## A few formulas for sets

$$A \subseteq B \quad \text{means} \quad x \in A \Longrightarrow x \in B. \tag{1}$$

$$x \in A \bigcup B \quad \text{means} \quad (x \in A \ \vee \ x \in B) \tag{2}$$

$$x \in A - B \quad \text{means} \quad (x \in A \ \wedge \ x \notin B) \tag{3}$$

$$x \in \bigcap_{i \in I} A_i \quad \text{means} \quad \forall_{i \in I} \ x \in A_i \tag{4}$$

$$x \in \bigcup_{i \in I} A_i \quad \text{means} \quad \exists_{i \in I} \ x \in A_i \tag{5}$$

## Writing Proofs.

1. **Direct proof for $p \Longrightarrow q$.**
   Assume: $p$. To prove: $q$.

2. **Proving $p \Longrightarrow q$ by contrapositive**.
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving $S$ by contradiction**.
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving $p \Longrightarrow q$ by contradiction**.
   Assume: $p$ and $\neg q$. To prove: a contradiction.

5. **Direct proof for a $\forall_{x \in A} P(x)$ statement**.
   To ensure you prove $P(x)$ for *all* (rather than for *some*) $x$ in $A$, do this:
   **Start your proof with:** Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for $\exists_{x \in A} P(x)$ statement**.
   Take $x :=$ [write down an expression that is in $A$, and satisfies $P(x)$].

7. **Proving $\forall_{x \in A} P(x)$ by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving $\exists_{x \in A} P(x)$ by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving $S$ by cases**.
   Suppose for example a statement $p$ can help to prove $S$. Write two proofs:
   Case 1: Assume $p$. To prove: $S$.
   Case 2: Assume $\neg p$. To prove $S$.

10. **Proving $p \wedge q$**
    Write two separate proofs: To prove: $p$. To prove: $q$.

11. **Proving $p \Longleftrightarrow q$**
    Write two proofs. To prove: $p \Longrightarrow q$ To prove: $q \Longrightarrow p$.

12. **Proving** $p \vee q$
    Method (1): Assume $\neg p$. To prove: $q$.
    Method (2): Assume $\neg q$. To prove: $p$.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using** $p \vee q$ **to prove another statement** $r$.
    Write two proofs:
    Assume $p$. To prove $r$.
    Assume $q$. To prove $r$.

14. **How to use a for-all statement** $\forall_{x \in A} P(x)$.
    You need to produce an element of $A$, then use $P$ for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* $x$. All you know is $x \in A$ and $P(x)$.

**List of facts on cardinal numbers**

1. $o(A) = o(B)$ means $\exists f : A \to B$ with $f$ bijection.

2. $o(A) \leq o(B)$ means $\exists f : A \to B$ with $f$ one-to-one.

3. $\aleph_0$ is short notation for $o(\mathbb{N}^*)$.

4. $c$ is short notation for $o(\mathbb{R})$.

5. The set $A$ is *countably infinite* when: $o(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with $f$ bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of $A$ fit into one sequence $f(1), f(2), \ldots$.

6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.

7. $o(A) < o(P(A))$.

8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $o(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!

9. If $f : A \to B$ is onto then $o(B) \leq o(A)$.

10. $A$ is *countable* when either: $A$ is countably infinite (defined in item 5) or $A$ is finite.

11. $A$ is countable when $o(A) \leq \aleph_0$.

12. A subset of a countable set is again countable.

13. If $A \subseteq B$ then $o(A) \leq o(B)$.

14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
You might remember that the proof was not easy!

15. The ordering $\leq$ on cardinal numbers is a *total ordering*. So given any two cardinals $d, e$ we have $d \leq e$ or $d \geq e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.

16. Set $A$ is uncountable when $o(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: $A$ is uncountable when $o(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*))$.

21. If $d = o(D)$ and $e = o(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$.
$d^e$ is the cardinality of $D^E$ where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If $d, e$ are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $o(P(A)) = o(\{0,1\}^A) = o(\{0,1\})^{o(A)} = 2^{o(A)}$.

24. $c = o(\mathbb{R}) = o(P(\mathbb{N}^*)) = o(\{0,1\}^{\mathbb{N}^*}) = 2^{o(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have $d$ sets, and each of these sets has cardinality $e$, and if $A$ is the union of all those sets, then $o(A) \leq de$ (if the $d$ sets are disjoint, then you may replace the $\leq$ by $=$). Now if $d$ or $e$ is infinite, and both are non-zero, then we can also replace $de$ by max($d$,$e$), see item 22.

List of facts for Chapter 4.

1. A **metric space** $M$ is set with a distance function with the following properties (for all $a, b, c \in M$): $D(a, a) = 0$, $D(a, b) > 0$ whenever $a \neq b$, $D(b, a) = D(a, b)$, and the triangle inequality: $D(a, c) \leq D(a, b) + D(b, c)$.

2. $S_r(x)$ is the **open ball** with radius $r$ and center $x$.
   $S_r(x) = \{p \in M | D(x, p) < r\}$. So this is the set of all points you can reach if you start from $x$ and then travel a distance that is *less than* $r$.

3. We say that $p$ and $x$ are $r$**-close** when $D(p, x) < r$.
   So $S_r(x)$ is the set of all points that are $r$-close to $x$.

4. Any set that contains $S_r(x)$ for some $r > 0$ is called a **neighborhood** of $x$. So a set $U$ is a neighborhood of $x$ when there exists some positive $r$ such that all points that are $r$-close to $x$ are in the set $U$.

5. Let $U$ be a subset of $M$. The following statements are **equivalent**:

   (a) $\exists_{r>0}\ S_r(x) \subseteq U$

   (b) $U$ is a neighborhood of $x$

   (c) $U$ contains a neighborhood of $x$.

6. A set $U \subseteq M$ is **open** when property 5(a)(b)(c) is true for every $x$ in $U$.

7. Note: a neighborhood of $x$ is **not the same** as an open set, because if we want to check that $U$ is an open set then we need to check property 5(a) for *every* element of $U$. Whereas to check if $U$ is a neighborhood of $x$, we only have to check property 5(a) for one element (namely $x$).

8. The sets $\emptyset$ and $M$ are **always open** (even if $M$ does not "look" open. To understand this, selecting $M$ means selecting *all points* to be considered. Then all $r$-close points to any $x$ in $M$ are automatically in $M$).

9. An **open neighborhood** is (these conditions are equivalent):

   (a) A neighborhood of $x$ that happens to be an open set.

   (b) An open set that happens to contain $x$.

10. **Any** union of open sets is always open (even infinitely many sets!).

11. The intersection of **finitely many** open sets is again open.

12. $x$ is an **isolated point** when:

    (a) $\{x\}$ is open

    (b) There is a neighborhood of $x$ that contains just $x$ and no other elements.

    (c) $\exists_{r>0}\, S_r(x) = \{x\}$

    (d) A sequence $x_1, x_2, \ldots$ in $M$ can only converge to $x$ when there is some $N$ such that all $x_i = x$ for all $i \geq N$. In other words, when there is some tail $x_N, x_{N+1}, \ldots$ of your sequence that equals $x, x, \ldots$.

13. $x$ is **not isolated** when

    (a) $\{x\}$ is not open.

    (b) Every neighborhood of $x$ will contain more elements than just $x$.

    (c) For every $r > 0$ the set $S_r(x)$ contains more than just $x$.

    (d) There exists a sequence $x_1, x_2, \ldots$ in $M$ that converges to $x$ but where $x_n \neq x$ for every $n$
    (To produce such a sequence, do the following: for every $n$, the set $S_{\frac{1}{n}}(x) - \{x\}$ is not empty by part (c), so we can choose some $x_n$ in $S_{\frac{1}{n}}(x) - \{x\}$. Then $x_n \neq x$ but $D(x_n, x) < \frac{1}{n}$ and therefore $x_1, x_2, \ldots$ converges to $x$.)

14. Let $x_1, x_2, \ldots$ be a sequence. A **tail** is what you get when you throw away the first … (finitely many) elements. So a tail is a subsequence of the form $x_N, x_{N+1}, \ldots$ for some $N$ (here we threw away the first $N - 1$ elements).

15. $x_1, x_2, \ldots$ **converges** to $x$ when

    (a) For every $\epsilon > 0$ the sequence has a tail contained in $S_\epsilon(x)$.

    (b) $\forall_{\epsilon>0} \, \exists_N \, \forall_{i \geq N} \, D(x_i, x) < \epsilon$

    When these equivalent properties hold then we say that $x$ is the limit of the sequence $x_1, x_2, \ldots$.
    The most boring convergent sequences are those that have a tail that is constant. Such a sequence obviously converges. If $x$ is isolated, then item 12(d) says that only boring sequences can converge to $x$.
    However, if $x$ is not isolated, then there are more interesting sequences that converge to $x$, see item 13(d).

16. $M$ is **discrete** when

    (a) Every $x$ in $M$ is isolated.

    (b) $\{x\}$ is open for every $x \in M$.

    (c) Every set $U \subseteq M$ is open.

17. A set $F \subseteq M$ is **closed** when

    (a) If a sequence $x_1, x_2, \ldots$ in $F$ converges to $x$ then $x$ must be in $F$.

    (b) If $S_r(x) \bigcap F$ is not empty for every $r > 0$ then $x \in F$.

    (c) If $F \bigcap U \neq \emptyset$ for every neighborhood $U$ of $x$ then $x \in F$.

    (d) If every neighborhood of $x$ intersects $F$ (if every neighborhood of $x$ has element(s) in common with $F$) then $x \in F$.

    (e) The complement of $F$ is open, i.e. $F^c = M - F$ is open.

    (f) $F$ contains all of its limit points ($x$ is a limit point of $F \Longrightarrow x \in F$).

18. A point $x$ is called a **limit point** of $A$ if there is a sequence in $A - \{x\}$ that converges to $x$.

19. $\overline{A}$ is called the **closure** of the set $A$.

    (a) $\overline{A}$ is the union of $A$ and all of its limit points.

    (b) $\overline{A}$ is the smallest closed set that contains $A$.

    (c) $\overline{A}$ is the intersection of all closed sets that contain $A$.

    (d) $x \in \overline{A} \iff$ every neighborhood of $x$ intersects $A$.

    (e) $x \in \overline{A} \iff \exists$ a sequence $x_1, x_2, \ldots \in A$ that converges to $x$.

    (f) $x \in \overline{A} \iff \forall_{\epsilon>0}$ there is a point in $A$ that is $\epsilon$-close to $x$.

20. $x$ is a **limit point** of $A$ if $x$ is in the closure of $A - \{x\}$.

21. If $x_1, x_2, \ldots$ converges to $x$ and $y_1, y_2, \ldots$ converges to $y$, then $D(x_1, y_1), D(x_2, y_2), \ldots$ converges to $D(x, y)$.

22. The diameter of a set $A$ is the supremum of $\{D(x, y) | x, y \in A\}$.

23. If $A$ is a set, then the diameter of $A$ equals the diameter of $\overline{A}$. To prove this, you need item 21.

24. The union of *finitely many* closed sets is again closed.

25. The intersection of closed sets (even if you take infinitely many closed sets!) is again closed.