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Chaos in the Hodgkin–Huxley Model*

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2. Evidence for chaos in the Hodgkin–Huxley system. A stringent definition of chaos in a discrete dynamical system is that there is an invariant subset on which the transformation is hyperbolic and topologically equivalent to a subshift of finite type [10, 20]. Continuous time dynamical systems are reduced to discrete time maps through the introduction of cross-sections and Poincaré return maps [10].

Poincare Return Maps

$$IVP : \begin{cases} \dot{\vec{x}} = f(x) \text{ where } \vec{x} = (x_1, \ldots, x_N)^T \\ \vec{x}(0) = (x_1(0), \ldots, x_N(0))^T \end{cases}$$

$$t_0 = \min_{x_i(t)=c} t > 0, \qquad i, c \text{ fixed}$$

$$t_{n+1} = \min\{t > t_n | x_i(t) = c\}$$

$$a_n = (x_1(t_n), \ldots, x_N(t_n))^T \text{ where } x_i(t_n) = c \text{ for all } n \in \mathbb{N}$$

Random Events and Trajectories

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$$f(x) := 2x \mod 1$$

$$a_{n+1} = \begin{cases} 0 & \text{if } f^n(x_0) \in (0, \frac{1}{2}) \\ 1 & \text{if } f^n(x_0) \in (\frac{1}{2}, 1) \end{cases}$$

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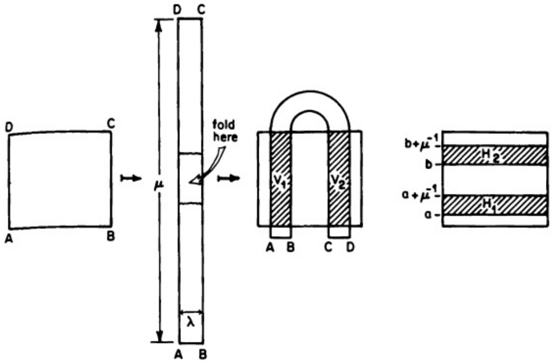

Figure 5.1.1. The Smale horseshoe.

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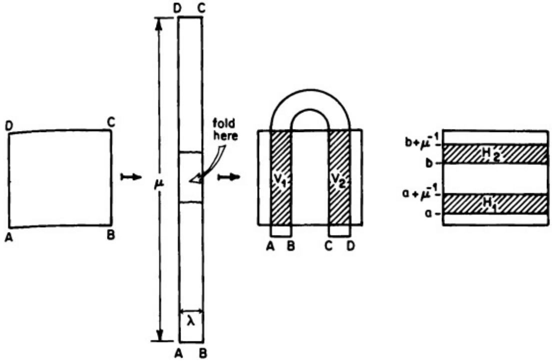

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Goal: Find a set invariant under Smale horseshoe mapping.

Smale Horseshoe

$f :=$ horseshoe mapping

$f : S \to \mathbb{R}^2$ where $S = [0,1] \times [0,1]$

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$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(S) = \text{Cantor set} \times \text{Cantor set}$$

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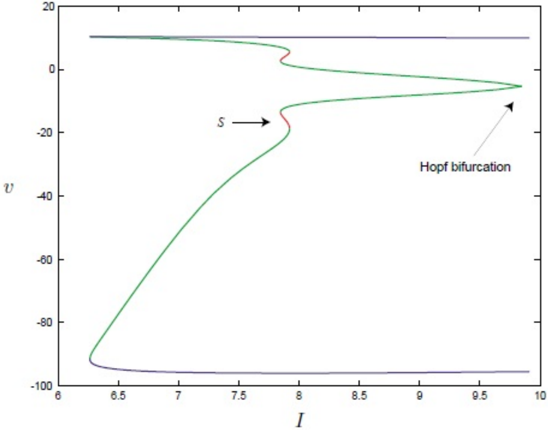
That is,

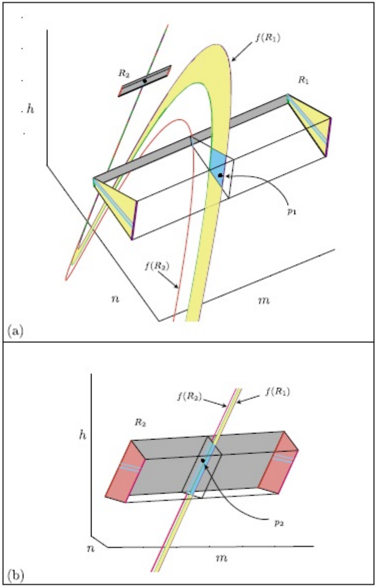
$$\phi(f(x)) = S(\phi(x)) \text{ where } S := \text{shift operator}$$
$$\phi \circ f = S \circ \phi$$
$$f = \phi^{-1} \circ S \circ \phi$$

"f is topologically equivalent to a subshift of finite type"

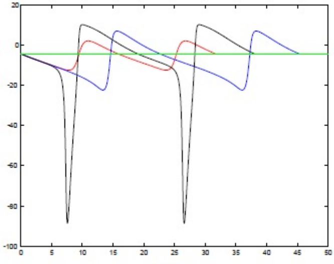
Bif. diagram of HH Model



HH Horsehoe

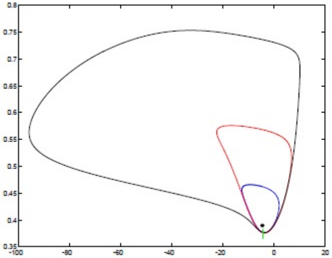


HH Chaos



20
0
-20
-40
-60
-80
-100
0
5
10
15
20
25
30
35
40
45
50
time

(a)



0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
-100
-80
-60
-40
-20
0
20

Biological Significance

Threshold function should depend on system variables i.e. $v = v_t(m, n, h)$ is a threshold function if initial states with $v > v_t(m, n, h)$ produce action potentials, but initial states with $v < v_t(m, n, h)$ don't

Such a function may be discontinuous due to presence of invariant Chaotic set