Summary of the PhD Defense Presentation

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Notation:

∂	$\frac{d}{dx}$
L_{inp}	Input differential operator; a second order linear differential operator
$H_{c,x}^{a,b}$	Gauss hypergeometric differential operator
$S(x) = {}_2F_1(a, b; c \mid x)$	Gauss hypergeometric function; a solution of $H_{c,x}^{a,b}$
(e_0, e_1, e_∞)	Exponent differences of $H_{c,x}^{a,b}$ at $(0, 1, \infty)$
$\xrightarrow{f}_C$	Change of variables
$\xrightarrow{r_0,r_1}_G$	Gauge transformation
$\xrightarrow{r}_E$	Exponential product
$[g_0, g_1, \ldots, g_k]$	k-constellation; $g_i \in S_n$ (details in the slides)

Given:

A second order linear differential operator L_{inp} with rational function coefficients (see slides for motivation);

1. irreducible and has no Liouvillian solutions,
2. has five regular singularities where at least one of the singularities is *logarithmic*, and
3. has arithmetic monodromy group.

Goal:

Solve L_{inp} in terms of ${}_2F_1(a, b; c \mid f)$, i.e, find a solution y of the following form (also called the *closed form solution*):

$$y = \exp\Big(\int r\, dx\Big) \Big(r_0 S(f) + r_1 S(f)'\Big) \neq 0$$

where $S(x) = {}_2F_1(a, b; c \mid x)$, $r, r_0, r_1, f \in \mathbb{C}(x)$.

Problem Discussion:

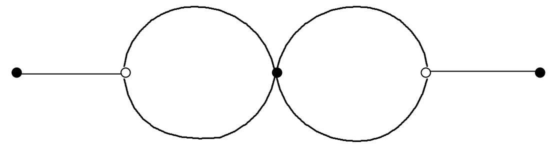
For the solver program to be complete we need the complete tables of Belyi, Belyi-1 and Belyi-2 maps. We use the following correspondence to prove the completeness:

$$\begin{aligned} \text{Belyi maps} &\longleftrightarrow \text{dessins, i.e, equivalence classes of 3-constellations } [g_0, g_1, g_\infty] \text{ mod conjugation} \\ \text{Belyi-1 maps} &\longleftrightarrow \text{near dessins, i.e, equivalence classes of 4-constellations } [g_0, g_1, g_t, g_\infty] \text{ mod conjugation} \\ \text{Belyi-2 maps} &\longleftrightarrow \text{we have algorithms to compute such maps} \end{aligned}$$

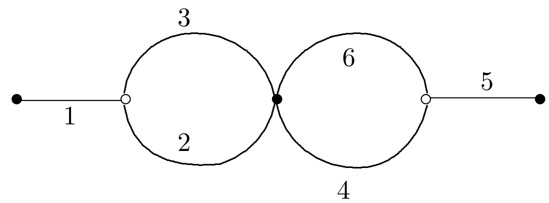
- We compute Belyi, Belyi-1 maps solving polynomial equations and using other techniques.
- We compute dessins, near dessins using combinatorial search plus various techniques to prevent computational explosion.

Proof: Compare!

Dessin:



“Labelled dessin”, i.e, 3-constellation:



Read off permutations:

black vertices:	$x = -\sqrt{3}, x = 0, x = \sqrt{3}$	$g_0 = (1)\,(2\,4\,6\,3)\,(5)$
white vertices:	$x = -1, x = 1$	$g_1 = (1\,2\,3)\,(4\,5\,6)$
faces:	$x = -\frac{1}{\sqrt{3}}, x = \frac{1}{\sqrt{3}}, x = \infty$	$g_\infty = (2)\,(6)\,(1\,3\,5\,4)$
		(follow outside face, $x = \infty$, clockwise)

Note:

Since the labelling was done arbitrarily, we should expect the result from “`algcurves[monodromy]`” to match only up to conjugation (same dessin, not the same 3-constellation). We have a program that can find the conjugation.