo kpr.

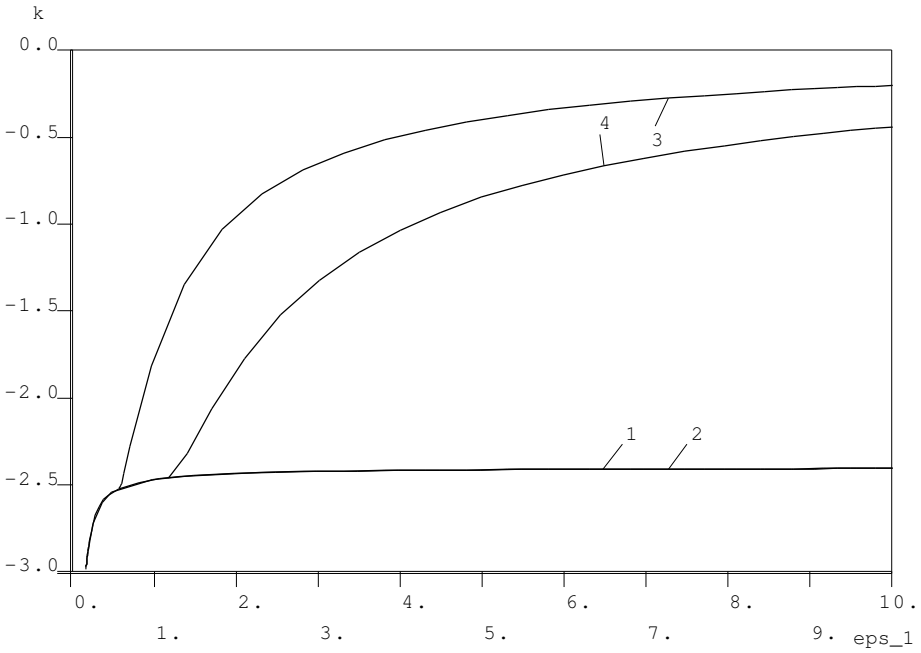


Figure 23.8: Projection onto the (PAR(3),PAR(2))-plane of the non-central saddle-node homoclinic orbit curves (labeled 1 and 2) and the inclination-flip curves (labeled 3 and 4)
.

AUTO-COMMAND	ACTION
`r7=run(r6('UZ1'),c='kpr.7',sv='7')` `sv('7')`	recompute the family of homoclinic orbits save output-files as b.7, s.7, d.7
`r8=run(r7('UZ1'),c='kpr.8',sv='8')`	continue central saddle-node homoclinics save output-files as b.8, s.8, d.8
`r9=run(r8('UZ1'),c='kpr.9',sv='9')`	continue homoclinics from codim-2 point save output-files as b.9, s.9, d.9
`r10=run(r3('UZ1'),c='kpr.10',` `sv='10')`	3-parameter curve of inclination-flips save output-files as b.10, s.10, d.10
`r11=run(r3('UZ2'),c='kpr.11',` `sv='11')`	another curve of inclination-flips save output-files as b.11, s.11, d.11
`r12=run(r7('UZ1'),c='kpr.12',` `sv='12')`	continue non-central saddle-node homoclinics save output-files as b.12, s.12, d.12
`r12=r12+run(r8('UZ1'),c='kpr.13',` `ap='12')`	continue non-central saddle-node homoclinics append output-files to b.12, s.12, d.12

Table 23.2: Detailed AUTO-Commands for running demo kpr.

Chapter 24

HomCont Demo : cir.

24.1 Electronic Circuit of Freire *et al.*

Consider the following model of a three-variable electronic circuit (Freire, Rodríguez-Luis, Gamero & Ponce 1993)

$$\begin{cases} \dot{x} &= [-(\beta+\nu)x + \beta y - a_3 x^3 + b_3(y-x)^3]/r, \\ \dot{y} &= \beta x - (\beta+\gamma)y - z - b_3(y-x)^3, \\ \dot{z} &= y. \end{cases} \tag{24.1}$$

These autonomous equations are also considered in the AUTO demo tor.

First, we copy the demo into a new directory and compile

```
dm('cir')
```

The system is contained in the equation-file cir.f90 and the initial run-time constants are stored in c.cir.1. We begin by starting from the data from cir.dat for a saddle-focus homoclinic orbit at $\nu = -0.721309$, $\beta = 0.6$, $\gamma = 0$, $r = 0.6$, $A_3 = 0.328578$ and $B_3 = 0.933578$, which was obtained by shooting over the time interval $2T =$PAR(11)$= 36.13$. We wish to follow the family in the (β, ν)-plane, but first we perform continuation in (T, ν) to obtain a better approximation to a homoclinic orbit.

```
r1=run('cir',c='cir.1')
```

yields the output

```
 BR  PT  TY LAB     PERIOD       L2-NORM    ...   PAR(1)
  1  21  UZ   2  1.000000E+02  1.286637E-01 ... -7.213093E-01
  1  42  UZ   3  2.000000E+02  9.097899E-02 ... -7.213093E-01
  1  50  EP   4  2.400000E+02  8.305208E-02 ... -7.213093E-01
```

Note that $\nu =$PAR(1) remains constant during the continuation as the parameter values do not change, only the length of the interval over which the approximate homoclinic solution is computed. Note from the eigenvalues, stored in d.1 that this is a homoclinic orbit to a saddle-focus with a one-dimensional unstable manifold.

We now restart at LAB=3, corresponding to a time interval $2T = 200$, and change the principal continuation parameters to be (ν, β). The new constants defining the continuation are given in c.cir.2. We also activate the test functions pertinent to codimension-two singularities which may be encountered along a family of saddle-focus homoclinic orbits, viz. ψ_2, ψ_4, ψ_5, ψ_9 and ψ_{10}. This must be specified in three ways: by appropriate IPSI in c.cir.2, by adding the corresponding parameter labels to the list of continuation parameters ICP(I) in c.cir.2 (recall that these parameter indices are 20 more than the corresponding ψ indices), and finally adding UZR functions defining zeros of these parameters in c.cir.2. Running

```
r2=run(r1('UZ2'),c='cir.2',sv='2')
```

results in

```
BR  PT  TY LAB    PAR(1)      ...    PAR(2)      ...    PAR(25)       PAR(29)
1   17  UZ   5 -7.256925E-01 ...  4.535645E-01 ... -1.765251E-05 -2.888436E-01
1   75  UZ   6 -1.014704E+00 ...  9.998966E-03 ...  1.664509E+00 -5.035997E-03
1   78  UZ   7 -1.026445E+00 ... -2.330391E-05 ...  1.710804E+00  1.165176E-05
1   81  UZ   8 -1.038012E+00 ... -1.000144E-02 ...  1.756690E+00  4.964621E-03
1  100  EP   9 -1.164160E+00 ... -1.087732E-01 ...  2.230329E+00  5.042736E-02
```

with results saved in b.2, s.2, d.2. Upon inspection of the output, note that label 5, where PAR(25)$\approx$ 0, corresponds to a neutrally-divergent saddle-focus, $\psi_5 = 0$. Label 7, where PAR(29)$\approx$ 0 corresponds to a local bifurcation, $\psi_9 = 0$, which we note from the eigenvalues stored in d.2 corresponds to a *Shil'nikov-Hopf* bifurcation. Note that PAR(2) is also approximately zero at label 7, which accords with the analytical observation that the origin of (24.1) undergoes a Hopf bifurcation when $\beta = 0$. Labels 6 and 8 are the user-defined output points, the solutions at which are plotted in Fig. 24.1. Note that solutions beyond label 7 (e.g., the plotted solution at label 8) do not correspond to homoclinic orbits, but to *point-to-cycle* heteroclinic orbits (c.f. Section 2.2.1 of Champneys et al. (1996)).

We now continue in the other direction along the family. It turns out that starting from the initial point in the other direction results in missing a codim 2 point which is close to the starting point. Instead we start from the first saved point from the previous computation (label 5 in s.2):

```
r3=run(r2('UZ1'),c='cir.3',ap='2')
```

The output

```
 BR  PT  TY LAB    PAR(1)      ...    PAR(2)       PAR(22)       PAR(24)
  1   9  UZ  10 -7.204001E-01 ...  5.912315E-01 -1.725669E+00 -3.295862E-05
  1  18  UZ  11 -7.590583E-01 ...  7.428734E-01  3.432139E-05 -2.822988E-01
  1  26  UZ  12 -7.746686E-01 ...  7.746147E-01  5.833163E-01  1.637611E-07
  1  28  EP  13 -7.746628E-01 ...  7.746453E-01  5.908902E-01  1.426214E-04
```

contains a neutral saddle-focus (a *Belyakov* transition) at LAB=10 ($\psi_4 = 0$), a double real leading eigenvalue (saddle-focus to saddle transition) at LAB =11 ($\psi_2 = 0$) and a neutral saddle at LAB=12 ($\psi_4 = 0$). Data at several points on the complete family are plotted in Fig. 24.2. If we had continued further (by increasing NMX), the computation would end at a no convergence

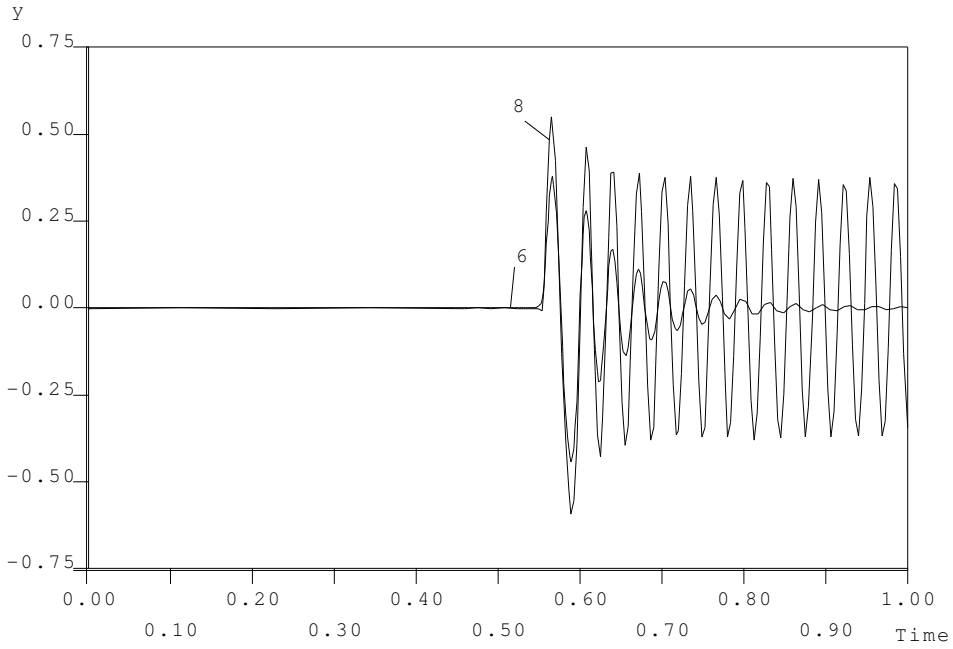


Figure 24.1: Solutions of the boundary value problem at labels 6 and 8, either side of the Shil'nikov-Hopf bifurcation

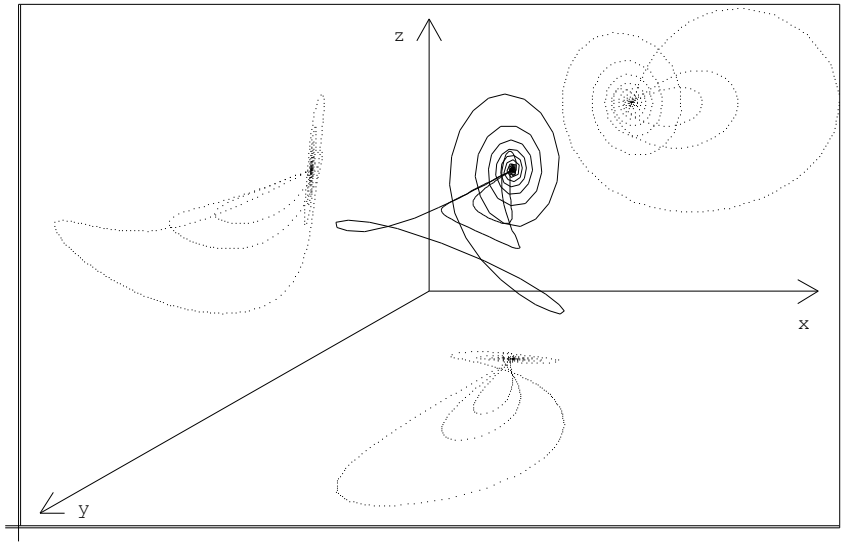


Figure 24.2: Phase portraits of three homoclinic orbits on the family, showing the saddle-focus to saddle transition

error TY=MX owing to the homoclinic family approaching a Bogdanov-Takens singularity at small amplitude. To compute further towards the BT point we would first need to continue to a higher value of PAR(11).

24.2 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION
mkdir cir cd cir demo('cir')	create an empty work directory change directory copy the demo files to the work directory
r1=run(c='cir.1',sv='1')	increase the truncation interval; restart from cir.dat save output-files as b.1, s.1, d.1
r2=run(r1('UZ2'),c='cir.2',sv='2')	continue saddle-focus homoclinic orbit; restart from r1 save output-files as b.2, s.2, d.2
r3=run(r2('UZ1'),c='cir.3',ap='2')	generate adjoint variables ; restart from r2 append output-files as b.2, s.2, d.2

Table 24.1: Detailed AUTO-Commands for running demo cir.

Chapter 25

HomCont Demo : she.

25.1 A Heteroclinic Example.

The following system of five equations Rucklidge & Mathews (1995)

$$\begin{aligned} \dot{x} &= \mu\, x + x\, y - z\, u, \\ \dot{y} &= -y - x^2, \\ \dot{z} &= \mu\, z + x\, u - 9\sigma\, z/4(1+\sigma) \\ \dot{u} &= -\sigma u/4 - \sigma Q v/4\pi^2 + 3(1+\sigma) xz/4\sigma \\ \dot{v} &= \zeta u/4 - \zeta v/4 \end{aligned} \tag{25.1}$$

has been used to describe shearing instabilities in fluid convection. The equations possess a rich structure of local and global bifurcations. Here we shall reproduce a single curve in the (σ, μ)-plane of codimension-one heteroclinic orbits connecting a non-trivial equilibrium to the origin for $Q = 0$ and $\zeta = 4$. The defining problem is contained in equation-file she.f90[1], and starting data for the orbit at $(\sigma, \mu) = (0.5, 0.163875)$ are stored in she.dat, with a truncation interval of `PAR(11)=85.07`.

We begin by computing towards $\mu = 0$ with the option `IEQUIB=-2` which means that both equilibria are solved for as part of the continuation process.

```
demo('she')
r1=run('she',c='she.1',sv='1')
```

This yields the output

```
 BR    PT  TY  LAB    PAR(3)        L2-NORM     ...   PAR(1)
  1     1  EP    1   5.00000E-01   4.05950E-01 ...  1.63875E-01
  1     5        2   4.52847E-01   3.72688E-01 ...  1.36505E-01
  1    10        3   3.94351E-01   3.30390E-01 ...  1.04419E-01
  1    15        4   3.35908E-01   2.87331E-01 ...  7.51623E-02
  1    20        5   2.77287E-01   2.43351E-01 ...  4.95320E-02
  1    25        6   2.18210E-01   1.98147E-01 ...  2.84629E-02
  1    30  EP    7   1.58178E-01   1.51246E-01 ...  1.29327E-02
```

[1]The last parameter used to store the equilibria (`PAR(21)`) is overlaped here with the first test-function. In this example, it is harmless since the test functions are irrelevant for heteroclinic continuation.

Alternatively, for this problem there exists an analytic expression for the two equilibria. This is specified in the subroutine **PVLS** of she.f90. Re-running with `IEQUIB=-1`

```
r2=run('she',c='she.2')
```

we obtain the output

```
 1     1  EP    1   5.00000E-01   4.05950E-01 ...  1.63875E-01
 1     5        2   4.43202E-01   3.65772E-01 ...  1.31056E-01
 1    10        3   3.72309E-01   3.14244E-01 ...  9.30098E-02
 1    15        4   3.00884E-01   2.61156E-01 ...  5.93397E-02
 1    20        5   2.28665E-01   2.06219E-01 ...  3.17994E-02
 1    25        6   1.55541E-01   1.49165E-01 ...  1.23990E-02
 1    30  EP    7   8.10746E-02   9.14311E-02 ...  2.38662E-03
```

This output is similar to that above, but note that it is obtained slightly more efficiently because the extra parameters `PAR(12-21)` representing the coordinates of the equilibria are no longer part of the continuation problem. Also note that AUTO has chosen to take slightly larger steps along the family. Finally, we can continue in the opposite direction along the family from the original starting point (again with `IEQUIB=-1`).

```
r3=run(r2(2),c='she.3')
    save(r2+r3,'2')
```

```
 BR    PT  TY  LAB    PAR(3)        L2-NORM    ...   PAR(1)
  1     5        8   4.99759E-01   4.06015E-01 ...  1.63732E-01
  1    10        9   5.70530E-01   4.55187E-01 ...  2.06526E-01
  1    15       10   6.41644E-01   5.03184E-01 ...  2.50783E-01
  1    20       11   7.13330E-01   5.50067E-01 ...  2.95934E-01
  1    25       12   7.85769E-01   5.95871E-01 ...  3.41549E-01
  1    30       13   8.59097E-01   6.40618E-01 ...  3.87300E-01
  1    35  EP   14   9.33416E-01   6.84317E-01 ...  4.32927E-01
```

The results of both computations are presented in Figure 25.1, from which we see that the orbit shrinks to zero as `PAR(1)`$=\mu \to 0$.

25.2 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION

`mkdir she`
`cd she`
`demo('she')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run('she',c='she.1',sv='1')`	continue heteroclinic orbit; start from she.dat save output-files as b.1, s.1, d.1
`r2=run('she',c='she.2')`	repeat with `IEQUIB=-1`

`r3=run(r2(2),c='she.3')`
 	
`save(r2+r3,'2')`
	continue in reverse direction ;
	restart from label 2 of `r2`
	Save appended results to b.2, s.2, d.2

Table 25.1: Detailed AUTO-Commands for running demo she.

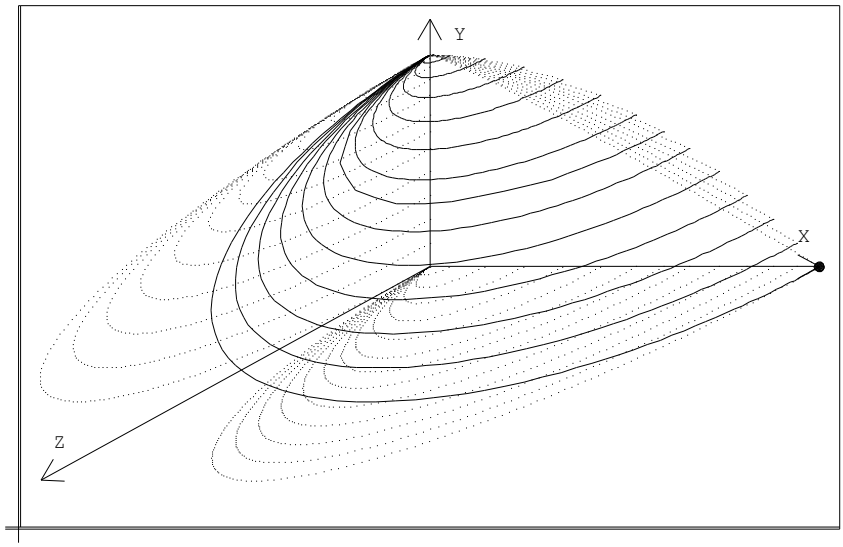


Figure 25.1: Projections into (x, y, z)-space of the family of heteroclinic orbits.

Chapter 26

HomCont Demo : rev.

26.1 A Reversible System.

The fourth-order differential equation

$$u'''' + Pu'' + u - u^3 = 0$$

arises in a number of contexts, e.g., as the travelling-wave equation for a nonlinear-Schrödinger equation with fourth-order dissipation (Buryak & Akhmediev 1995) and as a model of a strut on a symmetric nonlinear elastic foundation (Hunt, Bolt & Thompson 1989). It may be expressed as a system

$$\begin{cases} \dot{u_1} &= u_2 \\ \dot{u_2} &= u_3 \\ \dot{u_3} &= u_4 \\ \dot{u_4} &= -Pu_3 - u_1 + u_1^3 \end{cases} \tag{26.1}$$

Note that (26.1) is invariant under two separate reversibilities

$$R_1 : (u_1, u_2, u_3, u_4, t) \mapsto (u_1, -u_2, u_3, -u_4, -t) \tag{26.2}$$

and

$$R_2 : (u_1, u_2, u_3, u_4, t) \mapsto (-u_1, u_2, -u_3, u_4, -t) \tag{26.3}$$

First, we copy the demo into a new directory

```
demo('rev')
```

For this example, we shall make two separate starts from data stored in equation and data files rev.c.1, rev.dat.1 and rev.c.3, rev.dat.3 respectively. The first of these contains initial data for a solution that is reversible under R_1 and the second for data that is reversible under R_2.

26.2 An R_1-Reversible Homoclinic Solution.

The first run

```
r1=run('rev',c='rev.1',sv='1')
```

starts by using the file rev.dat.1 via the 'dat' AUTO-constant in c.rev.1. The orbit contained in the data file is a "primary" homoclinic solution for $P = 1.6$, with truncation (half-)interval PAR(11) = 39.0448429. which is reversible under R_1. Note that this reversibility is specified in c.rev.1 via IREV=[0,1,0,1]. Note also, from c.rev.1 that we only have one free parameter PAR(1) because symmetric homoclinic orbits in reversible systems are generic rather than of codimension one. The first run results in the output

```
 BR    PT  TY  LAB    PAR(1)         L2-NORM        MAX U(1)    ...
  1     1  EP    1   1.60000E+00   2.85704E-01   3.62232E-01
  1     8  UZ    2   1.70000E+00   2.90288E-01   4.18225E-01
  1    11  UZ    3   1.80000E+00   2.95723E-01   4.80604E-01
  1    14  UZ    4   1.90000E+00   2.74864E-01   4.43069E-01
  1    20  EP    5   1.99678E+00   1.13379E-01   9.59430E-02
```

which is consistent with the theoretical result that the solution tends uniformly to zero as $P \to 0$. Note, by plotting the data saved in s.1 that only "half" of the homoclinic orbit is computed up to its point of symmetry. See Figure 26.1.

The second run continues in the other direction of PAR(1), with the test function ψ_2 activated for the detection of saddle to saddle-focus transition points

```
r1=r1+run(r1('UZ1'),c='rev.2',ap='1')
```

The output

```
 BR  PT  TY  LAB    PAR(1)         L2-NORM        MAX U(1)    ...    PAR(22)
  1  11  UZ    6   1.00001E+00   2.81700E-01   1.76625E-01 ... -3.00001E+00
  1  22  UZ    7  -1.00743E-07   2.89421E-01   4.69706E-02 ... -2.00000E+00
  1  33  UZ    8  -1.00000E+00   3.02208E-01   4.32654E-03 ... -1.00000E+00
  1  44  UZ    9  -2.00000E+00   3.16798E-01   1.22616E-11 ...  2.66362E-08
  1  55  EP   10  -3.09920E+00   3.32927E-01  -4.00188E-10 ...  1.09920E+00
```

shows a saddle to saddle-focus transition (indicated by a zero of PAR(22)) at PAR(1)=-2. Beyond that label the first component of the solution is negative and (up to the point of symmetry) monotone decreasing. See Figure 26.2.

26.3 An R_2-Reversible Homoclinic Solution.

```
r3=run('rev',c='rev.3',sv='3')
```

starts by using the file rev.dat.3 via the 'dat' AUTO-constant in c.rev.3, and runs them with the constants stored in c.rev.3. The orbit contained in the data file is a "multi-pulse" homoclinic solution for $P = 1.6$, with truncation (half-)interval PAR(11) = 47.4464189. which is reversible under R_2. This reversibility is specified in c.rev.1 via IREV=[1,0,1,0]. The output

```
 BR    PT  TY  LAB    PAR(1)         L2-NORM        MAX U(1)    ...
  1     1  EP    1   1.60000E+00   3.69766E-01   3.83942E-01
  1     7  UZ    2   1.70000E+00   3.83640E-01   4.89066E-01
```

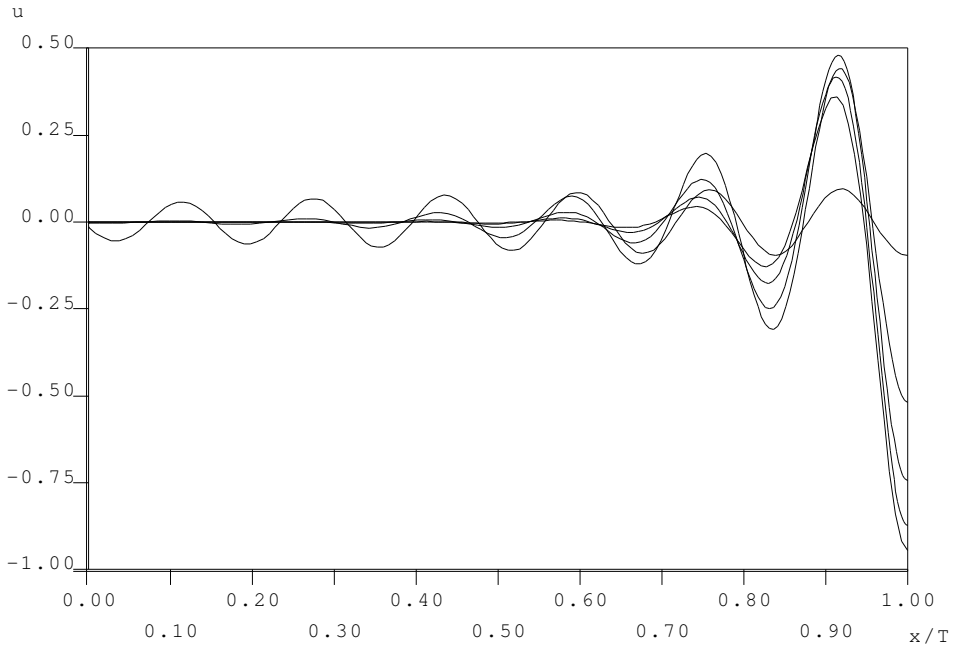


Figure 26.1: R_1-Reversible homoclinic solutions on the half-interval $x/T \in [0,1]$ where $T = 39.0448429$ for P approaching 2 (solutions with labels `1-5` respectively have decreasing amplitude)

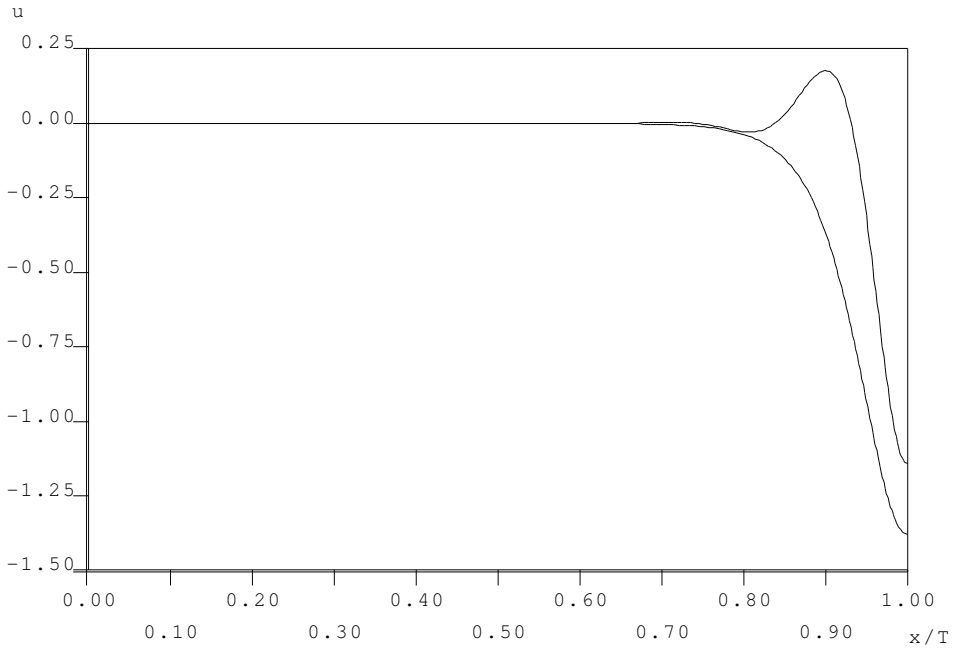


Figure 26.2: R_1-reversible homoclinic orbits with oscillatory decay as $x \to -\infty$ (corresponding to label `6`) and monotone decay (at label `10`)

```
 1     9  LP    3   1.71157E+00   3.92475E-01   5.46080E-01
 1    11  UZ    4   1.69884E+00   4.04207E-01   6.10428E-01
 1    14  UZ    5   1.60000E+00   4.32940E-01   7.77395E-01
 1    26  UZ    6   1.00000E+00   4.80849E-01   1.08252E+00
 1    49  UZ    7  -5.38706E-08   5.15846E-01   1.25863E+00
 1   128  MX    8  -9.15462E-01   5.44202E-01   1.32395E+00
```

contains the label of a limit point (`ILP` was set to `1` in c.rev.3), which corresponds to a "coalescence" of two reversible homoclinic orbits. The two solutions on either side of this limit point are displayed in Figure 26.3. The computation ends in a no-convergence point. The solution here is depicted in Figure 26.4. The lack of convergence is due to the large peak and trough of the solution rapidly moving to the left as $P \to -2$ (cf. Champneys & Spence (1993)).

Continuing from the initial solution in the other parameter direction

```
r3=r3+run(r3('UZ1'),c='rev.4',ap='3')
```

we obtain the output

```
 BR    PT  TY  LAB    PAR(1)        L2-NORM       MAX U(1)    ...
  1     7  UZ    9   1.60000E+00   3.70171E-01   3.84045E-01
  1    33  UZ   10   9.99998E-01   3.61440E-01   1.77504E-01
  1    94  UZ   11  -5.14775E-08   3.71301E-01   4.69706E-02
  1   153  MX   12  -2.54464E-01   3.75071E-01   3.00627E-02
```

which again ends at a no convergence error for similar reasons.

26.4 Detailed AUTO-Commands.

AUTO-COMMAND	ACTION

`mkdir rev`
`cd rev`
`demo('rev')`
	create an empty work directory
	change directory
	copy the demo files to the work directory

`r1=run('rev',c='rev.1',sv='1')`	use the starting data in rev.dat.1 increase `PAR(1)` save output-files as b.1, s.1, d.1
`r1=r1+run(r1('UZ1'),c='rev.2',ap='1')`	continue in reverse direction; restart: 1st `UZ` append output-files to b.1, s.1, d.1
`r3=run('rev',c='rev.3',sv='3')`	use the starting data in rev.dat.3 restart with different reversibility save output-files as b.3, s.3, d.3
`r3=r3+run(r3('UZ1'),c='rev.4',ap='3')`	continue in reverse direction; restart: 1st `UZ` append output-files to b.3, s.3, d.3

Table 26.1: Detailed AUTO-Commands for running demo rev.

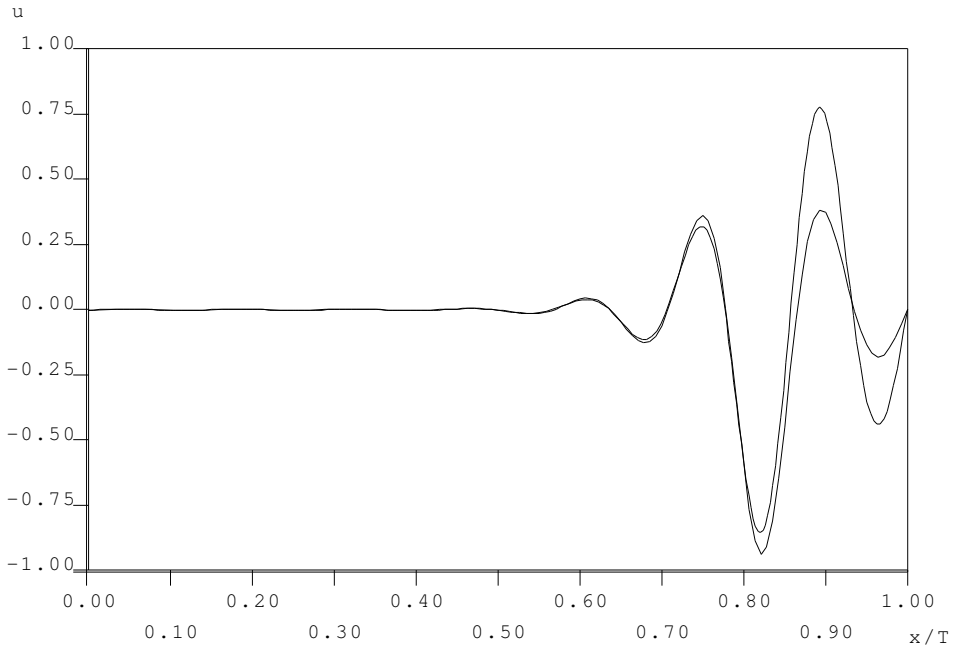


Figure 26.3: Two R_2-reversible homoclinic orbits at $P = 1.6$ corresponding to labels `1` (smaller amplitude) and `5` (larger amplitude)

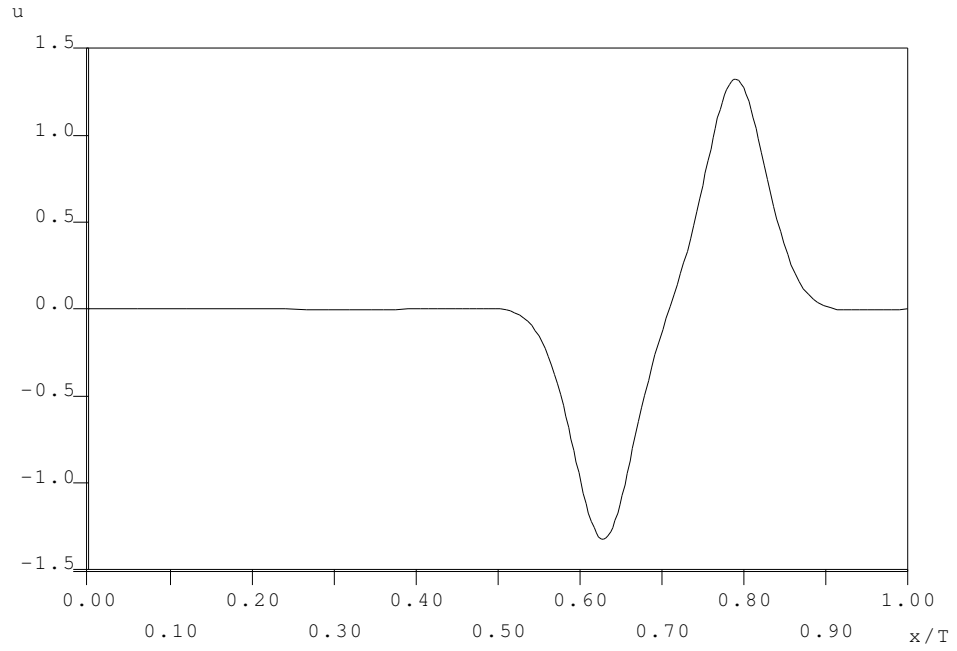


Figure 26.4: An R_2-reversible homoclinic orbit at label `8`

Chapter 27

HomCont Demo : Homoclinic branch switching.

This demo illustrates homoclinic branch switching, which is an implementation of Lin's method (Lin 1990, Sandstede 1993, Yew 2001) as described in Oldeman et al. (2003). We use a direct branch switching method to switch from 1- to 2- and 3-homoclinic orbits near an inclination flip bifurcation in a model due to Sandstede, which was introduced in Chapter 21. This also shows how to obtain a homoclinic orbit through continuation of a periodic orbit born at a Hopf bifurcation. Thereafter, we illustrate homoclinic branch switching for the FitzHugh-Nagumo equations and a 5th-order Korteweg-De Vries model.

The equation files in these demos are written in C.

27.1 Branch switching at an inclination flip in Sandstede's model.

Consider the system (Sandstede 1995*a*)

$$\begin{aligned} \dot{x} &= ax + by - ax^2 - \alpha zx(2-3x), \\ \dot{y} &= bx + ay - \tfrac{3}{2}x(bx+ay) + \alpha z 2y, \\ \dot{z} &= cz + \mu x + 3xz + \alpha(x^2(1-x) - y^2). \end{aligned} \tag{27.1}$$

as given in the file sib.c, where for simplicity we have set $\tilde{\mu} = 0$, $\beta = 1$ and $\gamma = 3$.

We study an inclination flip that exists for $a = 0.375$, $b = 0.625$ and $c = -0.75$. This corresponds to the situation where the eigenvalues of the equilibrium at the origin are $a+b = 1$, $a - b = -0.25$ and $c = -0.75$. Hence, the corresponding bifurcation diagram consists of a complicated structure involving a fan of infinitely many n-periodic and n-homoclinic orbits for arbitrary n and a region with horseshoe dynamics; see also Homburg & Krauskopf (2000) and the references therein.

This computation starts from an equilibrium at $(2/3, 0, 0)$, which exists for $a = \mu = \alpha = 0$. Also, b is set to 0.625 (the value we would like it to be) and c is set to -2.5 in **stpnt**. Choosing $c = -2$ at this stage leads to convergence problems. This equilibrium is not the one corresponding to the homoclinic orbit, but it is an equilibrium with complex eigenvalues, that

we can follow until it reaches a Hopf bifurcation. A periodic orbit emanates from this Hopf bifurcation and can be followed to the homoclinic orbit. However, first we need to change a from 0 to 0.375.

All the following commands, except for `demo('sib')` are contained within the file `'sib.auto'` which you can either execute in a batch mode by entering

```
> auto sib.auto
```

or step by step using

```
AUTO> demofile('sib.auto').
```

We start by copying the demo to the current work directory and running the first step

```
demo('sib')
r1=run(e='sib',c='sib')
save(r1,'1')
```

The equilibrium is followed in a until a (or `PAR(1)`) is at our desired value, 0.375.

```
 BR    PT  TY  LAB    PAR(1)      ...    U(1)          U(2)          U(3)
  1     1  EP    1   0.00000E+00 ... 6.66667E-01   0.00000E+00   0.00000E+00
  1     5  UZ    2   3.75000E-01 ... 6.66667E-01  -1.33333E-01   0.00000E+00
```

The output is saved in the files `b.1`, `s.1` and `d.1`. Next we continue in α (`PAR(4)`) until a Hopf bifurcation is found:

```
r2=run(r1,ICP=[4])
save(r2,'2')
```

or, alternatively,

```
rn(c='sib.2',s='1')
sv('2')
```

```
 BR    PT  TY  LAB    PAR(4)      ...    U(1)          U(2)          U(3)
  1     6  HB    3   3.18429E-01 ... 6.54375E-01  -1.34754E-01   7.70102E-02
```

The output is saved in the files `b.2`, `s.2` and `d.2`. This Hopf bifurcation can then be continued into a periodic orbit. The periodic orbit eventually reaches a homoclinic bifurcation. We continue in μ=`PAR(5)` and `PAR(11)`, which corresponds to the period, and stop when the period is equal to 35.

```
r3=run(r2('HB1'),IPS=2,ICP=[5,11],NMX=200,DS=0.01,DSMAX=0.01,UZR={-11:35})
save(r3,'3')
```

```
 BR    PT  TY  LAB    PAR(5)        L2-NORM     ...    PERIOD
  3    10        5  -2.41881E-03   6.70569E-01 ...  1.08975E+01
                                                ...
  3    40        8  -1.29495E-02   6.14547E-01 ...  1.41297E+01
                                                ...
  3    81  UZ   13  -1.04657E-04   4.01829E-01 ...  3.50000E+01
```

The output is saved in the files b.3, s.3 and d.3. Note that μ first decreases and then increases towards 0, which is precisely what we expect in this model, as homoclinic orbits occur on the line $\mu = 0$ in the (α, μ)-plane. It is now instructive to look at a phase space diagram to see what is going on.

```
plot(r3)
```

Selecting 'solution' for Type, [5,6,7,8,9,10,11,12,13] for Label, [U(1)] for X and [U(2)] for Y, we obtain the diagram depicted in Figure 27.1(a), where the periodic orbit grows from the Hopf equilibrium to a homoclinic orbit.

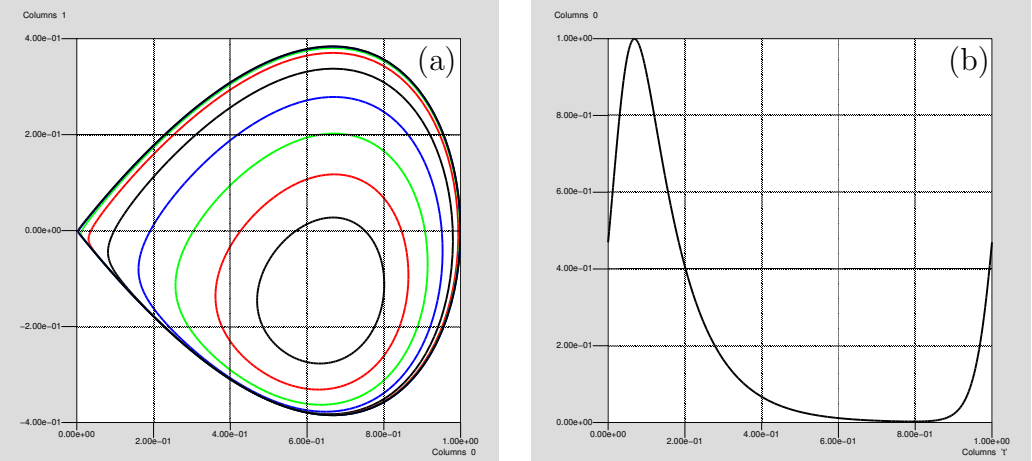


Figure 27.1: Periodic orbit growing from a Hopf bifurcation to a homoclinic orbit (a). The unshifted homoclinic orbit (b).

Note however, that the homoclinic orbit has the wrong left-hand and right-hand end points. This can be seen by plotting the solution corresponding to Label [13] using 't' vs. 'x' (coordinate [U(1)]), as depicted in Figure 27.1(b).

Hence, in order to continue this as a real homoclinic we have to give HomCont special instructions, to do a phase-shift in time. This can be done by setting `ISTART=4`. Moreover, since we have not specified the value of the equilibrium at the origin in sib.c, we need to set `IEQUIB=1` (this is the default value) to let HomCont detect the equilibrium. Note that in this case this is not strictly necessary; however, we do this for instructional purposes.

Now we use HomCont to continue the homoclinic orbit in c and μ (`PAR(3)`, `PAR(5)`), to get the desired value $c = -2.0$.

```
r4=run(r3,IPS=9,ICP=[3,5],NPR=60,JAC=1,UZR={-3:-2.0},ISTART=4)
save(r4,'4')
```

```
 BR    PT  TY  LAB    PAR(3)         L2-NORM       ...    PAR(5)
  3    51  UZ   14  -2.00000E+00   4.01890E-01    ...  2.66146E-09
```

The output is saved in the files b.4, s.4 and d.4. Note that PAR(5)=μ remains zero, which is exactly what we expect.

Next we want to add a solution to the adjoint equation to this solution. This is achieved by making the change ITWIST = 1. Also, we set ISTART to 1 to tell HomCont that it should not try to shift the orbit anymore.

```
r5=run(r4,ICP=[5,8],NMX=2,ITWIST=1,ISTART=1)
save(r5,'5')
```

or, alternatively,

```
rn(c='sib.5',s='4')
sv('5')
```

The output is stored in b.5, s.5 and d.5.

```
 BR    PT  TY  LAB    PAR(5)         L2-NORM     ...     PAR(8)
  3     2  EP   15   2.66146E-09   4.01890E-01 ...   1.00000E-02
```

Here PAR(8) is a dummy (unused) parameter and μ just stays where it is. Now that we have obtained the solution of the adjoint equation, we are able to detect inclination flips. This can be achieved by setting IPSI to [13] and monitoring PAR(33).

```
r6=run(r5,ICP=[4,5,33],NMX=30,DS=-0.01,DSMAX=1.0,UZR={33:0,-4:0},IPSI=[13])

 BR    PT  TY  LAB    PAR(4)         L2-NORM     ...  PAR(5)         PAR(33)
  3    19  UZ   16   7.11774E-02   4.01890E-01 ... 1.24376E-11  -2.36702E-07
```

The output is stored in the Python variable r6. Hence an inclination flip was found at $\alpha = 0.711774$.

Now we are ready to perform homoclinic branch switching, using the techniques described in (Oldeman et al. 2003). Our first aim is to find a 2-homoclinic orbit. The ingredients we need are: a homoclinic orbit where n-homoclinic orbits are close by, and the solution to the adjoint equation to obtain the Lin vector. Since both ingredients are there, we can now continue in μ, ε_1 and T_1, to obtain the initial Lin gap. Recall from Chapter 20 that the Lin gaps ε_i correspond to PAR(20+i*2) and the time intervals T_i correspond to PAR(21+i*2). We stop when $\varepsilon_1 = 0.2$. We need to specify ITWIST=2, to tell HomCont we aim to find a 2-homoclinic orbit, so that it will split it up in three parts with two potential Lin gaps. We effectively have a 9-dimensional system at this point.

```
r7=run(r6('UZ1'),ICP=[21,22,5],NMX=300,NPR=10,UZR={-22:0.2},ISTART=-2,IPSI=[])
save(r7,'7')
```

```
 BR    PT  TY  LAB    PAR(21)        L2-NORM     ...  PAR(22)        PAR(5)
  3    10       18   3.45897E+01   4.46818E-01 ... 7.87712E-07  -1.55885E-11
  3    20       19   2.73699E+01   4.46818E-01 ... 2.91119E-05  -1.63974E-09
  3    30       20   1.73720E+01   4.46817E-01 ... 4.42273E-03  -3.10167E-05
  3    38  UZ   21   1.01451E+01   4.46796E-01 ... 2.00000E-01  -1.48615E-02
```

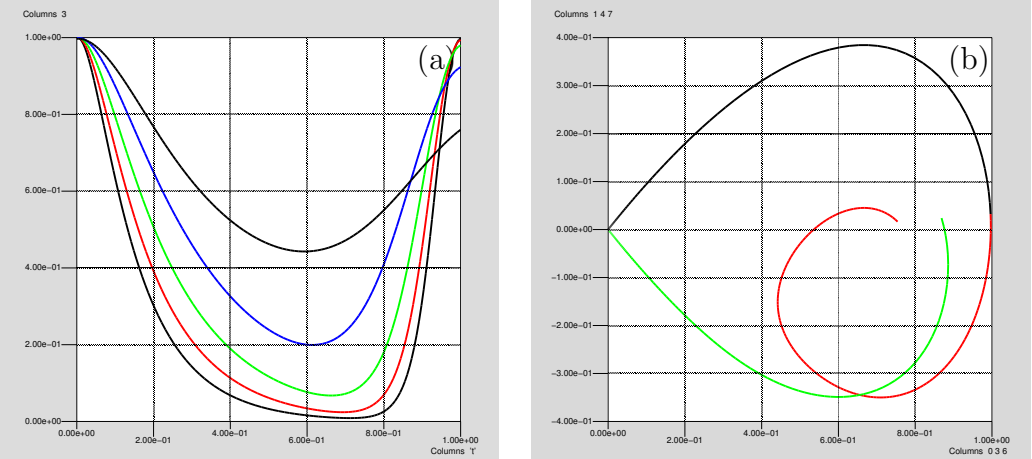


Figure 27.2: Behaviour of the second piece of the 'broken homoclinic orbit' when creating a Lin gap (a). Projection of the "broken homoclinic orbit" onto the (x, y)-plane, where $\varepsilon_1 = 0.2$. To include all the pieces necessary to obtain this figure, the "X" box must contain [U(1),U(4),U(7)] and the "Y" box must contain [U(2),U(5),U(8)] (b).

The output is stored in b.7, s.7 and d.7. Here we see that T_1, the time it takes to make the first loop with respect to the Poincaré section, decreases. This is illustrated in Figure 27.2. Next we are ready to close this gap, by continuing in α, μ, and ε_1, while keeping T_1 at a constant value.

```
r8=run(r7,ICP=[4,5,22],NPR=310,DS=0.01,DSMAX=0.01,UZR={-22:0.0,4:0.074})
                                        r6=r6+r8
```

```
 BR     PT  TY  LAB     PAR(4)          L2-NORM      ...    PAR(5)          PAR(22)
  3      3  UZ   22   7.40000E-02    4.46781E-01  ... -1.43162E-02    1.93746E-01
  3     32  UZ   23   1.98414E-01    4.46590E-01  ... -6.05495E-03    2.29300E-06
```

The output is appended to the Python variable r6. Now we have obtained a 2-homoclinic orbit at label 23. However, the homoclinic orbit is still split in three parts. We can switch back to a normal orbit by setting ITWIST back to 0 and continuing in the usual way. Here we continue back to the inclination flip point in α and μ.

```
r9=run(r8,ICP=[4,5],NMX=30,DS='-',DSMAX=0.1,UZR={4:0.15},ISTART=1,ITWIST=0)
                                        r6=r6+r9
```

```
 BR     PT  TY  LAB     PAR(4)          L2-NORM      ...    PAR(5)
  3      7  UZ   24   1.50000E-01    4.94490E-01  ... -3.60248E-03
  3     30  EP   25   7.61403E-02    4.98746E-01  ... -2.64847E-06
```

So the 2-homoclinic orbit converges back to the 1-homoclinic orbit at the inclination flip bifurcation. The output is appended to the python variable r6. The resulting 2-homoclinic orbits can be seen using

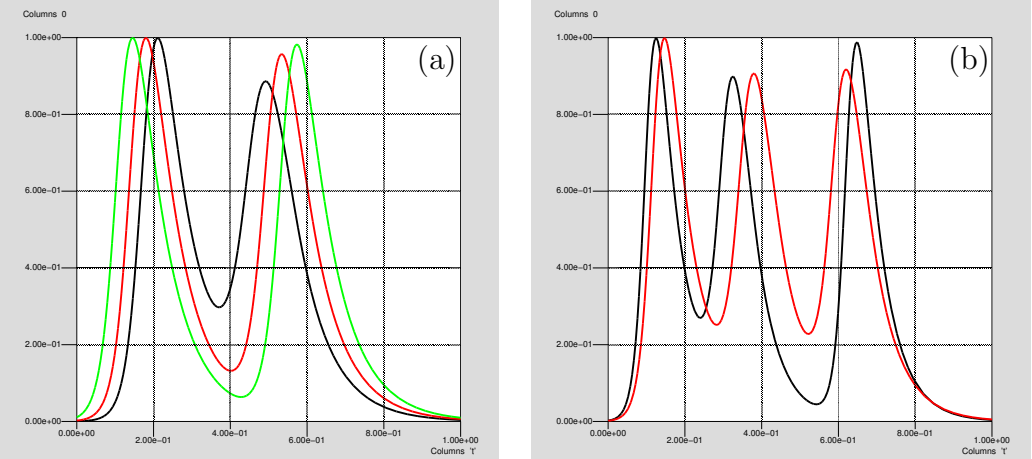


Figure 27.3: The 2-homoclinic orbit as a is changed (a). The two different 3-homoclinic orbits (b).

```
plot(r6)
```

and is depicted in Figure 27.3(a).

Next, we aim to find a 3-homoclinic orbit. To do so, we restart at the inclination flip point at label 16 and set `ITWIST=3`. Moreover, we need to continue in one more gap, ε_2=`PAR(24)` and, once again, stop when ε_1=`PAR(22)=0`. Note that the dimension of the boundary value problem we continue is now equal to 12. This is not to be confused with the setting of the AUTO constant `NDIM=3`, because HomCont handles this internally.

```
r10=run(r6('UZ1'), ICP=[21,22,24,5], NMX=300, NPR=10, UZR={-22:0.2}, ISTART=-3,
                                    IPSI=[])
                                save(r10,'10')
```

```
  BR     PT  TY  LAB     PAR(21)     ...    PAR(22)        PAR(24)        PAR(5)
   3     10       26   3.45896E+01 ...   7.87894E-07    6.42157E-07  -1.06346E-11
   3     20       27   2.73699E+01 ...   2.91126E-05    6.51591E-07  -1.63655E-09
   3     30       28   1.73719E+01 ...   4.42289E-03    1.44090E-04  -3.10188E-05
   3     38  UZ   29   1.01451E+01 ...   2.00000E-01    6.97445E-02  -1.48615E-02
```

The output is stored in `b.10`, `s.10` and `d.10`. Now we need to subsequently close the Lin gaps. Our strategy is to keep T_1 fixed. We first continue in α, μ, ε_1 and ε_2 until $\varepsilon_1 = 0$.

```
r11=run(r10,ICP=[4,5,22,24],NPR=310,DS=0.01,DSMAX=0.01,UZR={-22:0.0,4:0.082})
                                r6=r6+r11
```

```
  BR     PT  TY  LAB     PAR(4)      ...    PAR(5)        PAR(22)        PAR(24)
   3      6  UZ   30   8.20000E-02 ... -1.29790E-02    1.76995E-01    6.37184E-02
   3     32  UZ   31   1.98414E-01 ... -6.05495E-03    2.30717E-06    3.62449E-02
```

The output is appended to the Python variable r6. Note that this continuation is very similar to the one where we found a 2-homoclinic orbit. In fact we have now found a 2-homoclinic orbit (numerically) followed by a 'broken' 1-homoclinic orbit; only the mesh is not aligned.

The next step is to close the gap corresponding to ε_2 to obtain a 3-homoclinic orbit. We replace the continuation parameter ε_1 by T_2, because T_2 (PAR(23)) still has to be decreased from its high value (35) and ε_1 needs to stay at 0.

```
r12=run(r11,ICP=[4,5,23,24],NMX=32,NTST=40,DS=-1,DSMAX=1, UZR={24:0,4:0.18})
                                  r6=r6+r12

 BR    PT  TY  LAB    PAR(4)      ...   PAR(5)         PAR(23)        PAR(24)
  3    16  UZ   32   1.98395E-01 ... -6.05536E-03   2.01311E+01   1.82491E-08
  3    24  UZ   33   1.80000E-01 ... -6.50293E-03   1.27554E+01  -3.14294E-02
  3    30  UZ   34   1.66990E-01 ... -6.89269E-03   9.41745E+00  -1.03179E-06
  3    32  EP   35   1.78172E-01 ... -6.55364E-03   9.50300E+00  -7.20367E-02
```

The output is appended to the Python variable r6. Note that we have found two zeros of PAR(24), at labels 32 and 34, respectively. The two zeros correspond to two different 3-homoclinic orbits, which, when followed from periodic orbits, both emanate from from the same saddle-node bifurcation. These two 3-homoclinic orbits are depicted in Figure 27.3(b). We can follow both of these back to the inclination flip point, by setting ITWIST back to 0:

```
   r13=run(r6('UZ7'), ICP=[4,5], NMX=30, DS=-0.01, DSMAX=0.1, UZR={4:0.13},
                              ISTART=1, ITWIST=0)
                                   r6=r6+r13

 BR    PT  TY  LAB    PAR(4)        L2-NORM     ...   PAR(5)
  3    13  UZ   36   1.29999E-01   5.04807E-01 ... -2.33902E-03
  3    30  EP   37   9.27258E-02   5.06560E-01 ... -2.76788E-04

  r14=run(r6('UZ9'), ICP=[4,5], NMX=30, DS=-0.01, DSMAX=0.1, UZR={4:0.145},
                               ISTART=1, ITWIST=0)
                             r6=r6+r14 save(r6,'6')

 BR    PT  TY  LAB    PAR(4)        L2-NORM     ...   PAR(5)
  3     7  UZ   38   1.45000E-01   5.47347E-01 ... -4.79400E-03
  3    30  EP   39   8.39399E-02   5.52605E-01 ... -7.36611E-05
```

All the combined appended output is saved to b.6, s.6 and d.6. The bifurcation diagram and the paths we followed when closing the Lin gaps are depicted in Figure 27.4. It is possible and straightforward to obtain $4, 5, 6, \ldots$-homoclinic orbits by extending the above strategy.

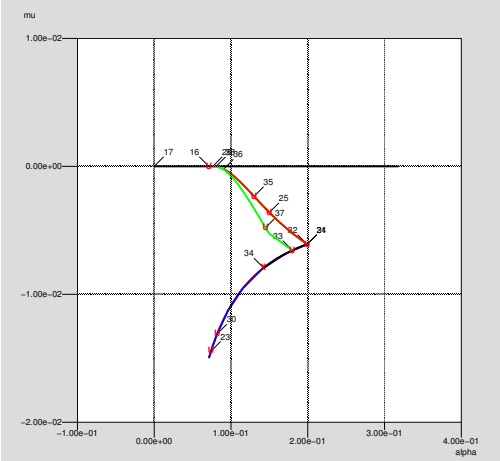


Figure 27.4: Parameter space diagram near an inclination flip. The curve through label 17 corresponds to a 1-homoclinic orbit. The opening of the Lin gaps occurs along the vertical line from label 16 to label 23. The curves through labels 23 and 30 denote the path that is followed when closing the Lin gaps. The (approximately overlaid) curves though labels 25 and 35 correspond to the 2- and one of the 3-homoclinic orbits. Finally, the curve through label 37 corresponds to the other 3-homoclinic orbit, which was obtained for `PAR(23)`$=T_2 = 12.03201$.

27.2 Branch switching for a Shil'nikov type homoclinic orbit in the FitzHugh-Nagumo equations.

The FitzHugh-Nagumo (FHN) equations (FitzHugh 1961, Nagumo, Arimoto & Yoshizawa 1962) are a simplified version of the Hodgkin-Huxley equations (Hodgkin & Huxley 1952). They model nerve axon dynamics and are given by

$$\begin{aligned} u_t &= u_{xx} - f_a(u) - w, \\ w_t &= \epsilon(u - \gamma w), \end{aligned} \tag{27.2}$$

where

$$f_a(u) = u(u-a)(u-1).$$

Travelling wave solutions of the form $(u,w)(x,t) = (u,w)(\xi)$, where $\xi = x + ct$ are solutions of the following ODE system:

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= cv + f_a(u) + w, \\ \dot{w} &= \frac{\epsilon}{c}(u - \gamma w). \end{aligned} \tag{27.3}$$

In particular we consider solitary wave solutions of (27.2). These correspond to orbits homoclinic to $(u,v,w) = 0$ in system (27.3). In our numerical example we keep $\gamma = 0$.

We aim to find a 2-homoclinic orbit at a Shil'nikov bifurcation. All the commands given here are in the file fnb.auto. First we obtain a homoclinic orbit using a homotopy technique (see Friedman, Doedel & Monteiro (1994)), using `ISTART=3`, for the parameter values $c = 0.21, a = 0.2, \epsilon = 0.0025$.

```
demo('fnb')
r1 = run('fhn',sv='1')
```

Among the output we see:

```
BR    PT  TY  LAB     PERIOD       L2-NORM    ...   PAR(17)
 1    21  UZ    4   2.91921E+01   2.38053E-01 ...  2.37630E-11
```

and a zero of `PAR(17)` means that a zero of an artificial parameter has been located and the right-hand end point of the corresponding solution belongs to the plane that is tangent to the stable manifold at the saddle. This point still needs to come closer to the equilibrium, which we can achieve by further increasing the period to 300, while keeping `PAR(17)` at 0:

```
r2 = run(r1('UZ1'),c='fhn.2',sv='2')
```

```
BR    PT  TY  LAB     PERIOD       L2-NORM    ...   PAR(2)
 1   189  UZ   11   3.00000E+02   7.37932E-02 ...  1.79286E-01
```

Next we stop using the homotopy technique and increase the period even further, to 1000.

```
r3 = run(r2('UZ1'),c='fhn.3',sv='3')
```

```
BR    PT  TY  LAB     PERIOD       L2-NORM    ...   PAR(2)
 1    80  UZ   14   1.00000E+03   4.04183E-02 ...  1.79286E-01
```

A continuation in `PAR(2)`=a and `PAR(1)`=c needs to be performed to arrive at the place where we wish to find a 2-homoclinic orbit: $a = 0$. At the same time we monitor `PAR(22)` to locate Belyakov points.

```
r4 = run(r3('UZ1'),c='fhn.4',sv='4')
```

```
BR    PT  TY  LAB    PAR(2)        L2-NORM    ...   PAR(1)        PAR(22)
 1     6  UZ   16   1.31812E-01   3.28710E-02 ...  2.17166E-01  -6.31253E-06
 1    23  UZ   20  -8.55398E-08   1.56158E-02 ...  2.74218E-01  -9.88772E-02
```

Hence, there exists a Belyakov point at $(a, c) = (0.131812, 0.21766)$. At label 19 we have a lower value of a than at the Belyakov point, and by inspection of the file `d.4` we can observe that the equilibrium has one positive eigenvalue and a complex conjugate pair of eigenvalues with negative real part, and conclude that this orbit is of Shil'nikov type. Before starting the homoclinic branch switching, we calculate the adjoint to obtain a 'Lin vector':

```
r5 = run(r4('UZ5'),c='fhn.5',sv='5')
```

```
  BR    PT  TY  LAB    PAR(9)         L2-NORM     ...   PAR(3)
   1     2  EP   29  -1.00000E+00   1.56158E-02 ...  2.50000E-03
```

Next, we continue in the time T_1 (`PAR(21)`), the gap ε_1 (`PAR(22)`) and c (`PAR(1)`), and by setting `ISTART`=-2 we try to locate a 2-homoclinic orbit:

```
r6 = run(r5,c='fhn.6',sv='6')
```

In fact we find many of them, exactly as is predicted by the theory:

```
  BR    PT  TY  LAB    PAR(21)     ...   PAR(1)         PAR(22)
...
   1   174  UZ   45   1.64799E+02 ...  2.74218E-01  -3.44422E-11
   1   178  UZ   46   1.44799E+02 ...  2.74218E-01   3.29142E-14
   1   182  UZ   47   1.24854E+02 ...  2.74218E-01   1.70138E-15
   1   187  UZ   48   1.04789E+02 ...  2.74218E-01  -8.57896E-14
   1   191  UZ   49   8.49517E+01 ...  2.74218E-01   1.93804E-13
   1   196  UZ   50   6.45145E+01 ...  2.74218E-01  -2.26551E-09
```

Each of these homoclinic orbits differ by about 20 in the value T_1. This is about the time it takes to make one half-turn close to and around the equilibrium, so that orbits differ by the number of half turns around the equilibrium before a big excursion in phase space. Note that the variation of c is so small that it does not appear.

A plot of T_1 vs. ε_1 gives insight into how the gap is opened and closed in the continuation process. This is depicted in Figure 27.5. We are now in a position to continue each of these

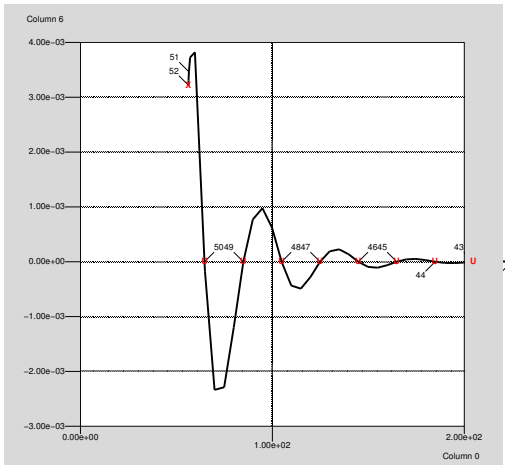


Figure 27.5: A plot of ε_1 as a function of T_1 during our computation of Shil'nikov-type two-homoclinic orbits. Each zero corresponds to a different orbit.

orbits as a normal homoclinic orbit by setting `ISTART=1` and `ITWIST=0`. We leave this as an exercise to the reader.

27.3 Branch switching to a 3-homoclinic orbit in a 5th-order Korteweg-De Vries model

In Champneys & Groves (1997) the following water wave model was considered:

$$\frac{2}{15}r'''' - br'' + ar + \frac{3}{2}r^2 - \frac{1}{2}(r')^2 + [rr']' = 0. \tag{27.4}$$

It represents solitary-wave solutions $r(x+at)$, $r \to 0$ as $x \to \pm\infty$ of the 5th-order PDE

$$r_t + \frac{2}{15}r_{xxxx} - br_{xxx} + 3rr_x + 2r_xr_{xx} + rr_{xxx=0},$$

where a is the wave speed. The ODE corresponds to a Hamiltonian system with Hamiltonian

$$H = -\frac{1}{2}q_1^3 - \frac{1}{2}aq_1^2 + p_1q_2 - \frac{1}{2}bq_2^2 + \frac{15}{4}p_2^2 + \frac{1}{2}q_2^2q_1$$

and

$$q_1 = r, \quad q_2 = r', \quad p_1 = -\frac{2}{15}r''' + br' - rr', \quad p_2 = \frac{2}{15}r''.$$

System (27.4) is also reversible under the transformation

$$t \mapsto -t, (q_1, q_2, p_1, p_2) \mapsto (q_1, -q_2, -p_1, p_2),$$

but we do not exploit the reversible structure (`IREV=0`), and instead use it as an example of Hamiltonian system. This system exhibits an orbit flip for a reversible Hamiltonian system. In Hamiltonian systems, homoclinic orbits are codimension-zero phenomena, and we have to add an additional parameter λ that breaks the Hamiltonian structure in this system, by introducing artificial friction. Thus, the actual system of equations that is used for continuation is

$$\dot{x} = (\lambda I + J)\nabla H(x),$$

where $x = (q_1, q_2, p_1, p_2)$ and J is the usual skew symmetric matrix in $\mathbb{R}^4$. It is now possible to continue a homoclinic orbit in HomCont in two parameters (λ and either a or b); see also Beyn (1990).

An explicit solution exists for $a = 3/5(2b+1)(b-2), b \geq -1/2$, and it is given by

$$r(t) = 3(b + \frac{1}{2})\text{sech}^2\left([\frac{3}{4}(2b+1)]^{1/2}t\right).$$

It corresponds to a reversible orbit flip for $b > 2$ ($a > 0$) We start from this explicit solution, using `ISTART=2`, for $a = 3$ and $b = (\sqrt{65}+3)/4$:

```
demo('kdv')
r1=run('kdv',sv='1')
```

```
 BR    PT  TY  LAB    PAR(1)        L2-NORM     ...   PAR(3)
  1     1  EP    1   3.00000E+00   5.56544E+00 ...  0.00000E+00
  1     2  EP    2   3.04959E+00   5.49141E+00 ... -4.53380E-18
```

Here PAR(1)=a, PAR(2)=b, and PAR(3)=λ. We have only done a very small continuation to give AUTO a chance to create a good mesh and avoid convergence problems later. Next, we set ITWIST=1 and calculate the adjoint:

```
r2=run(r1,c='kdv.2',sv='2')
```

```
 BR    PT  TY  LAB    PAR(2)         L2-NORM     ...    PAR(9)
  1     2  EP    3   2.76557E+00   5.49141E+00 ... -3.12500E-04
```

We now need to move back to the orbit flip at $a = 3$:

```
r3=run(r2,c='kdv.3',sv='3')
```

```
 BR    PT  TY  LAB    PAR(1)         L2-NORM     ...    PAR(3)
  1    14  UZ    5   3.00000E+00   5.47613E+00 ...  1.47725E-09
```

Now all preparations are done to start homoclinic branch switching. This is very similar to the technique used in Sandstede's model in Section 27.1; to find a 3-homoclinic orbit, we open 2 Lin gaps, until $T_1 = 3.5$, while also varying λ=PAR(3).

```
r4=run(r3('UZ2'),c='kdv.4',sv='4')
```

```
 BR    PT  TY  LAB    PAR(3)      ...    PAR(21)        PAR(22)        PAR(24)
  1    13        8   5.85315E-10 ...  1.65474E+01  -9.20183E-08  -6.11537E-07
  1    23  UZ    9   1.52986E-09 ...  9.85223E+00  -6.68578E-12   2.01956E-07
  1    26       10   4.09273E-09 ...  6.87525E+00   2.68679E-07   7.64502E-07
  1    33  UZ   11   2.15483E-06 ...  3.49999E+00   7.94022E-04   3.99104E-04
```

We then look for an orbit with $a < 3$ and close the gap corresponding to ε_1=PAR(22), for decreasing a.

```
r5=run(r4,c='kdv.5',sv='5')
```

```
 BR    PT  TY  LAB    PAR(2)      ...    PAR(3)         PAR(22)        PAR(24)
  1    10       12   2.57977E+00 ...  2.15713E-06   7.65450E-04   3.82670E-04
  1    13  UZ   13   2.32044E+00 ...  3.86701E-11   1.13817E-10   1.58675E-08
  1    20  EP   14  -1.47788E-01 ... -9.46232E-04  -7.53666E-01  -3.43203E-01
```

and finally close the gap corresponding to ε_2=PAR(24),

```
r6=run(r5('UZ1'),c='kdv.6',sv='6')
```

```
 BR    PT  TY  LAB    PAR(2)      ...    PAR(3)         PAR(23)        PAR(24)
  1    23  UZ   15   2.32044E+00 ...  3.30393E-12   1.48758E+01  -2.30540E-10
  1    35       16   2.31894E+00 ... -2.15192E-08   7.69389E+00  -1.07760E-05
  1    51  UZ   17   2.33846E+00 ...  2.57829E-07   3.48152E+00   1.29755E-04
  1    58  UZ   18   3.08085E+00 ...  2.28299E-12   3.50004E+00  -1.62266E-10
```

so that a three-homoclinic orbit is found. Here the zero at label 17 is the one we are looking for. Label 15 is a false positive since $T_2 = PAR(23)$ is still too high. At label 18, a=`PAR(1)` has changed considerably to the extend that $a > 3$ and a second 3-homoclinic orbit is found. Note that for all zeros of `PAR(24)`=ε_2, the parameter λ=`PAR(3)` is also zero (within AUTO accuracy), which it has to be to remain within the original Hamiltonian system. Setting `ISTART=1`, a normal "trivial" continuation (with `NMX=1`) of the orbit corresponding to label 17 lets HomCont produce a proper concatenated 3-homoclinic orbit:

```
r7=run(r6('UZ2'),c='kdv.7',sv='7')
```

```
 BR    PT  TY  LAB    PAR(2)         L2-NORM     ...    PAR(3)
  1     1  EP   20   2.33846E+00   7.50835E+00  ...  2.57829E-07
```

This 3-homoclinic orbit is depicted in Figure 27.6.

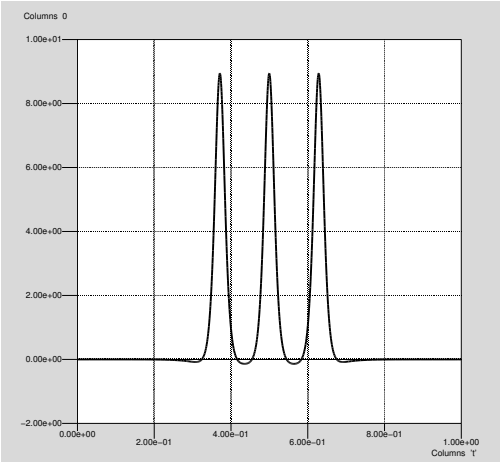


Figure 27.6: A 3-homoclinic orbit in a 5th-order Hamiltonian Korteweg-De Vries model.

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