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Proposed by David Beckwith, Sag Harbor, NY. Let $a(n)$ be the number of ways to place n identical balls into a sequence of urns $U_1, U_2, \ldots$ in such a way that U_1 receives at least one ball, and while any balls remain, each successive urn receives at least as many balls as in all the previous urns combined. Let $b(n)$ denote the number of partitions of n into powers of 2, with repeated powers allowed. (Thus, $a(6) = 6$ because the placements are 114, 123, 15, 24, 33, and 6, while $b(6) = 6$ because the partitions are 111111, 11112, 1122, 114, 222, and 24.) Prove that $a(n) = b(n)$ for all $n \in \mathbb{N}$.

Let i and k be integers with $1 \le i \le k$ and define $\pi_i : \mathbb{Z}^k \to \mathbb{Z}$ to be the function that maps a k-tuple to its i^{th} coordinate. Let j be an integer with $0 \le j \le k$ and define the vectors $\mathbf{1_j}, \boldsymbol{t_j} \in \mathbb{Z}^k$ by

$$\pi_i(\mathbf{1_j}) = \begin{cases} 1 & \text{if } 1 \le i \le j \le k \\ 0 & \text{otherwise} \end{cases}, \qquad \pi_i(\boldsymbol{t_j}) = \begin{cases} 2^{j-i} & \text{if } 1 \le i \le j \le k \\ 0 & \text{otherwise.} \end{cases}$$

We use but omit proof of the identity $\mathbf{1_i} + \boldsymbol{t_1} + \boldsymbol{t_2} + \cdots + \boldsymbol{t_{i-1}} = \boldsymbol{t_i}$. Let n and m be positive integers with $2^{m-1} \le n < 2^m$ and let A_n and B_n be the set of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{Z}^k$ that satisfy the three properties

$$\begin{array}{lcl} \pi_1(\boldsymbol{a}) \ge 1 & & \pi_1(\boldsymbol{b}) \ge 1 \\ \pi_i(\boldsymbol{a}) \ge \boldsymbol{a} \cdot \mathbf{1_{i-1}} & \text{and} & \pi_i(\boldsymbol{b}) \ge 0 \\ \boldsymbol{a} \cdot \mathbf{1_k} = n & & \boldsymbol{b} \cdot \boldsymbol{t_k} = n \end{array}$$

respectively, for $1 \le i \le k \le m$. The cardinality of A_n and B_n are $a(n)$ and $b(n)$, resp. It will suffice to prove $A_n \cap \mathbb{Z}^k \cong B_n \cap \mathbb{Z}^k$ for $1 \le k \le m$; we do so by constructing an isomorphism. Let $f_k : A_n \cap \mathbb{Z}^k \to B_n \cap \mathbb{Z}^k$ be the function defined by $\pi_i(\boldsymbol{u}) \mapsto \pi_i(\boldsymbol{u}) - \boldsymbol{u} \cdot \mathbf{1_{i-1}}$. Let $\boldsymbol{u_1}, \boldsymbol{u_2} \in A_n \cap \mathbb{Z}^k$ and assume $f_k(\boldsymbol{u_1}) = f_k(\boldsymbol{u_2})$. By the definition of f_k we have $\pi_1(\boldsymbol{u_1}) = \pi_1(\boldsymbol{u_2})$ and $\pi_i(\boldsymbol{u_1}) = \pi_i(\boldsymbol{u_2})$ if and only if $\boldsymbol{u_1} \cdot \mathbf{1_{i-1}} = \boldsymbol{u_2} \cdot \mathbf{1_{i-1}}$. Therefore $\boldsymbol{u_1} = \boldsymbol{u_2}$ and f_k is one-to-one. Let $\boldsymbol{v} \in B_n \cap \mathbb{Z}^k$ and define $\boldsymbol{u} \in \mathbb{Z}^k$ by $\pi_i(\boldsymbol{u}) = \pi_i(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_{i-1}}$. First we prove $\boldsymbol{u} \in A_n$. We have

$$\pi_1(\boldsymbol{u}) = \pi_1(\boldsymbol{v}) \ge 1$$

by the definition of $\boldsymbol{u}$ and the assumption $\boldsymbol{v} \in B_n$, resp. Before proceeding we prove a useful fact, we have

$$\begin{aligned} \boldsymbol{u} \cdot \mathbf{1_i} &= [\pi_1(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_0}] + [\pi_2(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_1}] + \cdots + [\pi_i(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_{i-1}}] \\ &= \boldsymbol{v} \cdot (\mathbf{1_i} + \boldsymbol{t_1} + \cdots + \boldsymbol{t_{i-1}}) = \boldsymbol{v} \cdot \boldsymbol{t_i} \end{aligned} \tag{1}$$

by the definition of $\boldsymbol{u}$, algebra, and the identity, resp. Continuing, we have

$$\pi_i(\boldsymbol{u}) = \pi_i(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_{i-1}} = \pi_i(\boldsymbol{v}) + \boldsymbol{u} \cdot \boldsymbol{1_{i-1}} \geq \boldsymbol{u} \cdot \boldsymbol{1_{i-1}}$$

by the definition of $\boldsymbol{u}$, (1), and the assumption $\boldsymbol{v} \in B_n$, resp., and

$$\boldsymbol{u} \cdot \boldsymbol{1_k} = \boldsymbol{v} \cdot \boldsymbol{t_k} = n$$

by (1) and the assumption $\boldsymbol{v} \in B_n$, resp. Therefore, $\boldsymbol{u} \in A_n$. Last we prove $\boldsymbol{u} \overset{f_k}{\mapsto} \boldsymbol{v}$ to conclude that f_k is onto and hence an isomorphism. We have

$$\pi_i(\boldsymbol{u}) \overset{f_k}{\mapsto} \pi_i(\boldsymbol{u}) - \boldsymbol{u} \cdot \boldsymbol{1_{i-1}} = \pi_i(\boldsymbol{v}) + \boldsymbol{v} \cdot \boldsymbol{t_{i-1}} - \boldsymbol{u} \cdot \boldsymbol{1_{i-1}} = \pi_i(\boldsymbol{v})$$

by the definition of f_k, the definition of $\boldsymbol{u}$, and (1), resp.