Final Exam, MGF 3301, send in by 9:30 am EST.

At 9:30 am EST you must submit everything you have solved at that point.

If the test turns out to be too long, then Ex 9 and 10 can be submitted after 9:30, so make sure to finish Ex 1–8 before solving Ex 9 and 10.

In Ex 1 and 2 no proofs are needed:

1. (2 pt each). Compute the cardinality (simplified to $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$)

   (a) $P(P(\{1\}))$

   (b) $\mathbb{Z} - \mathbb{N}$

   (c) $\mathbb{R}^{\mathbb{N}}$

   (d) $\mathbb{N}^{\mathbb{R}}$

2. (1.5 pt each). Take the metric space $M = \mathbb{R}$ with the usual distance function. Let $A = \mathbb{R} - \mathbb{Z}$ and let $B = (0, 1) \bigcap \mathbb{Q}$.

   (a) Is A open? Yes/No

   (b) Is A closed? Yes/No

   (c) What is the closure of A?

   (d) Is B open? Yes/No

   (e) Is B closed? Yes/No

   (f) What is the closure of B?

   (g) Is M open? Yes/No

   (h) Is M closed? Yes/No

3. (10 pt). Let A, B, C be sets with $C \subseteq A \bigcup B$. Prove that $C - A \subseteq B$. (Hint: A few set formulas are in the included handout).

4. (10 pt). Let $f : A \to B$ and $g : B \to C$. Suppose that $g \circ f$ is surjective. Prove that g is surjective (first write down the definition of surjective).

5. Suppose that A and B are non-empty sets and that there is no injective function from $A \times B$ to $A \bigcup B$. Show that A and B are finite.

6. Let M be a metric space and let $p, x \in M$. Let S be the following statement:

$$S: \quad (\forall_{r>0} \;\; x \in S_r(p)\,) \implies p = x$$

(a) (3 pt). Write down the *contrapositive* of S.

(b) (5 pt). Write a proof for the statement you wrote in part (a).

(c) (2 pt). Let

$$A := \bigcap_{r>0} S_r(p) \quad \text{and} \quad B := \bigcup_{r>0} S_r(p)$$

Statement S says something about one of these two sets. Do you see which one, and what S says about that set?

7. (10 pt). Let M be a discrete metric space (definition of "discrete" is in item 16). Must every subset S of M be closed? Explain.

8. (10 pt). Let A be a subset of a metric space M. Suppose that $x \in \overline{A}$ and $x \notin A$. Prove that x is not isolated.

If you run out of time, make sure to turn in Ex 1–8 by 9:30 am EST, so only work on Ex 9 and 10 after completing Ex 1–8.

9. (10 pt). Let M be a metric space, let $a \in M$ and let

$$U = \{x \in M \mid 1 < D(a, x) < 2\}.$$

Show that U is open.

10. (10 pt). Let x be a positive real number. Use induction to prove that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.