Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients

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Main Question

- Given a second order homogeneous differential equation $a_2 y'' + a_1 y' + a_0 = 0$, where a_i's are rational functions, can we find solutions in terms of Bessel functions?
- A homogeneous equation corresponds a second order differential operator $L := a_2 \partial^2 + a_1 \partial + a_0$.

An Analogy

- $\frac{I_\nu(x)\sqrt{x}}{e^x}$ converges when $x \to +\infty$.
  $I_\nu(x)$ and e^x have similar asymptotic behavior when $x \to +\infty$.
- The idea behind finding closed form solutions is to reconstruct them from the asymptotic behavior at the singular points.
- Before studying how to find Bessel type solutions, let's see how this strategy works for exponential solutions $e^{f(x)}$.

Generalized Exponents

- To find exponential solutions $y = e^{f(x)}$, we need to know the asymptotic behavior of y at each singularity.
- Generalized exponents (up to equivalence) effectively determine asymptotic behavior up to a meromorphic function.

Finding Exponential Solutions

Let $L \in \mathbb{C}(x)[\partial]$. Suppose $y = e^{f(x)}$ is a solution of L, where $f \in \mathbb{C}(x)$. Question: How to find f?

Poles of f

Let $p \in \mathbb{C} \cup \{\infty\}$.

$$\begin{aligned} p \text{ is a pole of } f \quad &\Longrightarrow \quad p \text{ is an essential singularity of } y \\ &\Longrightarrow \quad p \text{ is an irregular singularity of } L. \end{aligned}$$

Finding Exponential Solutions

Suppose L has order n and p is an irregular singularity of L (notation $p \in S_{irr}$).

- L has n generalized exponents at p, one of which gives the polar part of f at $x = p$.
- There are finitely many combinations of generalized exponents at all irregular singularities. Each combination give us a candidate for f.
- Try all candidate f's will give us the exponential solutions.

Finding Bessel type Solutions

1. The same process as finding $e^{f(x)}$ will give us all solutions of the form $I_\nu(f)$, $f \in \mathbb{C}(x)$.
2. We want to find *all solutions of L that can be expressed in terms of Bessel functions.*
3. As we shall see, (1) $\not\Longrightarrow$ (2).

Finding Bessel Type Solutions-Challenges

1. Let $g \in \mathbb{C}(x)$ and $f = \sqrt{g}$. Then $I_\nu(f)$ satisfies an equation in $\mathbb{C}(x)[\partial]$.
2. So it is not sufficient to only consider $f \in \mathbb{C}(x)$. We need to allow for f's with $f^2 \in \mathbb{C}(x)$.
3. As for $e^{f(x)}$ solutions, we find at each $p \in S_{irr}$:

$$\begin{aligned} \text{Polar part of } f &\Longrightarrow \text{half of polar part of } g \\ &\Longrightarrow \text{half of } g \text{ (half of } f). \end{aligned}$$

An Example

If

$$f = 1x^{-3} + 2x^{-2} + 3x^{-1} + O(x^0),$$

then

$$g = x^{-6} + 4x^{-5} + 10x^{-4} + ?x^{-3} + O(x^{-2}).$$

Find Bessel type Solutions–Challenges

- Let $r \in \mathbb{C}(x)$, then $\exp(\int r) I_\nu(\sqrt{g(x)})$ also satisfies an equation in $\mathbb{C}(x)[\partial]$.
- Let $r_0, r_1 \in \mathbb{C}(x)$, then $r_0 I_\nu(\sqrt{g(x)}) + r_1(I_\nu(\sqrt{g(x)}))'$ satisfies an equation in $\mathbb{C}(x)[\partial]$ too.
- So to solve L "in terms of " Bessel functions, we also need to allow sums, products, differentiations, exponential integrals.
- Note: our "in terms of" is the same as that in Singer's (1985) definition. (more on that later.)

Find Bessel type Solutions

To summarize the three cases, when we say solve equations *in terms of* Bessel Functions we mean find solutions which have the form

$$e^{\int r dx}(r_0 B_\nu(\sqrt{g}) + r_1(B_\nu(\sqrt{g}))')$$

where $B_\nu(x)$ is one of the Bessel functions, and $r, r_0, r_1, g \in \mathbb{C}(x)$. (Later in the talk: completeness theorem regarding this form.)

Differential Fields

- Let C_K be a number field with characteristic 0.
- Let $K = C_K(x)$ be the rational function field over C_K.
- Let $\partial = \frac{d}{dx}$.
- Then K is a differential field with derivative ∂ and $C_K := \{c \in K | \partial(c) = 0\}$ is the constant field of K.

Differential Operators

- $L := \sum_{i=0}^{n} a_i \partial^i$ is a differential operator over K, where $a_i \in K$.
- $K[\partial]$ is the ring of all differential operators over K.
- L corresponds to a homogeneous differential equation $Ly = 0$.
- We say y is a solution of L, if $Ly = 0$.
- Denote $V(L)$ as the vector space of solutions. (Defined inside a so-called *universal extension*).
- p is a singularity of L, if p is a root of a_n or p is a pole of $a_i, i \neq n$.

Bessel Functions

- The two linearly independent solutions $J_\nu(x)$ and $Y_\nu(x)$ of $L_{B1} = x^2\partial^2 + x\partial + (x^2 - \nu^2)$ are called Bessel functions of first and second kind, respectively.
- Solutions $I_\nu(x)$ and $K_\nu(x)$ of $L_{B2} = x^2\partial^2 + x\partial - (x^2 + \nu^2)$ are called the modified Bessel functions of first and second kind, respectively.
- The change of variables $x \to x\sqrt{-1}$ sends $V(L_{B1})$ to $V(L_{B2})$ and vice versa. So we can start our algorithm with $L_B := L_{B2}$. And let $B_\nu(x)$ refer to one of the Bessel functions.
- If $\nu \in \frac{1}{2} + \mathbb{Z}$, then L_B is reducible.

Questions

- Given an irreducible second order differential operator $L = a_2\partial^2 + a_1\partial + a_0$, with $a_0, a_1, a_2 \in K$. Can we solve it in terms of Bessel Functions?
- More precisely can we find solutions which have the form

$$e^{\int r dx}(r_0 B_\nu(\sqrt{g}) + r_1(B_\nu(\sqrt{g}))')$$

where $B_\nu(x)$ is one of the Bessel functions.

Why Second Order?

Why Second Order?

- Definition (Singer 1985): $L \in \mathbb{C}(x)[\partial]$, and if a solution y can be expressed in terms of solutions of second order equations, then y is a *eulerian solution*.
- Note: any solution of $L \in \mathbb{C}(x)[\partial]$ that can be expressed in terms of Bessel functions is a eulerian solution.

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- Singer proved that solving such L can be reduced to solving second order L's
- van Hoeij developed an algorithm that reduces to order 2.
- such reduction to order 2 is valuable, *if* we can actually solve such second order equations.
- In summary, to solve n's order equation in terms of Bessel, we need an algorithm that solve 2nd order equations in terms of Bessel functions.

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If we can find a Bessel Solver, then we can find all ${}_pF_q$ type solutions of second order equations excepts $(p, q) = (2, 1)$

- ${}_0F_1$ and ${}_1F_1$ functions can be written in terms of either Whittaker functions or Bessel functions.
- Whittaker functions has already been handled. (Debeerst, van Hoeij, and Koepf)
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Why Irreducible?

If the second order operator is reducible, it has Liouvillian solutions. Kovacic's algorithm can find such solutions.

Completeness

Questions

For Bessel type solutions, is it sufficient to consider solutions with form

$$e^{\int rdx}(r_0 B_\nu(\sqrt{g}) + r_1(B_\nu(\sqrt{g}))')$$

where $B_\nu(x)$ is one of the Bessel functions, and $r, r_0, r_1, g \in K$?

To answer that, we need to answer:

1. what about $B''_\nu, B'''_\nu, \ldots$?
2. what about sums, products, derivatives, exponential integrals?
3. what about $r, r_0, r_1, g \in \overline{K}$?

Completeness

Theorem of Completeness

Let $K = C_K(x) \subseteq \mathbb{C}(x)$. Let $L \in K[\partial]$. Let $r, f, r_0, r_1 \in \overline{\mathbb{C}(x)}$ and

$$e^{\int r dx}(r_0 B_\nu(f) + r_1(B_\nu(f))')$$

be a non-zero solution of f. Then $\exists \widetilde{r}, \widetilde{r_0}, \widetilde{r_1}, \widetilde{f}, \widetilde{\nu}$ with $\widetilde{f}^2 \in K$ such that

$$e^{\int \widetilde{r} dx}(\widetilde{r_0} B_{\widetilde{\nu}}(\widetilde{f}) + \widetilde{r_1}(B_{\widetilde{\nu}}(\widetilde{f}))')$$

is a non-zero solution of L.
Moreover, $\left(\nu - \frac{n}{2}\right)^2 \in C_K$ for some $n \in \mathbb{Z}$, and $\widetilde{r}, \widetilde{r_0}, \widetilde{r_1} \in K(\nu^2)$.
(If $n \in 2\mathbb{Z}$, we may assume $\nu^2 \in C_K$)

Transformations

There are three types of transformations that preserve order 2:

1. change of variables $\xrightarrow{f}_C$: $y(x) \mapsto y(f(x))$, $\quad f(x) \in K$.
   (for L_B, $f^2 \in K$)
2. exp-product $\longrightarrow_E$: $y \mapsto \exp(\int r\, dx) \cdot y$, $\quad r \in K$.
3. gauge transformation $\longrightarrow_G$: $y \mapsto r_0 y + r_1 y'$, $\quad r_0, r_1 \in K$.

L can be solved in terms of Bessel functions when $L_B \longrightarrow_{CEG} L$.
Where $\longrightarrow_{CEG}$ is any combination of $\longrightarrow_C$, $\longrightarrow_E$, $\longrightarrow_G$.

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Note

- The composition of 2 & 3 is an equivalence relation ($\sim_{EG}$). And there exist some algorithms to find such relations.
- If $L_1 \longrightarrow_{CEG} L_2$, then there exist an operator $M \in K[\partial]$ such that $L_1 \xrightarrow{f}_C M \sim_{EG} L$.

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Rephrase the Main Problem

Given an irreducible second linear order differential operator $L \in K[\partial]$, find f and ν with $f^2 \in K$ and $(\nu + \frac{n}{2})^2 \in C_K$ s.t there exist M and $L_B \xrightarrow{f}_C M \sim_{EG} L$

Related Work

$\longrightarrow_C$, $\longrightarrow_E$

- Bronstein, M., and Lafaille, S. (ISSAC 2002) solve using only $\longrightarrow_C$ and $\longrightarrow_E$.
- An analogy about $\longrightarrow_C$ and $\longrightarrow_E$: Suppose you solve polynomial equations using only $x \mapsto c \cdot x$ and $x \mapsto x + c$. then $x^6 - 24x^3 - 108x^2 - 72x + 132$ will not be solved in terms of solutions of $x^6 - 12$, even though it does have a solution in $\mathbb{Q}(\sqrt[6]{12})$. Likewise omitting $\longrightarrow_G$ means not solving the non-trivial case!

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No Square Root

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Invariant Under $\sim_{EG}$

Assume the input is L, and $L_B \xrightarrow{f}_C M \sim_{EG} L$:

If M were known, it would be easy to compute f from M. However, the input is not M, but an operator $L \sim_{EG} M$. So we must compute f not from M, but only from the portion of M that is invariant under $\sim_{EG}$. The portion is **exponent difference** $(mod \mathbb{Z})$.

Generalized Exponents

Assume $L \in K[\partial]$ with order 2:

- Define
$$t_p := \begin{cases} x - p & \text{if } p \neq \infty \\ \frac{1}{x} & \text{if } p = \infty \end{cases}$$
- there are two generalized exponents $e_1, e_2 \in \mathbb{C}[t_p^{-\frac{1}{2}}]$ at each point $x = p$.
- We can think of e_1, e_2 as truncated Puiseux series. They determine the asymptotic behavior of solutions.
- If a solution contains $\ln(t_p)$, then we say L is **logarithmic** at $x = p$. (only occurs when $e_1 - e_2 \in \mathbb{Z}$)
- $\Delta(L, p) := \pm(e_1 - e_2)$ is the **exponent difference**.

Singularities

A singularity p of $L \in K[\partial]$ is:

- *removable singularity* if and only if $\Delta(L,p) \in \mathbb{Z}$ and L is not logarithmic at $x = p$.
- *non-removable regular singularity* (denoted by S_{reg}) if and only if $\Delta(L,p) \in \mathbb{C} \setminus \mathbb{Z}$ or L is logarithmic at $x = p$.
- *irregular singularity* (denoted by S_{irr}) if and only if $\Delta(L,p) \in \mathbb{C}[t_p^{-\frac{1}{2}}] \setminus \mathbb{C}$.

Exponent Difference

- $L_B \xrightarrow{f}_C M$ then:
  1. if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an removable singularity or $p \in S_{reg}$, and $\Delta(M,p) = m_p \cdot 2\nu$.
  2. p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(M,p) = 2\sum_{i<0} i \cdot f_i t_p^i$.
- $\Delta(L,p)$ is invariant under $\longrightarrow_E$.
- $\longrightarrow_G$ shifts $\Delta(L,p)$ by integers.
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Local Information

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$ and $g = f^2 = \frac{A}{B}$, where A, B are polynomials. Exponent difference will give us the following information:

- some (not necessarily all!) zeroes of A from S_{reg}.
- the polar parts of f (from S_{irr}), then by squaring that we know the polar parts of g partially. (as a truncated Laurent series at each irregular singularity).
- B
- an upper bound for the degree of A (denoted by d_A).
- Now we need to compute A.

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Bessel Parameter ν

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$.

- The exponent differences of L give us whether $\nu \in \mathbb{Z}$, $\nu \in \mathbb{Q} \setminus \mathbb{Z}$, $\nu \in C_K \setminus \mathbb{Q}$ or $\nu \notin C_K$.
- if $\nu \notin \mathbb{Q}$, we first compute candidates for f, and use them to compute candidates for ν.
- If $\nu \in \mathbb{Q}$, then exponent differences give a list of the candidates for the denominator of ν.
- It is sufficient to consider only $Re(\nu) \in [0, \frac{1}{2}]$, because $\nu \mapsto \nu + 1$ and $\nu \mapsto 1 - \nu$ are special case of $\longrightarrow_G$

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- It is sufficient to consider only $Re(\nu) \in [0, \frac{1}{2}]$, because $\nu \mapsto \nu + 1$ and $\nu \mapsto 1 - \nu$ are special case of $\longrightarrow_G$

An Example

$L := \partial^2 - \frac{1}{x-1}\partial + \frac{1}{18}\frac{18-23x+4x^2-20x^3+12x^4}{(x-1)^4x^3}$

From generalized exponent, we can obtain the following:

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From generalized exponent, we can obtain the following:

- $S_{reg} = \emptyset$, so no known zeroes.
- the polar part of f is $\frac{\pm 2i}{\sqrt{t_0}}$ at $x = 0$, and $\frac{\pm 1}{\sqrt{2}\cdot t_1}$ at $x = 1$.
- the polar part of g is $\frac{-4}{t_0}$ at $x = 0$, and $\frac{1}{2t_1^2} + \frac{?}{t_1}$ at $x = 1$
- $B = x(x-1)^2$, $d_A = 3$.
- $\nu \in \{\frac{1}{3}\}$

How to compute A?

Linear Equations

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$ and $g = f^2 = \frac{A}{B}$ and $A = \sum_{i=0}^{d_A} a_i x^i$.

Roots

$$\begin{aligned} p \in S_{reg} \quad &\Longrightarrow \quad p \text{ is a root of } A \\ &\Longrightarrow \quad \text{one linear equation of } a_i\text{'s.} \end{aligned}$$

Poles

$$\begin{aligned} \text{If } p \in S_{irr} \quad &\Longrightarrow \quad p \text{ is a pole of } g \text{ (assume } m_p \text{ is the pole order)} \\ &\Longrightarrow \quad \left\lceil \frac{m_p}{2} \right\rceil \text{ linear equations of } a_i\text{'s.} \end{aligned}$$

We get at least $\#S_{reg} + \frac{1}{2}d_A$ linear equations in total.

Continuation of the Example

In our example we can assume

$$g = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{x(x-1)^2}$$

Roots

$S_{reg} = \emptyset \implies$ no linear equations from regular singularities.

Poles

- polar part of g at $x = 0$ is $\frac{a_0}{t_0} + O(t_0^0) \implies a_0 = -4$.
- polar part of g at $x = 1$ is $\frac{a_0+a_1+a_2+a_3}{t_1^2} + O(t_1^{-1}) \implies a_0 + a_1 + a_2 + a_3 = \frac{1}{2}$.

The First Difficulty

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$, $g = f^2 = \frac{A}{B}$.

Not enough equations to compute A

- Only know about half of polar parts of g
- Only have about $\frac{1}{2}d_A$ linear equations from irregular singularities to get A.
- With disappearing singularities, we do not have enough equations to get A.

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the Reason for the First difficulty

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$ and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

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Reason: Regular singularities may become removable under $\xrightarrow{f}_C$, thus may disappear under $\sim_{EG}$

Note: If $f \in K$, this is not a problem, because we do not need as many equations in that case.

the Solution for the First Difficulty

Assume $L_B \xrightarrow{f}_C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$.
Let d be the denominator of ν and m_p be the multiplicity of f at p.

Solution:

- Singularity p disappears only if $\nu \in \mathbb{Q} \setminus \mathbb{Z}$ and $d \mid 2m_p$.
- We can write $A = C \cdot A_1 \cdot A_2^d$. Here A_1 contains all known roots, A_2 is the disappeared part.
- Now we need to compute A_2.
- Since $d \geq 3$, so we only need roughly $\frac{1}{3} d_A$ equations to get A_2.

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Continuation of the Example

In our example: assume $A = C \cdot A_1 \cdot A_2^3$

- $S_{reg} = \emptyset \implies A_1 = 1$;
- Fix $C = -4$. (We will discuss how to find C later.)
- Assume $A_2 = a_0 + a_1 x$.

Now we get

$$g = \frac{-4(a_0 + a_1 x)^3}{x(x-1)^2}$$

- polar part of g at $x = 0$ is $\frac{-4a_0^3}{t_0} + O(t_0^0) \implies -4a_0^3 = -4$.
- polar part of g at $x = 1$ is $\frac{-4(a_0+a_1)^3}{t_1^2} + O(t_1^{-1}) \implies -4(a_0 + a_1)^3 = \frac{1}{2}$.
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Non-linear equations

- To get enough equations, we write $A = C \cdot A_1 \cdot A_2^d$.
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the Solution:

From power series of A_2^d, try to get a power series of A_2, then we will have linear equations.

Continuation of the Example

Assume $A = -4(a_0 + a_1 x)^3$, $\mu_3 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

- the power series of $g = \frac{CA_2^3}{B}$ at 0 is $\frac{-4}{t_0} + O(t_0^0)$.

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- the power series of $g = \frac{CA_2^3}{B}$ at 1 is $\frac{1}{2t_1^2} + O(t_1^{-1})$.
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- We get $a_0 + a_1 = -\frac{1}{2}$.
- solve both equations we get $A_2 = 1 - \frac{3}{2}x$.

Solution

By computing the relation under $\sim_{EG}$, we find two independent solutions:

$$\sqrt{x(3x-2)}(x-1)I_{\frac{1}{3}}\left(\sqrt{\frac{(3x-2)^3}{2x(x-1)^2}}\right)$$

and

$$\sqrt{x(3x-2)}(x-1)K_{\frac{1}{3}}\left(\sqrt{\frac{(3x-2)^3}{2x(x-1)^2}}\right)$$

Fix A_1

$\nu \in \mathbb{Q}$, $A = C \cdot A_1 \cdot A_2^d$.
We can fix A_1 this way:

- If we don't have regular singularities, then $A_1 = 1$
- Each $p \in S_{reg}$ corresponds to each root of A_1.
- Exponent differences and d will give a set of candidates for the multiplicities. (Diophantine equations)
- Try all candidates.

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For our example, $S_{reg} = \emptyset$, so $A_1 = 1$.

About C

- We know that no algebraic extension of C_K is needed for g.
- However without the right value for C in $g = \frac{CA_1A_2^d}{B}$, an algebraic extension of C_K will be needed in A_2.
- Define $C_1 \sim C_2$ if $C_1 = c^d \cdot C_2$, where $c \in C_K$.
- C is unique (up to $\sim$) if there exist $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$.
- If $p \in \overline{C_K} \setminus C_K$ then finding all C's up to $\sim$ involves a number theoretical problem.

Fix C

Pick $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$. If no such p exists, pick any $p \in S_{irr}$ and consider everything over $C_K(p)$

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$\Rightarrow$ the series of $CA_2^d = \frac{gB}{A_1}$.

$\Rightarrow$ Let C equal the coefficient of the first term of this series.

For our examples, we can fix $C = -4$ (if we start with $p = 0$) or $\frac{1}{2}$ (if we start with $p = 1$). There are equivalent, since $-4 = \frac{1}{2} \cdot (-2)^3$.

Uniqueness

Theorem 1

If L has a solution $\exp(\int r)(r_0 B_\nu(f_1) + r_1(B_\nu(f_1))')$ and $\exp(\int \hat{r})(\hat{r}_0 B_\nu(f_2) + \hat{r}_1(B_\nu(f_2))')$ where $r, r_0, r_1, \hat{r}, \hat{r}_0, \hat{r}_1, f_1, f_2 \in \overline{\mathbb{Q}(x)}$, then $f_1 = \pm f_2$.

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Why Need Uniqueness

- Theoretically, it to prove the completeness of our algorithm.
- Practically, if we get a candidate of f and $f^2 \notin K$, we can discard f without further computation, which increases the speed of algorithm significantly.

(Note: In our example, it reduced the number of combinations from 9 to 1.)

Theory Requirement

To prove the theorem, we need to use

- Classification of differential operators mod p (p-curvature).
- Number theory (Chebotarev's density theorem).
- Differential Galois theory.

the Sketch of the proof

- If $\nu \in \frac{1}{2} + \mathbb{Z}$ (non-interesting case in algorithm), then L_B has exponential solutions.
- Use Chebotarev's density theorem, there are infinitely many p, for which ν reduces to an element in $\mathbb{F}_p$.
- Thus $\nu \equiv \frac{1}{2} \mod p$.
- So we know the solutions mod such p in these cases.
- by classification theory (p-curvature), we get $\pm f' \equiv 1 \mod p$.
- Since there exist infinity many such p, we get $\pm f$ is unique up to a constant.
- The rest of the proof is based on the differential Galois theory.

Conclusions

Our contribution in the thesis:

- Developed a complete Bessel solver for second order differential equations.
- Combine Bessel Solver with Whittaker/Kummer solver to get a solver for ${}_0F_1$, ${}_1F_1$ functions.
- Proved the completeness of our algorithm.
- As an application, found relations between Heun functions and Bessel functions.

Acknowledgement

- Thanks to my advisor Mark van Hoeij for his support, patience, and friendship.
- Thanks to the members of my committee for their time and efforts.
- Thanks to my family and friends for their support.