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Proposed by Michael Woltermann, Washington & Jefferson College, Washington, PA. A block fountain of coins is an arrangement of n identical coins in rows such that the coins in the first row form a contiguous block, and each row above that forms a contiguous block. If a_n denotes the number of block fountains with exactly n coins in the base, then $a_n = F_{2n-1}$, where F_k denotes the kth Fibonacci number. (Wilf, generatingfunctionology, 1994.) How many block fountains are there if two fountains that are mirror images of each other are considered to be the same?

Let B_n denote the set of block fountains that have exactly n coins in the base and that possess mirror symmetry. We prove $|B_n| = F_{n+1}$ for $n \geq 1$ by strong induction. The case $n = 1$ is trivial. Assume $|B_n| = F_{n+1}$ for $1 \leq n < k$. A contiguous block of k coins is an element of B_k, and centering an element of B_j atop a contiguous block of k coins, for $1 \leq j < k$ with j and k of opposite parity, forms an element of B_k. Conversely, deleting the base from an element of B_k reveals either the empty block fountain or an element of B_j, for j as above. It follows that

$$|B_k| = 1 + \sum_{j=1}^{\lfloor k/2 \rfloor} |B_{k-2j+1}| = 1 + \sum_{j=1}^{\lfloor k/2 \rfloor} F_{k-2(j-1)} = F_{k+1}.$$

Now the set of block fountains that have exactly n coins in the base and that do not possess mirror symmetry may be partitioned into sets of mirror pairs. We conclude that $a_n = |B_n| + \frac{F_{2n-1} - |B_n|}{2} = \frac{F_{n+1} + F_{2n-1}}{2}$.