Problem: 1843; Mathematics Magazine Vol. 83, No. 2, Apr. 2010
Solver: Joe McKenna
Address: Peace Corps, PO Box 5796, Accra-North, Ghana
Email: joepatmckenna@gmail.com

Proposed by Jose Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela. For every positive integer n, let S_n denote the set of permutations of the set $N_n = \{1, 2, \dots, n\}$. For every $1 \le j \le n$, the permutation $\sigma \in S_n$ has a *left to right maximum* (LRM) at position j, if $\sigma(i) < \sigma(j)$ whenever $i < j$. Note that all $\sigma \in S_n$ have a LRM at position 1. Let M be a subset of N_n. Prove that the number of permutations in S_n with LRMs at exactly the positions in M is equal to

$$\prod_{k \in N_n \setminus M} (k-1),$$

where an empty product is equal to 1.

We prove the theorem by induction on n. In S_1, there is $\Pi_{k\in N_1\setminus N_1}(k-1) = 1$ permutation with an LRM and $\Pi_{k\in N_1\setminus\emptyset}(k-1) = 0$ without. Assume the theorem is true of S_n for positive integer n. Define $L : \cup_{k\in\mathbb{Z}^+} S_k \to \mathcal{P}(\mathbb{N})$ so that each $\sigma \in S_k$ has LRMs at exactly the positions in $L(\sigma) \subseteq N_k$. For $1 \notin M_0 \subset N_{n+1}$ there are $\Pi_{k\in N_{n+1}\setminus M_0}(k-1) = 0$ permutations $\sigma \in S_{n+1}$ such that $L(\sigma) = M_0$ because all $\sigma \in S_{n+1}$ have a LRM at position 1. Now for $1 \in M \subseteq N_{n+1}$ let $m = \max(M)$, let $X = \{\sigma \in S_{n+1} : L(\sigma) = M\}$ and let $Y = \{\hat\sigma \in S_n : L(\hat\sigma) \cap N_{m-1} = M \cap N_{m-1}\}$. We seek the cardinality of X but first prove $f_m : X \to Y$ defined by

$$f_m(\sigma)(i) = \begin{cases} \sigma(i) & \text{if } 1 \le i < m \\ \sigma(i+1) & \text{if } m \le i \le n \end{cases}$$

is bijective. For $\sigma_1,\ \sigma_2 \in X$ we have $\sigma_1(m) = \sigma_2(m) = n+1$, and if $f_m(\sigma_1) = f_m(\sigma_2)$ then $\sigma_1(i) = \sigma_2(i)$ for $1 \le i \le n$, $i \ne m$; that is, f_m is injective. For $\hat\sigma \in Y$ there exists $\sigma \in S_{n+1}$ defined by

$$\sigma(i) = \begin{cases} \hat\sigma(i) & \text{if } 1 \le i < m \\ n+1 & \text{if } i = m \\ \hat\sigma(i-1) & \text{if } m < i \le n, \end{cases}$$

such that $L(\sigma) = (L(\hat\sigma) \cap N_{m-1}) \cup \{m\} = M$ and $f_m(\sigma) = \hat\sigma$; that is, f_m is surjective. Therefore, f_m is bijective. Notice permutations in Y may have LRMs at the positions in $N_n \setminus N_{m-1}$. By the induction hypothesis, the cardinality of Y, and hence X, is

$$\prod_{k \in N_m \setminus M} (k-1) \sum_{J \subseteq N_n \setminus N_{m-1}} \prod_{j \in J} (j-1). \tag{1}$$

We have

$$\sum_{J \subseteq N_n \setminus N_{m-1}} \prod_{j \in J} (j-1) = \sum_{\substack{J \subseteq N_n \setminus N_{m-1} \\ m \in J}} \prod_{j \in J} (j-1) + \sum_{\substack{J \subseteq N_n \setminus N_{m-1} \\ m \notin J}} \prod_{j \in J} (j-1) =$$

$$m \sum_{J \subseteq N_n \setminus N_m} \prod_{j \in J} (j-1) = \cdots = \prod_{k \in N_n \setminus N_m} (k-1) \sum_{J \subseteq \{n\}} \prod_{j \in J} (j-1) = \prod_{k \in N_{n+1} \setminus N_m} (k-1)$$

and $M \cap (N_{n+1} \setminus N_m) = \emptyset$, therefore (1) is equal to $\prod_{k \in N_{n+1} \setminus M} (k-1)$. This completes the proof.