Subanalytic Solutions of Linear Difference Equations and Multidimensional Hypergeometric Sequences

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Abstract

We consider linear difference equations with polynomial coefficients over $\mathbb{C}$ and their solutions in the form of sequences indexed by the integers (sequential solutions). We investigate the $\mathbb{C}$-linear space of subanalytic solutions, i.e., those sequential solutions that are the restrictions to $\mathbb{Z}$ of some analytic solutions of the original equation. It is shown that this space coincides with the space of the restrictions to $\mathbb{Z}$ of entire solutions and the dimension of this space is equal to the order of the original equation.

We also consider d-dimensional ($d \geq 1$) hypergeometric sequences, i.e., sequential resp. subanalytic solutions of consistent systems of first-order difference equations for a single unknown function. We show that the dimension of the space of subanalytic solutions is always at most 1, and that this dimension may be equal to 0 for some systems (although the dimension of the space of all sequential solutions is always positive).

Subanalytic solutions have applications in computer algebra. We show that some implementations of certain well-known summation algorithms in existing computer algebra systems work

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correctly when the input sequence is a subanalytic solution of an equation or a system, but can give incorrect results for some sequential solutions.

Key words: Entire Solutions of Linear Difference Equations, Subanalytic Solutions of Linear Recurrence Equations, Hypergeometric Sequences

1. Introduction

Power series are a convenient tool to investigate analytic solutions of equations of different kinds, in particular difference equations. It turns out that such series are also useful to work with sequential solutions of difference equations, i.e., solutions in the form of sequences indexed by the integers.

In this paper we deal with solutions of linear equations with polynomial coefficients:

$$a_d(z)y(z+d) + \cdots + a_1(z)y(z+1) + a_0(z)y(z) = 0; \tag{1}$$

the polynomials $a_0(z), a_1(z), \ldots, a_d(z)$ will be often considered as polynomials over $\mathbb{C}$. Then a *sequential solution* of equation (1) is a sequence of complex numbers $(c_n)_{n\in\mathbb{Z}}$ such that $a_d(n)c_{n+d} + \cdots + a_1(n)c_{n+1} + a_0(n)c_n = 0$ for all $n \in \mathbb{Z}$. The dimension of the $\mathbb{C}$-linear space of such solutions cannot be less than d, but is not necessarily equal to d. In Section 2 we show that for any integer $m \geq 0$ there exists an equation of the form (1) of order d such that the dimension of the space of sequential solutions of this equation is $d+m$. But the situation is different if we consider those sequential solutions that are the restrictions to $\mathbb{Z}$ of single-valued analytic solutions of (1) which are defined for all integer values of the argument. Such sequential solutions will be called *subanalytic* [1]. In Section 4 we show that the dimension of the $\mathbb{C}$-linear space of subanalytic solutions is always equal to d.

It is known that any equation of order d of the form (1) has a fundamental system of entire solutions (Praagman, 1986, Th. 5). We strengthen this result. In Section 5 we show that the space of subanalytic sequential solutions coincides with the space of those sequences that are the restrictions to $\mathbb{Z}$ of entire solutions of (1). This implies that the restrictions to $\mathbb{Z}$ of entire solutions make up a $\mathbb{C}$-linear space of dimension d. If I is an arbitrary segment $\{k, k+1, \ldots, l\}$ of integers such that $a_0(z) \neq 0$ for $z = k-1, k-2, \ldots$, $a_d(z-d) \neq 0$ for $z = l+1, l+2, \ldots$, and $l-k+1 \geq d$, then any sequential solution is uniquely defined by the values of those elements whose indices belongs to I. We show that a basis of the restrictions to I of all entire solutions of (1) can be found algorithmically (Sections 3.3, 5).

In Section 6 we consider d–dimensional ($d \geq 1$) hypergeometric sequences, i.e., sequential resp. subanalytic solutions of consistent systems of first-order difference equations for a single unknown function:

$$p_i(z_1, \ldots, z_d)y(z_1, \ldots, z_{i-1}, z_i+1, z_{i+1}, \ldots, z_d) = q_i(z_1, \ldots, z_d)y(z_1, \ldots, z_d),$$

where $(z_1, \ldots, z_d) \in \mathbb{C}^d$, and p_i, q_i are non-zero polynomials which are relatively prime for each $i \in \{1, 2, \ldots, d\}$. We show that the dimension of the space of subanalytic solutions is always at most 1, and that this dimension may be equal to 0 for some systems (although the dimension of the space of all sequential solutions is always positive).

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[1] No direct relation to the *subanalytic functions* of real algebraic geometry.

Subanalytic solutions have applications in computer algebra. In Section 7 we show that some implementations of certain well-known summation algorithms (Gosper's algorithm (Gosper, 1978; Petkovšek, Wilf, Zeilberger, 1996), Zeilberger's algorithm (Zeilberger, 1991; Petkovšek, Wilf, Zeilberger, 1996), Accurate Summation algorithm (Abramov, van Hoeij, 1999)) in existing computer algebra systems work correctly when the input sequence is a subanalytic solution of an equation or a system, but can give incorrect results for some sequential solutions.

In addition to subanalytic solutions we also consider the so-called *subformal solutions*, whose values are derived from formal power series instead of from convergent ones. This allows us to consider an arbitrary field K of characteristic zero as the ground field in (1), and, moreover, we show that if $K = \mathbb{C}$ then the $\mathbb{C}$-linear spaces of subanalytic and subformal solutions of equation (1) coincide (Section 4.2). This simplifies proofs of some statements in the case $K = \mathbb{C}$ since we do not need to treat convergent and divergent series separately.

Short reviews of some of the results of this paper were given in (Abramov, 2008; Abramov et al., 2008).

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2. Sequential solutions

Let K (the *ground field*) be an arbitrary field of characteristic zero. We start with linear difference equations of the form (1) where $a_1(z), a_2(z), \ldots, a_{d-1}(z) \in K[z]$, $a_0(z), a_d(z) \in K[z] \setminus \{0\}$, $\gcd(a_0(z), \ldots, a_d(z)) = 1$. We associate with equation (1) the linear difference operator

$$L = a_d(z)E^d + \cdots + a_1(z)E + a_0(z) \in K[z, E], \tag{2}$$

where E is the shift operator: $E(y(z)) = y(z+1)$. Equation (1) can be rewritten in the form $L(y) = 0$.

In the rest of this paper, L will denote an operator of the form (2) ($K = \mathbb{C}$ in Sections 4, 5). By a *solution of* L we will mean a solution of the equation $L(y) = 0$.

Definition 1. A sequence of elements of K indexed by the integers

$$c : \mathbb{Z} \to K, \quad c = (c_n),$$

is a *sequential* solution of operator (2) if

$$a_d(n)c_{n+d} + \cdots + a_1(n)c_{n+1} + a_0(n)c_n = 0$$

for all $n \in \mathbb{Z}$. The K-linear space of sequential solutions of L will be denoted by $V(L)$.

A segment of integer numbers

$$I = \{k, k+1, \ldots, l\}, \quad k, l \in \mathbb{Z}, \quad k \le l,$$

is an *essential segment* of (2) if

- the polynomial $a_d(z - d)$ has no integer roots greater than l,
- the polynomial $a_0(z)$ has no integer roots smaller than k,

- $l - k + 1 \geq d$.

If I is an essential segment of operator (2) then any sequential solution c is uniquely determined by the values c_n, $n \in I$. Therefore to describe $V(L)$ it is sufficient to find a basis of the restriction of $V(L)$ to I.

Theorem 2. *Let L be an operator of the form (2) and $V(L)$ the K-linear space of its sequential solutions. Then:*

(i) for any L of the form (2) we have $\dim V(L) \geq d$*;*

(ii) for any integer $d > 0$, $m \geq 0$ there exists an operator L of order d such that $\dim V(L) = d + m$.

Proof. (i) Let I be an essential segment of L. The restriction of $V(L)$ to I consists of all the vectors $(c_k, c_{k+1}, \ldots, c_l)$ that satisfy

$$\sum_{i=0}^{d} a_i(n) c_{n+i} = 0, \quad \text{for } n = k, k+1, \ldots, l-d.$$

This is a system of $l - d - k + 1 = \#(I) - d \geq 0$ linear algebraic equations for the $l - k + 1$ unknowns $c_k, c_{k+1}, \ldots, c_l$. Let $A \in K^{(l-d-k+1)\times(l-k+1)}$ be the matrix of this system. Then $\dim V(L) = \text{nullity}(A)$, and from the rank-nullity theorem

$$\begin{aligned}\dim V(L) &= \#(\text{columns of } A) - \text{rank}(A)\\ &\geq \#(\text{columns of } A) - \#(\text{rows of } A)\\ &= (l-k+1) - (l-d-k+1) \;=\; d.\end{aligned}$$

(ii) The case $d = 1$ has been proven in (Abramov, Petkovšek, 2008). It can be generalized as follows: take the operator

$$L_{d,m} \;= (E^d - 1) \circ q_{d,m}(z) = q_{d,m}(z+d)E^d - q_{d,m}(z) \tag{3}$$

where $q_{d,m}(z) = \prod_{k=0}^{m}(z - (2k+1)d)$.

For any sequential solution (u_n) of the operator $E^d - 1$ one has, for all $n \in \mathbb{Z}$

$$u_n = \sum_{k=0}^{d-1} u_k \delta_{k,\bar{n}}$$

where $\bar{n}$ denotes the remainder of n modulo d, δ being the Kronecker delta.

If (c_n) is a sequential solution of $L_{d,m}$ then the sequence $(q_{d,m}(n)c_n)$ is a solution of $E^d - 1$ and hence

$$q_{d,m}(n)c_n = \sum_{k=0}^{d-1} q_{d,m}(k) c_k \delta_{k,\bar{n}}$$

for all $n \in \mathbb{Z}$.

Putting $n = d$ (or $n =$ any root of $q_{d,m}(z)$) in the above equality shows that $q_{d,m}(0)c_0 = 0$. It then follows that $c_0 = 0$ and

$$c_n = \frac{1}{q_{d,m}(n)} \sum_{k=1}^{d-1} q_{d,m}(k) c_k \delta_{k,\bar{n}},$$

for all $n \in \mathbb{Z} \setminus \{d, 3d, \ldots, (2m+1)d\}$. Hence (c_n) is uniquely determined by the $d + m$ constants: $c_1, \ldots, c_{d-1}, c_d, c_{3d}, \ldots, c_{(2m+1)d}$.

A basis of $V(L_{d,m})$ is given by the following $m + d$ sequences:

$$c_n^{(k)} = \begin{cases} 0, & \text{if } n \in \{d, 3d, \ldots, (2m+1)d\}, \\ \frac{q_{d,m}(k)}{q_{d,m}(n)} \delta_{k,\bar{n}}, & \text{otherwise} \end{cases} \quad \text{for } k = 1, \ldots d-1,$$

$$c_n^{(d+j)} = \delta_{n,(2j+1)d} \qquad \text{for } j = 0, \ldots m.$$

□

Example 1. $(d = 1, m = 2)$ *Let*

$$L = z(z-2)(z-4)E - (z-1)(z-3)(z-5). \tag{4}$$

The segment $I = \{1,\ 2,\ 3,\ 4,\ 5\}$ is an essential segment of L. A basis for the restriction of $V(L)$ to I (i.e., a basis of fragments $(c_1, c_2, c_3, c_4, c_5)$ of sequential solutions) is

$$(1, 0, 0, 0, 0), \quad (0, 0, 1, 0, 0), \quad (0, 0, 0, 0, 1).$$

There are three sequential solutions

$$c^{(1)}, \quad c^{(2)}, \quad c^{(3)}$$

corresponding to the elements of this basis. It is easy to show that

$$c_n^{(1)} = \delta_{n,1}, \quad c_n^{(2)} = \delta_{n,3}, \quad c_n^{(3)} = \delta_{n,5}, \tag{5}$$

$n \in \mathbb{Z}$.

When $d = 1, m = 2$, the operator (3) coincides with the operator $L_{1,2}$ from Example 1. In the following example $d = 1, m = 1$, but the operator does not coincide with $L_{1,1}$.

Example 2. *Let*

$$L = 2(z+1)(z-2)E - (2z-1)(z-1). \tag{6}$$

The segment $I = \{1,\ 2,\ 3\}$ is an essential segment of L. A basis for the restriction of $V(L)$ to I (i.e., a basis of fragments (c_1, c_2, c_3) of sequential solutions) is

$$\left(-\frac{1}{8},\ 0,\ \frac{1}{64}\right), \quad \left(-\frac{1}{4},\ 0,\ \frac{1}{64}\right).$$

There are two sequential solutions

$$c_n^{(1)} = \lim_{v\to n} \frac{\Gamma(2v-2)}{\Gamma(v+1)\Gamma(v-2)4^v} = \frac{(n-2)\Gamma(n-\frac{1}{2})}{8\sqrt{\pi}\Gamma(n+1)}, \quad n \in \mathbb{Z},$$

and

$$c_n^{(2)} = \frac{\binom{2n-3}{n}}{4^n}, \quad n \in \mathbb{Z},$$

corresponding to the elements of this basis. The sequences $c^{(1)}$ and $c^{(2)}$ coincide when $n > 1$ or $n < 0$, but in combinatorics $\binom{2n-3}{n}$ is usually defined to be -1 when $n = 1$ and 1 when $n = 0$, while $\lim_{v\to 1} \frac{\Gamma(2v-2)}{\Gamma(v+1)\Gamma(v-2)} = -\frac{1}{2}$ and $\lim_{v\to 0} \frac{\Gamma(2v-2)}{\Gamma(v+1)\Gamma(v-2)} = \frac{1}{2}$.

3. Subformal solutions

3.1. Formal sequences of bounded altitude

As usual, we denote by $K[[\varepsilon]]$ the ring of formal power series in ε and by $K((\varepsilon))$ the field of formal Laurent series, i.e., the quotient field of the ring $K[[\varepsilon]]$ (here ε is a new variable, rather than a "small number"). If $s(\varepsilon) \in K((\varepsilon))$ and

$$s(\varepsilon) = t_m\varepsilon^m + t_{m+1}\varepsilon^{m+1} + \cdots, \quad t_m \neq 0 \tag{7}$$

for some $m \in \mathbb{Z}$, then we write $\nu(s) = m$, setting $\nu(0) = \infty$. It is well known that ν is a valuation, $\nu(st) = \nu(s) + \nu(t)$ and $\nu(s+t) \geq \min\{\nu(s), \nu(t)\}$ for all $s(\varepsilon), t(\varepsilon) \in K((\varepsilon))$. Write $[\varepsilon^k]s$ for the coefficient of ε^k in $s(\varepsilon)$. For any $s(\varepsilon) \in K((\varepsilon))$ and $m \in \mathbb{Z}$, define the *truncation of $s(\varepsilon)$ at m* as the Laurent polynomial

$$s(\varepsilon)\,|_m \;=\; \begin{cases} 0, & \text{if } \nu(s) > m, \\ \displaystyle\sum_{k=\nu(s)}^{m} \left([\varepsilon^k]s\right)\varepsilon^k, & \text{otherwise.} \end{cases}$$

For $s(\varepsilon), t(\varepsilon) \in K((\varepsilon))$ and $m \in \mathbb{Z}$, write $s \sim_m t$ if $\nu(s-t) > m$. Then $\sim_m$ is an equivalence relation in $K((\varepsilon))$ with the following properties:

Lemma 1. Let $s(\varepsilon), t(\varepsilon), s'(\varepsilon), t'(\varepsilon) \in K((\varepsilon))$ and $m, k \in \mathbb{Z}$. Then
- (i) $s|_m \sim_m s$,
- (ii) $s \sim_m s' \implies st \sim_{m+\nu(t)} s't$,
- (iii) $s \sim_m t,\ m \geq k \implies s \sim_k t$,
- (iv) $s \sim_m s',\ t \sim_m t' \implies s + t \sim_m s' + t'$.

Proof.
- (i) If $\nu(s) > m$ then $\nu(s|_m - s) = \nu(-s) = \nu(s) > m$. Otherwise $\nu(s|_m - s) = \nu(\sum_{k=m+1}^{\infty}\left([\varepsilon^k]s\right)\varepsilon^k) > m$.
- (ii) $\nu(st - s't) = \nu((s-s')t) = \nu(s-s') + \nu(t) > m + \nu(t)$.
- (iii) This is obvious.

(iv) $\nu(s+t-(s'+t')) = \nu((s-s')+(t-t')) \geq \min\{\nu(s-s'),\nu(t-t')\} > m.$ $\square$

A sequence $F : \mathbb{Z} \to K((\varepsilon))$ will be called a *formal sequence*. Set

$$B_K = \{F \mid F : \mathbb{Z} \to K((\varepsilon)),\ \min_{n\in\mathbb{Z}} \nu(F_n) > -\infty\}.$$

The set B_K is evidently a $K((\varepsilon))$-linear space. If $F \in B_K$ then the value of $\min_{n\in\mathbb{Z}}(\nu(F_n))$ is the *altitude* of F; we will use the notation $\mathrm{alt}(F)$ for this value. Let $\mathrm{alt}(F) = m < \infty$. Then we can consider the sequence $f : \mathbb{Z} \to K$, where f_n is the coefficient of ε^m in the series F_n. The sequence f is called the *bottom* of F and denoted by $\mathrm{bott}(F)$. Notice that the altitude of the zero sequence is positive infinity: $\mathrm{alt}(0) = \infty$. Set $\mathrm{bott}(0) = 0$.

For $F, G \in B_K$ and $m \in \mathbb{Z}$, write $F \sim_m G$ if $F_n \sim_m G_n$ for all $n \in \mathbb{Z}$.

Lemma 2. Let $F, G \in B_K$. If $F \sim_{\mathrm{alt}(G)} G$, then $\mathrm{alt}(F) = \mathrm{alt}(G)$ and $\mathrm{bott}(F) = \mathrm{bott}(G)$.

Proof. Write $F_n = \sum_{k=-\infty}^{\infty} a_{k,n}\varepsilon^k$ and $G_n = \sum_{k=-\infty}^{\infty} b_{k,n}\varepsilon^k$. By assumption, $\nu(F_n - G_n) > \mathrm{alt}(G)$ for all $n \in \mathbb{Z}$, so $a_{k,n} = b_{k,n}$ for all $k \leq \mathrm{alt}(G)$ and $n \in \mathbb{Z}$. Therefore

(1) $k < \mathrm{alt}(G) \implies a_{k,n} = 0$ for all $n \in \mathbb{Z}$,

(2) $k = \mathrm{alt}(G) \implies a_{k,n} \neq 0$ for some $n \in \mathbb{Z}$.

It follows that $\mathrm{alt}(F) = \mathrm{alt}(G)$, and $a_{\mathrm{alt}(F),n} = b_{\mathrm{alt}(G),n}$ for all $n \in \mathbb{Z}$. Hence $\mathrm{bott}(F) = \mathrm{bott}(G)$. $\square$

Proposition 1. If $F^{(1)}, F^{(2)}, \ldots, F^{(l)} \in B_K$ are $K((\varepsilon))$-linearly dependent then $\mathrm{bott}(F^{(1)}), \mathrm{bott}(F^{(2)}), \ldots, \mathrm{bott}(F^{(l)})$ are K-linearly dependent.

Proof. Let $\sum_{j=1}^{l} s_j(\varepsilon)F^{(j)} = 0$ where not all of $s_1(\varepsilon), s_2(\varepsilon), \ldots, s_l(\varepsilon) \in K((\varepsilon))$ are zero series. Set

$$m \;=\; \min_{1\leq j\leq l} \mathrm{alt}\left(s_j(\varepsilon)F^{(j)}\right) \;=\; \min_{1\leq j\leq l}\left(\nu(s_j) + \mathrm{alt}\left(F^{(j)}\right)\right).$$

If $m = \infty$ then $s_j(\varepsilon)F^{(j)} = 0$ for all j, hence $F^{(j)} = 0$ for some j, and the assertion holds. Suppose that $m < \infty$. For $j = 1, 2, \ldots, l$ define

$$t_j = \begin{cases} [\varepsilon^{\nu(s_j)}]s_j(\varepsilon), & \text{if } \mathrm{alt}\left(s_j(\varepsilon)F^{(j)}\right) = m, \\ 0, & \text{otherwise.} \end{cases}$$

Then not all t_j are zero and

$$\begin{aligned} \sum_{j=1}^{l} t_j \,\mathrm{bott}(F^{(j)}) &= \sum_{\mathrm{alt}\left(s_j(\varepsilon)F^{(j)}\right)=m} \left([\varepsilon^{\nu(s_j)}]s_j(\varepsilon)\right)\cdot \mathrm{bott}(F^{(j)}) \\ &= \mathrm{bott}\left(\sum_{j=1}^{l} s_j(\varepsilon)F^{(j)}\right) \;=\; \mathrm{bott}(0) \;=\; 0. \end{aligned}$$ $\square$

The converse of Proposition 1 is false as witnessed by Example 3 below. Nevertheless, if $F^{(1)}, F^{(2)}, \ldots, F^{(l)} \in B_K$ are $K((\varepsilon))$-linearly independent then there always exist

$K[\varepsilon, \varepsilon^{-1}]$-linear combinations $G^{(1)}, G^{(2)}, \ldots, G^{(l)}$ of $F^{(1)}, F^{(2)}, \ldots, F^{(l)}$ whose bottoms are K-linearly independent (see Proposition 2).

Example 3. *Define formal sequences $F^{(1)}, F^{(2)}, \ldots, F^{(l)}$ by setting*

$$F_n^{(j)} = \begin{cases} 1, \text{ if } n = 0, \\ \varepsilon, \text{ if } n = j, \\ 0, \text{ otherwise,} \end{cases}$$

for all $j \in \{1, \ldots, l\}$ and $n \in \mathbb{Z}$. These sequences are clearly $K((\varepsilon))$-linearly independent, but their bottoms are K-linearly dependent since they coincide: $\operatorname{bott}(F^{(j)}) = (\delta_{n,0})_{n\in\mathbb{Z}}$ *for $j = 1, \ldots, l$. Another such example is given in Section 4.1 (Example 7).*

Lemma 3. If $F^{(1)}, F^{(2)}, \ldots, F^{(l)} \in B_K$ are $K((\varepsilon))$-linearly independent then there are $n_1, \ldots, n_l \in \mathbb{Z}$ and $G^{(1)}, \ldots, G^{(l)} \in K((\varepsilon))F^{(1)} + \cdots + K((\varepsilon))F^{(l)}$ such that $\operatorname{alt}\left(G^{(j)}\right) = 0$ and $G_{n_k}^{(j)} = \delta_{j,k}$, for $j, k \in \{1, \ldots, l\}$.

Proof. By induction on l.

$l = 1$: $F^{(1)}$ is linearly independent, hence $F^{(1)} \neq 0$. Let $n_1 \in \mathbb{Z}$ be such that $\operatorname{alt}\left(F^{(1)}\right) = \nu\left(F_{n_1}^{(1)}\right)$. Set $G^{(1)} = F^{(1)}/F_{n_1}^{(1)}$. Then $\operatorname{alt}\left(G^{(1)}\right) = \operatorname{alt}\left(F^{(1)}\right) - \nu\left(F_{n_1}^{(1)}\right) = 0$ and $G_{n_1}^{(1)} = 1 = \delta_{1,1}$.

$l \to l + 1$: By inductive hypothesis there are $n_1, \ldots, n_l \in \mathbb{Z}$ and $H^{(1)}, \ldots, H^{(l)} \in K((\varepsilon))F^{(1)} + \cdots + K((\varepsilon))F^{(l)}$ such that $\operatorname{alt}\left(H^{(j)}\right) = 0$ and $H_{n_k}^{(j)} = \delta_{j,k}$, for $j, k \in \{1, \ldots, l\}$. Clearly $H^{(1)}, \ldots, H^{(l)}$ are $K((\varepsilon))$-linearly independent. We claim that $H^{(1)}, \ldots, H^{(l)}, F^{(l+1)}$ are $K((\varepsilon))$-linearly independent as well. Indeed, if $s_1(\varepsilon), \ldots, s_{l+1}(\varepsilon) \in K((\varepsilon))$ are such that $\sum_{j=1}^{l} s_j(\varepsilon)H^{(j)} + s_{l+1}(\varepsilon)F^{(l+1)} = 0$, then there are $t_1(\varepsilon), \ldots, t_{l+1}(\varepsilon) \in K((\varepsilon))$ such that $\sum_{j=1}^{l} t_j(\varepsilon)F^{(j)} + s_{l+1}(\varepsilon)F^{(l+1)} = 0$. It follows that $s_{l+1}(\varepsilon) = 0$, hence $\sum_{j=1}^{l} s_j(\varepsilon)H^{(j)} = 0$ and therefore $s_1(\varepsilon) = \cdots = s_l(\varepsilon) = 0$, proving the claim.

Let $\tilde{G} = F^{(l+1)} - \sum_{j=1}^{l} F_{n_j}^{(l+1)} H^{(j)}$ and $G = \varepsilon^{-\operatorname{alt}(\tilde{G})}\tilde{G}$. Then $G \neq 0$, $G \in K((\varepsilon))F^{(1)} + \cdots + K((\varepsilon))F^{(l+1)}$, $\operatorname{alt}(G) = 0$, and $G_{n_k} = \varepsilon^{-\operatorname{alt}(\tilde{G})}\left(F_{n_k}^{(l+1)} - \sum_{j=1}^{l} F_{n_j}^{(l+1)}\delta_{j,k}\right) = 0$ for $k \in \{1, \ldots, l\}$. Let $n_{l+1} \in \mathbb{Z}$ be such that $\nu\left(G_{n_{l+1}}\right) = \operatorname{alt}(G) = 0$. Define

$$\begin{aligned} G^{(j)} &= H^{(j)} - H_{n_{l+1}}^{(j)}\left(G_{n_{l+1}}\right)^{-1} G \quad \text{for } j = 1, \ldots, l, \\ G^{(l+1)} &= G/G_{n_{l+1}}. \end{aligned}$$

Then $G_{n_k}^{(j)} = H_{n_k}^{(j)} = \delta_{j,k}$ for $j, k \in \{1, \ldots, l\}$, $G_{n_{l+1}}^{(j)} = 0$ for $j \in \{1, \ldots, l\}$, $G_{n_k}^{(l+1)} = G_{n_k}/G_{n_{l+1}} = 0$ for $k \in \{1, \ldots, l\}$, and $G_{n_{l+1}}^{(l+1)} = 1$. So $G^{(j)} \in K((\varepsilon))F^{(1)} + \cdots + K((\varepsilon))F^{(l+1)}$, $G_{n_k}^{(j)} = \delta_{j,k}$, and $\operatorname{alt}\left(G^{(j)}\right) = 0$, for $j, k \in \{1, \ldots, l + 1\}$, proving the lemma. $\square$

Proposition 2. If $F^{(1)}, F^{(2)}, \ldots, F^{(l)} \in B_K$ are $K((\varepsilon))$-linearly independent then there are $G^{(1)}, G^{(2)}, \ldots, G^{(l)} \in K[\varepsilon, \varepsilon^{-1}]F^{(1)} + K[\varepsilon, \varepsilon^{-1}]F^{(2)} + \cdots + K[\varepsilon, \varepsilon^{-1}]F^{(l)}$ such that

$$\mathrm{bott}(G^{(1)}), \mathrm{bott}(G^{(2)}), \ldots, \mathrm{bott}(G^{(l)})$$

are K-linearly independent.

Proof. From Lemma 3 it follows that there are $s_{j,k}(\varepsilon) \in K((\varepsilon))$ for $j, k \in \{1, \ldots, l\}$ such that for the formal sequences

$$G^{(j)} = \sum_{k=1}^{l} s_{j,k}(\varepsilon) F^{(k)}, \quad j = 1, \ldots, l,$$

we have $\mathrm{alt}\left(G^{(j)}\right) = 0$ and $\mathrm{bott}\left(G^{(j)}\right)_{n_k} = \delta_{j,k}$, for $j, k \in \{1, \ldots, l\}$. Let

$$\tilde{G}^{(j)} = \sum_{k=1}^{l} s_{j,k}(\varepsilon)\,|_{-\alpha}\; F^{(k)}, \quad \text{for } j = 1, \ldots, l,$$

where $\alpha = \min_{1 \le k \le l} \mathrm{alt}\left(F^{(k)}\right)$. Then $\tilde{G}^{(j)} \in K[\varepsilon, \varepsilon^{-1}]F^{(1)} + \cdots + K[\varepsilon, \varepsilon^{-1}]F^{(l)}$. By using all the claims of Lemma 1 consecutively, we have

$$\begin{aligned}
& s_{j,k}(\varepsilon)\,|_{-\alpha} \;\sim_{-\alpha}\; s_{j,k}(\varepsilon) \quad \text{(by (i))} \\
\Longrightarrow \quad & s_{j,k}(\varepsilon)\,|_{-\alpha}\; F_n^{(k)} \;\sim_{-\alpha+\nu(F_n^{(k)})}\; s_{j,k}(\varepsilon) F_n^{(k)} \quad \text{(by (ii))} \\
\Longrightarrow \quad & s_{j,k}(\varepsilon)\,|_{-\alpha}\; F_n^{(k)} \;\sim_0\; s_{j,k}(\varepsilon) F_n^{(k)} \quad \text{(by (iii))} \\
\Longrightarrow \quad & \sum_{k=1}^{l} s_{j,k}(\varepsilon)\,|_{-\alpha}\; F_n^{(k)} \;\sim_0\; \sum_{k=1}^{l} s_{j,k}(\varepsilon) F_n^{(k)} \quad \text{(by (iv))}.
\end{aligned}$$

This holds for all $n \in \mathbb{Z}$, hence $\tilde{G}^{(j)} \sim_0 G^{(j)}$. Since $\mathrm{alt}(G^{(j)}) = 0$, Lemma 2 implies that $\mathrm{alt}\left(\tilde{G}^{(j)}\right) = \mathrm{alt}\left(G^{(j)}\right) = 0$ and $\mathrm{bott}\left(\tilde{G}^{(j)}\right) = \mathrm{bott}\left(G^{(j)}\right)$, for $j \in \{1, \ldots, l\}$. Hence the sequences $\mathrm{bott}\left(\tilde{G}^{(j)}\right)$, $j = 1, \ldots, l$, are K-linearly independent, proving the assertion. $\square$

It is easy to see that the statement of Proposition 2 can be formulated in a more general form (and the given proof will not need any changes) considering the ring $K[s(\varepsilon), (s(\varepsilon))^{-1}]$ instead of $K[\varepsilon, \varepsilon^{-1}]$, where $s(\varepsilon)$ is any fixed series in ε such that $\nu(s) = 1$. So the following proposition is valid.

Proposition 3. Let $l \in \mathbb{N}$ and $F^{(j)} \in B_K$, $j = 1, 2, \ldots, l$. Let $s(\varepsilon) \in K[[\varepsilon]]$, $\nu(s) = 1$, and formal sequences $F^{(1)}, F^{(2)}, \ldots, F^{(l)}$ are $K((\varepsilon))$-linearly independent. Then there exist $G^{(j)}$, $j = 1, 2, \ldots, l$, belonging to the set

$$K[s(\varepsilon), (s(\varepsilon))^{-1}]F^{(1)} + K[s(\varepsilon), (s(\varepsilon))^{-1}]F^{(2)} + \ldots + K[s(\varepsilon), (s(\varepsilon))^{-1}]F^{(l)}$$

such that

$$\mathrm{bott}(G^{(1)}), \mathrm{bott}(G^{(2)}), \ldots, \mathrm{bott}(G^{(l)}),$$

are K-linearly independent.

3.2. Subformal solutions

If $a(z)$ is a polynomial or a rational function then we set $\hat{a}(z) = a(z+\varepsilon)$ where ε is a variable, as in Section 3.1. We associate with L the operator

$$\hat{L} = \hat{a}_d(z)E^d + \cdots + \hat{a}_1 E(z) + \hat{a}_0(z) = a_d(z+\varepsilon)E^d + \cdots + a_1(z+\varepsilon)E + a_0(z+\varepsilon),$$

considering each of its coefficients $\hat{a}_i(z)$ as a formal sequence: for any integer value of z the value of $\hat{a}_i(z)$ belongs to $K[[\varepsilon]]$. This operator acts on formal sequences. The operator $\hat{L}$ is called the *deformation* of L.

A sequence $F : \mathbb{Z} \to K((\varepsilon))$ which satisfies $\hat{L}F = 0$ will be called a *formal sequential solution* of the operator $\hat{L}$. The set of formal sequential solutions of $\hat{L}$ is a $K((\varepsilon))$-linear space that will be denoted by $V(\hat{L})$.

An advantage of $\hat{L}$ over L is that neither the leading nor the trailing coefficient of $\hat{L}$ vanishes when z is any integer number. We can always divide by the value of such coefficient. This implies in particular that $\dim V(\hat{L}) = d = \operatorname{ord} L$ (for arbitrary given $F_0, F_1, \ldots, F_{d-1} \in K((\varepsilon))$ the element F_n, $n \in \mathbb{Z}$, of $F \in V(\hat{L})$ is uniquely defined).

Example 4. *Let $L = (z+1)E - z$. Then $\hat{L} = (z+1+\varepsilon)E - (z+\varepsilon)$. Set*

$$F_n = \begin{cases} \frac{1}{\varepsilon}, & \text{if } n = 0, \\ -\sum_{i=0}^{\infty} \left(-\frac{1}{n}\right)^{i+1} \varepsilon^i, & \text{otherwise.} \end{cases} \tag{8}$$

It is possible to check that $F \in V(\hat{L})$.

Note that the idea of the deformation of difference operators and computing truncated power series at each integer point is used in (van Hoeij, 1999) for computing hypergeometric solutions. Later this idea was used in (Abramov, van Hoeij, 1999), (Abramov, van Hoeij, 2003). A similar idea in the multidimensional case was used in (Abramov, Petkovšek, 2008).

Since coefficients of L are polynomials, we have $V(\hat{L}) \subset B_K$. Indeed, let $F \in V(\hat{L})$, and m_i be the sum of multiplicities of all integer roots of $a_i(z)$, then $\operatorname{alt}(F) \geq \min_{0 \leq j \leq d-1} \nu(F_j) - \max(m_0, m_d)$.

Proposition 4. (Abramov, Petkovšek, 2007) Let $F \in V(\hat{L})$. Then $\operatorname{bott}(F) \in V(L)$.

Definition 3. A formal solution $F : \mathbb{Z} \to K[[\varepsilon]]$ of $\hat{L}$ will be called a *Taylor formal solution*. A sequential solution f is a *subformal (sequential) solution* of L if $\hat{L}$ has a formal Taylor solution F such that f_n is the constant term of the series F_n, $n \in \mathbb{Z}$. The K-linear space of subformal solutions of L will be denoted by $V_{\text{sf}}(L)$.

The fact that $V(\hat{L})$ is a $K((\varepsilon))$-linear space and Proposition 4 imply that a sequential solution g of L belongs to $V_{\text{sf}}(L)$ iff $g = \operatorname{bott}(G)$, where $G \in V(\hat{L})$ (we can consider $F = \varepsilon^{-\operatorname{alt}(G)} G$).

Example 4 (continued) *We have* $\operatorname{alt}(F) = -1$, $\operatorname{bott}(F) = (\delta_{0,n})$, *and* $L((\delta_{0,n})) = 0$.

Theorem 4. $\dim V_{\text{sf}}(L) = \operatorname{ord} L$.

Proof. Let $\operatorname{ord} L = d \geq 1$. We will use the fact that $V(\hat{L})$ is a $K((\varepsilon))$-linear space of dimension $d = \operatorname{ord} L$.

$\dim V_{\mathrm{sf}}\ (L) \leq d$: Let

$$F^{(1)}, F^{(2)}, \ldots, F^{(d+1)} \in V(\hat{L}), \tag{9}$$

and $f^{(i)} = \operatorname{bott}(F^{(i)})$, $i = 1, 2, \ldots, d+1$. Formal sequences (9) are $K((\varepsilon))$-linearly dependent and by Proposition 1 the sequences $f^{(i)}$, $i = 1, 2, \ldots, d+1$, are linearly dependent too.

$\dim V_{\mathrm{sf}}\ (L) \geq d$: Let $F^{(i)}$, $i = 1, 2, \ldots, d$, be $K((\varepsilon))$-linearly independent elements of $V(\hat{L})$. By Proposition 2 the set

$$U_d = K[\varepsilon, \varepsilon^{-1}]F^{(1)} + K[\varepsilon, \varepsilon^{-1}]F^{(2)} + \ldots + K[\varepsilon, \varepsilon^{-1}]F^{(d)}$$

contains formal sequences $G^{(i)}, i = 1, 2, \ldots, d$, such that the sequences $g^{(i)} = \operatorname{bott}(G^{(i)})$, $i = 1, 2, \ldots, d$, are K-linear independent. Now the claim follows from $U_d \subset V(\hat{L})$. $\square$

Going back to Example 1 we see that it is impossible that all three sequential solutions (5) of the first order operator (4) could be subformal.

3.3. A basis of the space of subformal solutions

For our operator L and an essential segment I (see Section 2) we can construct a basis of the restrictions to I of all subformal solutions of L. The algorithm is based on the algorithm from (Abramov, van Hoeij, 2003) for finding values of subformal solutions, the idea of that algorithm is as follows.

Let $q \in \mathbb{Z}$ and $F_q, F_{q+1}, \ldots, F_{q+d-1}$ be given elements of $K[[\varepsilon]]$, then, theoretically speaking, by using the operator $\hat{L}$, we can compute any element F_p of the sequential solution $F = (F_n)$ of the equation $\hat{L}(y) = 0$. It may be that $F_p \in K((\varepsilon)) \setminus K[[\varepsilon]]$ for a given integer $p \notin \{q, q+1, \ldots, q+d-1\}$. Starting with p, q we can write down in advance a finite set $C_{q,p}$ of linear equations for a finite number of coefficients of power series $F_q, F_{q+1}, \ldots, F_{q+d-1}$ which guarantee that $F_p \in K[[\varepsilon]]$. Indeed, set

$$\begin{array}{rcccccccc} F_q & = & u_{q,0} & + & u_{q,1}\varepsilon & + & u_{q,2}\varepsilon^2 & + & \cdots, \\ F_{q+1} & = & u_{q+1,0} & + & u_{q+1,1}\varepsilon & + & u_{q+1,2}\varepsilon^2 & + & \cdots, \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ F_{q+d-1} & = & u_{q+d-1,0} & + & u_{q+d-1,1}\varepsilon & + & u_{q+d-1,2}\varepsilon^2 & + & \cdots, \end{array} \tag{10}$$

where series on the right are generic. When we compute F_p we get a series, and each of its coefficients is a linear form in a finite set of $u_{i,j}$. The series F_p may contain negative exponents of ε. We can find desired conditions on the coefficients $u_{i,j}$ in (10) after equating the corresponding coefficients to zero.

If $q \in \mathbb{Z}$ is fixed then the systems $C_{q,p}$ for any integer $p > q + d - 1$ can be found algorithmically using truncated series (taking into account the terms of power series (10) till ε^m where m is the sum of multiplicities of all integer roots of the polynomial $a_d(z-d)$).

It is also possible to find the linear form $l_{q,p}$ which represents the coefficient of ε^0 in the series F_p.

Now we are able to describe how to construct a basis of the restrictions to I of all subformal solutions of L. Let $I = \{k, k+1, \dots, l\}$ be an essential segment of L. If $l = k+d-1$ then we can take any basis of K^d and this will be a basis of the restriction of $V_{\rm sf}\ (L)$ to I. Suppose that $l > k+d-1$. Take $q = k$ and construct $C_{q,p}$, $l_{q,p}$ for $p = k+d, k+d+1, \dots, l$. Add to linear equations from all constructed $C_{q,p}$ the equations $u_{p,0} = l_{q,p}$, $p = k+d, k+d+1, \dots, l$. Denote by $\mathcal{A}$ the obtained system of linear algebraic equations. Let us construct a basis of the solution space of $\mathcal{A}$ and then construct the projection of each vector of this basis into the space of vectors $(u_{k,0}, u_{k+1,0}, \dots, u_{l,0})$. Taking any basis of the K-linear space generated by such projections we get a basis of the restrictions to I of all subformal solutions of the operator L.

Example 5. *Let*

$$L = z^2E^2 + (1+z^2)E - z$$

and $F \in V(\hat{L})$. The segment $I = \{0, 1, 2\}$ is an essential segment of L.

Write

$$F_0 = u_{0,0} + u_{0,1}\varepsilon + u_{0,2}\varepsilon^2 + O(\varepsilon^3),$$

$$F_1 = u_{1,0} + u_{1,1}\varepsilon + u_{1,2}\varepsilon^2 + O(\varepsilon^3).$$

We calculate using $\hat{L}$:

$$F_2 = -\frac{u_{1,0}}{\varepsilon^2} + \frac{-u_{1,1} + u_{0,0}}{\varepsilon} - u_{1,0} - u_{1,2} + u_{0,1} + O(\varepsilon).$$

We find $C_{0,2} = \{-u_{1,0} = 0,\ u_{0,0} - u_{1,1} = 0\}$ and $l_{0,2} = -u_{1,0} - u_{1,2} + u_{0,1}$. We get the system $\mathcal{A}$:

$$\begin{array}{rcccccl} & -\ u_{1,0} & & & & & = 0, \\ u_{0,0} & & & & -\ u_{1,1} & & = 0, \\ & u_{1,0} & +\ u_{2,0} & -\ u_{0,1} & & +\ u_{1,2} & = 0. \end{array}$$

A basis of the space of its solutions

$$(u_{0,0},\ u_{1,0},\ u_{2,0},\ u_{0,1},\ u_{1,1},\ u_{1,2})$$

is

$$(0,0,1,1,0,0),\ (0,0,-1,0,0,1),\ (1,0,0,0,1,0).$$

The projections of these vectors into the space of vectors $(u_{0,0}, u_{1,0}, u_{2,0})$ are

$$(0,0,1),\ (0,0,-1),\ (1,0,0),$$

and a basis of the space generated by these three vectors is

$$(0,0,1),\ (1,0,0). \tag{11}$$

It follows that the vectors (11) give a basis of subformal solutions restricted to I.

It can be shown that in the latter example the order of the space of all sequential solutions is equal to the order of L (the substitution $n = 0$ into $n^2 c_{n+2} + (1 + n^2)c_{n+1} - nc_n = 0$ gives $c_1 = 0$ for any sequential solution c of L). This means that any sequential solution of L is subformal. The situation is different in the following example.

Example 1 (continued) *We have*

$$L = z(z - 2)(z - 4)E - (z - 1)(z - 3)(z - 5),$$

and $I = \{1, 2, 3, 4, 5\}$ is an essential segment of L. We find $C_{1,2} = C_{1,3} = C_{1,4} = C_{1,5} = \emptyset$ and $l_{1,2} = 0$, $l_{1,3} = -2u_{1,0}$, $l_{1,4} = 0$, $l_{1,5} = u_{1,0}$. We get the system $\mathcal{A}$:

$$\begin{array}{rrrrrl} & u_{2,0} & & & & = 0, \\ 2u_{1,0} & & + \ u_{3,0} & & & = 0, \\ & & & u_{4,0} & & = 0, \\ -u_{1,0} & & & & + \ u_{5,0} & = 0. \end{array}$$

A basis of the space of its solutions

$$(u_{1,0},\ u_{2,0},\ u_{3,0},\ u_{4,0},\ u_{5,0})$$

consists of the single vector

$$(1,\ 0,\ -2,\ 0,\ 1).$$

Projection produces the same vector. It follows that this vector gives a basis of subformal solutions restricted to I.

In particular, the sequence

$$(\delta_{n,1} - 2\delta_{n,3} + \delta_{n,5})$$

is a subformal solution of L. This implies that none of sequential solutions

$$(\delta_{n,1}), \quad (\delta_{n,3}), \quad (\delta_{n,5})$$

is subformal.

4. Submeromorphic and subanalytic solutions

In this section we suppose that $K = \mathbb{C}$ in (1), (2).

4.1. Submeromorphic solutions

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. For each $n \in \mathbb{Z}$ expand

$$\varphi(z) = c_{n,\rho_n}(z - n)^{\rho_n} + c_{n,\rho_n+1}(z - n)^{\rho_n+1} + \cdots$$

with $\rho_n \in \mathbb{Z}$ and $c_{n,\rho_n} \neq 0$. Define the formal sequence

$$\hat{\varphi} : \mathbb{Z} \to \mathbb{C}((\varepsilon)), \quad \hat{\varphi} = (\hat{\varphi}_n),$$

where for each $n \in \mathbb{Z}$

$$\hat{\varphi}_n = c_{n,\rho_n}\varepsilon^{\rho_n} + c_{n,\rho_n+1}\varepsilon^{\rho_n+1} + \cdots .$$

Set

$$\sigma(z) = \frac{e^{2i\pi z} - 1}{2i\pi}, \quad s(\varepsilon) = \sum_{j=0}^{\infty} \frac{(2i\pi)^j \varepsilon^{j+1}}{(j+1)!}. \tag{12}$$

Notice that $\sigma(z)$ is a 1-periodic entire function with simple zeros at all $z \in \mathbb{Z}$, and $\hat{\sigma}$ is a constant formal sequence with each of its elements equal to $s(\varepsilon)$.

An arbitrary meromorphic solution $\varphi(z)$ of L evidently has the following properties:

(M1) $\hat{\varphi} \in V(\hat{L})$;

(M2) the functions $\lambda(z) = \sigma(z)\varphi(z)$, $\xi(z) = (\sigma(z))^{-1}\varphi(z)$ *are meromorphic solutions of* L, *and* $\hat{\lambda} = s(\varepsilon)\hat{\varphi}$, $\hat{\xi} = (s(\varepsilon))^{-1}\hat{\varphi}$;

(M3) the function $\chi(z) = (\sigma(z))^{-\operatorname{alt}(\hat{\varphi})}\varphi(z)$ *is a meromorphic solution of* L *such that* $\operatorname{alt}(\hat{\chi}) = 0$ *and* $\operatorname{bott}(\hat{\varphi})$, $\operatorname{bott}(\hat{\chi})$ *are sequential solutions of* L *which coincide with the restriction of* $\chi(z)$ *to* $\mathbb{Z}$.

Example 4 (continued) *We have* $L = (z+1)E - z$, $\hat{L} = (z+1+\varepsilon)E - (z+\varepsilon)$, $\varphi = -\frac{1}{z}$ *and* $\hat{\varphi} = F$, *where* F *is defined by (8). By (M1), the equality* $\hat{L}(\hat{\varphi}) = 0$ *follows from* $L(\varphi) = 0$.

Example 6. *The* Γ*-function* $\Gamma(z)$ *is a meromorphic solution of* $L = E - z$, *and* $\Gamma(z)$ *has finite values when* $z = 1, 2, \ldots$, *and has simple poles when* $z = 0, -1, -2, \ldots$ *We have*

$$\operatorname{alt}(\hat{\Gamma}) = -1,$$

and

$$\operatorname{bott}(\hat{\Gamma}) = \begin{cases} 0, & \text{if } n > 0, \\ \frac{(-1)^{-n+1}}{(-n+1)!}, & \text{if } n \leq 0. \end{cases} \tag{13}$$

In accordance with (M3) the sequence (13) is a sequential solution of L.

It turns out that for any $L \in \mathbb{C}[z, E]$, $\operatorname{ord} L = d$, there exist meromorphic solutions $\varphi_1(z), \varphi_2(z), \ldots, \varphi_d(z)$ of L such that the corresponding formal sequential solutions

$$\hat{\varphi_1}, \ \hat{\varphi_2}, \ \ldots \ , \hat{\varphi_d}$$

are $\mathbb{C}((\varepsilon))$-linearly independent. This is a consequence of the following theorem.

Theorem 5. *An operator* $L \in \mathbb{C}[z, E]$ *of order* d *has* d *linearly independent meromorphic solutions* $\varphi_1(z), \varphi_2(z), \ldots, \varphi_d(z)$ *such that for some integer* l

(a) $\varphi_i(l + j) = \delta_{i,j}$, $i, j = 1, 2, \ldots, d$;

(b) $\varphi_i(z)$ *has no poles in the half-plane* $\operatorname{Re} z > l$, $i = 1, 2, \ldots, d$.

Proof. In an unpublished paper ((Ramis, 1988); see (Barkatou, 1989, pp. 97–101) where a brief summary of Ramis' method with some additions is given, and (Immink, 1999) for a complete proof), Ramis showed that a difference operator L has a basis of solutions (in a suitable space of functions) consisting of d functions $\varphi_1, \varphi_2, \ldots, \varphi_d$ which are holomorphic in some half-plane $\operatorname{Re} z > l \geq 0$, for a sufficiently large integer l, and have integral representations of the form:

$$\varphi_i(z) = \Gamma(z)^p \int\limits_{\gamma_i} x^{-z-1} f_i(x) dx,$$

where:
- $p \in \mathbb{Z}$,
- f_i is a holomorphic function in a sector V_i of $\mathbb{C}$ with vertex at the origin, flat at 0 (this means that φ_i is asymptotic to 0 in V_i at 0) and is a solution of a differential operator with polynomial coefficients D which can be obtained from L,
- γ_i is a half-line (based at the origin) included in V_i.

Since $\varphi_1, \varphi_2, \ldots, \varphi_d$ is a fundamental system of solutions of L, the determinant of Casorati

$$\begin{vmatrix} \varphi_1(z) & \varphi_2(z) & \ldots & \varphi_d(z) \\ \varphi_1(z+1) & \varphi_2(z+1) & \ldots & \varphi_d(z+1) \\ \vdots & \vdots & & \vdots \\ \varphi_1(z+d-1) & \varphi_2(z+d-1) & \ldots & \varphi_d(z+d-1) \end{vmatrix},$$

is non-zero for all z such that $\operatorname{Re} z > l$. It then follows that the constant matrix $C = (\varphi_i(l+j))_{1 \leq i,j \leq d}$ is non-singular and hence $(\varphi_1(z), \ldots, \varphi_d(z))C^{-1}$ is a basis of meromorphic solutions satisfying (a) (the $\mathbb{C}((\varepsilon))$-linear independence of $\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_d$ follows from (a)). $\square$

Remark 6. There exists a finite (possibly empty) set of complex numbers $u_1, u_2, \ldots, u_k$ such that $\operatorname{Re} u_j \leq l$, $j = 1, 2, \ldots, k$, and the solutions $\varphi_1(z), \varphi_2(z), \ldots, \varphi_d(z)$ from Theorem 5 have no poles outside of the set

$$U = \bigcup_{j=1}^{k} (u_j - \mathbb{N}). \tag{14}$$

This follows from Theorem 5(b) and the fact that L has polynomial coefficients.

Definition 7. The restriction to $\mathbb{Z}$ of a meromorphic solution of L which has no poles in $\mathbb{Z}$ will be called a *submeromorphic* (sequential) solution of L. The $\mathbb{C}$-linear space of submeromorphic solutions of L will be denoted by $V_{\mathrm{sm}}(L)$.

Example 7. *For any $d \geq 1$ the operator $L = (E-1)^d \circ z^d$ has rational solutions*

$$\frac{1}{z}, \ \frac{1}{z^2}, \ \ldots \ , \frac{1}{z^d}. \tag{15}$$

If we multiply these solutions by $\sigma(z), (\sigma(z))^2, \ldots, (\sigma(z))^d$, respectively, where $\sigma(z)$ is defined in (12), then all restrictions to $\mathbb{Z}$ will be equal to the sequence $(\delta_{n,0})$. The first impression is that the restriction to $\mathbb{Z}$ of any meromorphic solution of L that has no poles in $\mathbb{Z}$ is a sequence of the form

$$c_n = \begin{cases} u, \text{ if } n = 0, \\ 0, \text{ if } n \neq 0, \end{cases}$$

$n \in \mathbb{Z}$, $u \in \mathbb{C}$, *since* $\mathrm{bott}(\hat{\varphi}) = (\delta_{n,0})$ *for all* $\varphi(z)$ *belonging to (15).*

But we can see that L *has besides (15), e.g., the entire solution*

$$\frac{1}{z} - \sigma(z)\frac{1}{z^2}$$

whose restriction to $\mathbb{Z}$ *is the sequence* h*:*

$$h_n = \begin{cases} -i\pi, & \text{if } n = 0, \\ \frac{1}{n}, & \text{if } n \neq 0, \end{cases}$$

$n \in \mathbb{Z}$.

It is easy to show that L *has* d *meromorphic solutions whose restrictions to* $\mathbb{Z}$ *are* $\mathbb{C}$*-linearly independent sequences* $h^{(1)}, h^{(2)}, \ldots, h^{(d)}$*:*

$$h_n^{(j)} = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n^j}, & \text{if } n \neq 0, \end{cases}$$

$j = 1, 2. \ldots, d-1$,

$$h_n^{(d)} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

$n \in \mathbb{Z}$.

Theorem 8. $\dim V_{\mathrm{sm}}(L) = \mathrm{ord}\, L$.

Proof. Let as usual $\mathrm{ord}\, L = d$. It follows from Theorem 5 that there exist meromorphic solutions $\varphi_1(z), \varphi_2(z), \ldots, \varphi_d(z)$ of L such that the formal sequences $\hat{\varphi}_1$, $\hat{\varphi}_2$, …, $\hat{\varphi}_d$ are $\mathbb{C}((\varepsilon))$-linearly independent. Using Proposition 3 and properties *(M1), (M2)* we derive that there exist meromorphic solutions $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ of L such that the sequences $\mathrm{bott}(\hat{\chi}_1)$, $\mathrm{bott}(\hat{\chi}_2)$, …, $\mathrm{bott}(\hat{\chi}_d)$ are $\mathbb{C}$-linearly independent. By property *(M3)* we get $\dim V_{\mathrm{sm}}(L) \geq d$. Since $V_{\mathrm{sm}}(L) \subset V_{\mathrm{sf}}(L)$ we have $\dim V_{\mathrm{sm}}(L) \leq d$. Finally $\dim V_{\mathrm{sm}}(L) = d$. $\square$

Remark 9. Since the function $\sigma(z) = \frac{e^{2i\pi z}-1}{2i\pi}$ vanishes only on $\mathbb{Z}$, the solutions $\chi_1(z), \chi_2(z), \ldots, \chi_d(z)$ of L such that the sequences $\mathrm{bott}(\hat{\chi}_1)$, $\mathrm{bott}(\hat{\chi}_2)$, …, $\mathrm{bott}(\hat{\chi}_d)$ are $\mathbb{C}$-linearly independent, can be taken such that their poles belong to the set $U \setminus \mathbb{Z}$ where U is the set (14).

4.2. Subanalytic solutions

By an *analytic function* we mean a single-valued analytic function of a single complex variable. If $\varphi(z)$ is an analytic function then we denote by $\mathrm{dom}(\varphi)$ its definition domain. Obviously $\mathrm{dom}(\varphi)$ is an open set, and $\varphi(z)$ is holomorphic in $\mathrm{dom}(\varphi)$.

Let again $L \in \mathbb{C}[z, E]$, $\mathrm{ord}\, L = d$, be of the form (2). An analytic function $\varphi(z)$ is a solution of L if

$$a_d(z)\varphi(z+d) + \cdots + a_1(z)\varphi(z+1) + a_0(z)\varphi(z) = 0$$

for any concrete $z \in \mathbb{C}$ such that $z, z+1, \dots, z+d \in \mathrm{dom}(\varphi)$.

Definition 10. A sequence c is a *subanalytic* (sequential) solution of L if there exists an analytic solution $\varphi(z)$ of L such that $\mathbb{Z} \subset \mathrm{dom}(\varphi)$ and $c_n = \varphi(n)$, $n \in \mathbb{Z}$. The $\mathbb{C}$-linear space of subanalytic solutions of L will be denoted by $V_{\mathrm{sa}}(L)$.

In other words, a subanalytic solution of L is a restriction to $\mathbb{Z}$ of an analytic solution $\varphi(z)$ such that $\mathbb{Z} \subset \mathrm{dom}(\varphi)$.

Theorem 11. $\dim V_{\mathrm{sa}}(L) = \mathrm{ord}\, L$.

Proof. It is evident that $V_{\mathrm{sa}}(L) \subset V_{\mathrm{sf}}\ (L)$. So it is sufficient to prove that

$$\dim V_{\mathrm{sa}}(L) \geq d, \tag{16}$$

where $d = \mathrm{ord}\, L$. This inequality follows from $V_{\mathrm{sm}}(L) \subset V_{\mathrm{sa}}(L)$ and Theorem 8. $\square$

Theorem 11 implies that we can use the algorithm from Section 3.3 to construct a basis of the restriction to an essential segment I of $V_{\mathrm{sa}}(L)$, which is also a basis of the restriction to I of $V_{\mathrm{sm}}(L)$.

Example 2 (continued) *We have*

$$L = 2(z+1)(z-2)E - (2z-1)(z-1),$$

$I = \{1, 2, 3\}$ *is an essential segment of* L*. The restriction of the generic subanalytic solution of* $L(y) = 0$ *to* I *is*

$$c_1 = -u_{0,0}/4, \quad c_2 = 0, \quad c_3 = u_{0,0}/32$$

with $C_{0,1} = C_{0,2} = C_{0,3} = \emptyset$*. The vector*

$$(-1/8,\ 0,\ 1/64)$$

forms a basis of the space of subanalytic solutions restricted to I*. In particular, it agrees with the restriction to* I *of the subanalytic solution generated by*

$$\frac{\Gamma(2z-2)}{\Gamma(z+1)\Gamma(z-2)4^z}.$$

Notice that the restriction to I *of the sequence*

$$\frac{\binom{2n-3}{n}}{4^n}, \quad n \in \mathbb{Z},$$

is $(-1/4,\ 0,\ 1/64)$*. This vector is not of the form*

$$C \cdot (-1/8,\ 0,\ 1/64)\,, \quad C \in \mathbb{C}.$$

Example 1 (continued) *We have*

$$L = z(z-2)(z-4)E - (z-1)(z-3)(z-5),$$

$I = \{1, 2, 3, 4, 5\}$ is an essential segment of L. The vector

$$(1,\ 0,\ -2,\ 0,\ 1)$$

forms a basis of the space of subformal solutions restricted to I (a basis of the sequence segments $(c_1, c_2, c_3, c_4, c_5)$). Observe that L has a meromorphic (rational) fundamental solution

$$\varphi(z) = \frac{1}{(z-1)(z-3)(z-5)}$$

and an entire fundamental solution

$$\chi(z) = \frac{e^{2i\pi z} - 1}{2i\pi(z-1)(z-3)(z-5)} \tag{17}$$

whose restriction to $\mathbb{Z}$

$$e_n = \chi(n) = \frac{1}{8}\delta_{n,1} - \frac{1}{4}\delta_{n,3} + \frac{1}{8}\delta_{n,5}, \quad n \in \mathbb{Z}$$

is a non-zero subanalytic solution of L. None of sequential solutions

$$(\delta_{n,1}),\ (\delta_{n,3}),\ (\delta_{n,5})$$

of L is subanalytic. Notice that each of these sequential solutions coincides with the restriction to $\mathbb{Z}$ of an analytic function. For example, for $(\delta_{n,1})$ this is

$$\frac{e^{2i\pi z} - 1}{2i\pi(z-1)}.$$

However those functions are not analytic solutions of L (unlike the function (17)).

5. Entire solutions

We again suppose that $K = \mathbb{C}$ in (1), (2).

Theorem 12. *The $\mathbb{C}$-linear space of restrictions to $\mathbb{Z}$ of entire solutions of $L \in \mathbb{C}[z, E]$ has dimension* $\operatorname{ord} L$*. A basis of the restrictions of these solutions to an essential segment can be found algorithmically.*

Proof. Let the meromorphic solutions $\chi_1(z), \chi_2(z), \dots, \chi_d(z)$ of L be as it was explained in Remark 9 and let N be the maximal value of the orders of the poles of $\chi_1(z), \chi_2(z), \dots, \chi_d(z)$ in the set $U \setminus \mathbb{Z}$. Consider the $1-$periodic function

$$\lambda(z) = \left(\prod_{u \in U \setminus \mathbb{Z}} \sigma(z-u) \right)^N$$

where U is the set (14). The entire functions

$$\gamma_1(z) = \lambda(z)\chi_1(z),\ \ \gamma_2(z) = \lambda(z)\chi_2(z),\ \ \dots\ \ , \gamma_d(z) = \lambda(z)\chi_d(z)$$

are solutions of L. Their restrictions to $\mathbb{Z}$ are $\mathbb{C}$-linearly independent, since up to the non-zero factor

$$\left(\prod_{u\in U\setminus\mathbb{Z}}\frac{e^{2i\pi u}-1}{2i\pi}\right)^N$$

they are equal to the restrictions of the functions $\chi_1(z),\chi_2(z),\dots,\chi_d(z)$ to $\mathbb{Z}$. Together with Theorem 8 this proves that the $\mathbb{C}$-linear space of restrictions to $\mathbb{Z}$ of entire solutions of L has dimension $d=\operatorname{ord}L$. It follows that this space coincides with $V_{\rm sf}\ (L)$ and we can use the algorithm from Section 3.3 to construct a basis of the restrictions to $\mathbb{Z}$. □

Example 8. *The meromorphic (rational) function $\varphi(z)=\frac{1}{z(2z+1)(3z+1)}$ is a solution of the operator $L=(z+1)(2z+3)(3z+4)E-z(2z+1)(3z+1)$. Multiplying $\varphi(z)$ by*

$$\sigma(z)\sigma(z+\frac{1}{2})\sigma(z+\frac{1}{3})$$

gives an entire solution $\gamma(z)$ of L which vanishes at any point z in $\mathbb{Z}\cup(-\frac{1}{2}+\mathbb{Z})\cup(-\frac{1}{3}+\mathbb{Z})\setminus\{0,-\frac{1}{2},-\frac{1}{3}\}$ and $(\gamma(n))=(\frac{i\sqrt{3}-3}{4\pi^2}\cdot\delta_{n,0})$.

As a consequence of Theorem 5 we have that any subanalytic solution of L coincides with the restriction to $\mathbb{Z}$ of some entire solution of L.

Denote the $\mathbb{C}$-linear space of restrictions to $\mathbb{Z}$ of entire solutions of L by $V_{\rm se}(L)$. Then evidently

$$V_{\rm se}(L)\subset V_{\rm sm}(L)\subset V_{\rm sa}(L)\subset V_{\rm sf}\ (L).$$

Using this and Theorems 4, 8, 11, 12 we get

Theorem 13. *In the case $K=\mathbb{C}$ the equalities*

$$V_{\rm se}(L)=V_{\rm sm}(L)=V_{\rm sa}(L)=V_{\rm sf}\ (L)$$

hold.

6. Multidimensional hypergeometric sequences

If $d=1$ then a sequential solution of (2) is a *hypergeometric sequence*. We also consider *multidimensional hypergeometric sequences*.

A d-dimensional *H-system* is a system of equations for a single unknown function which has the form

$$p_i(z_1,z_2,\dots,z_d)y(z_1,z_2,\dots,z_i+1,\dots,z_d)= \tag{18}$$

$$=q_i(z_1,z_2,\dots,z_d)y(z_1,z_2,\dots,z_i,\dots,z_d),$$

where p_i,q_i are relatively prime non-zero polynomials over K for $i=1,2,\dots,d$. (The prefix "H" refers to Jakob Horn and to the adjective "hypergeometric" as well.)

Rational functions $W_1,W_2,\dots,W_d\in\mathbb{C}(z_1,z_2,\dots,z_d)$ are *compatible* if

$$W_i(z_1,z_2,\dots,z_j+1,\dots,z_d)W_j(z_1,z_2,\dots,z_j,\dots,z_d)=$$

$$=W_j(z_1,z_2,\dots,z_i+1,\dots,z_d)W_i(z_1,z_2,\dots,z_i,\dots,z_d)$$

for all $1 \le i \le j \le d$. The H-system (18) is *consistent* if the rational functions

$$W_i = \frac{q_i(z_1, z_2, \dots, z_d)}{p_i(z_1, z_2, \dots, z_d)}, \quad i = 1, 2, \dots, d,$$

are compatible. This consistency condition of (18) is similar to the condition of the commutation of differentiations by independent variables in the differential case. In the rest of the paper we will consider only consistent H-systems.

A sequential solution of (18) is a sequence $c : \mathbb{Z}^d \to K$, $c = (c_{n_1,n_2,\dots,n_d})$, such that

$$p_i(n_1, n_2, \dots, n_d)c_{n_1,n_2,\dots,n_i+1,\dots,n_d} =$$
$$= q_i(n_1, n_2, \dots, n_d)c_{n_1,n_2,\dots,n_i,\dots,n_d},$$

$i = 1, 2, \dots, d$, for all $(n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$. A *$d$-dimensional hypergeometric sequence* is a sequential solution of some d-dimensional H-system.

The K-linear space of hypergeometric sequences that satisfy a given H-system $\mathcal{H}$ will be denoted by $V(\mathcal{H})$.

Theorem 14. *(Abramov, Petkovšek, 2008) The following statements on K-linear spaces of the form $V(\mathcal{H})$ hold:*

(i) $\dim V(\mathcal{H}) > 0$ *for every H-system $\mathcal{H}$;*

(ii) for arbitrary natural numbers d and m there exists a d-dimensional H-system $\mathcal{H}$ such that $\dim V(\mathcal{H}) = m$*;*

(iii) for any one-dimensional H-system $\mathcal{H}$ the equality $\dim V(\mathcal{H}) < \infty$ *is satisfied. However, for any integer $d > 1$ there exists a d-dimensional H-system $\mathcal{H}$ such that* $\dim V(\mathcal{H}) = \infty$.

We denote by $K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]]$ the ring of formal power series in $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d$ (here $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d$ are variables).

Let $G : \mathbb{Z}^d \to K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]]$. The sequence $g : \mathbb{Z}^d \to K$, such that $g_{n_1,n_2,\dots,n_d}$ is the constant term (i.e., the coefficient of $\mathrm{e}_1^0 \mathrm{e}_2^0 \dots \mathrm{e}_d^0$) of $G_{n_1,n_2,\dots,n_d}$ will be called the *constant terms sequence* of G and will be denoted by $\mathrm{cts}(G)$.

We define the lexicographic order (based on $\mathrm{e}_d >_{lex} \mathrm{e}_{d-1} >_{lex} \dots >_{lex} \varepsilon_1$) on the monomials of the form $\mathrm{e}_1^{k_1} \mathrm{e}_2^{k_2} \dots \mathrm{e}_d^{k_d}$, $(k_1, k_2, \dots, k_d) \in \mathbb{N}^d$. The minimal monomial of $s \in K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]] \setminus \{0\}$ will be denoted by $\mathrm{mm}(s)$, and $\mathrm{mc}(s)$ will denote the coefficient of $\mathrm{mm}(s)$ in s. If $\mathrm{mm}(r) = \mathrm{mm}(s)$ for $r, s \in K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]] \setminus \{0\}$, then we write $r \sim s$. If $r, s \in K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]] \setminus \{0\}$ then we write $r \ge_{lex} s$ if $\mathrm{mm}(r) >_{lex} \mathrm{mm}(s)$ or $\mathrm{mm}(r) = \mathrm{mm}(s)$. If $r \ge_{lex} s$ and $s \ge_{lex} r$ then evidently $r \sim s$.

We denote by $K((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d))$ the quotient field of the ring $K[[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d]]$. Any sequence $F : \mathbb{Z}^d \to K((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d))$ will be called a *formal (d-dimensional) sequence.*

If $a(z_1, z_2, \dots, z_d)$ is a polynomial or a rational function then we set $\hat{a}(z_1, z_2, \dots, z_d) = a(z_1 + \varepsilon_1, z_2 + \varepsilon_2, \dots, z_d + \mathrm{e}_d)$, and associate with each H-system $\mathcal{H}$ of the form (18) its *deformation* $\hat{\mathcal{H}}$:

$$\hat{p}_i(z_1, z_2, \dots, z_d)y(z_1, z_2, \dots, z_i + 1, \dots, z_d) =$$
$$= \hat{q}_i(z_1, z_2, \dots, z_d)y(z_1, z_2, \dots, z_i, \dots, z_d)$$

i.e., the system

$$p_i(z_1 + \varepsilon_1, z_2 + \varepsilon_2, \dots, z_d + \mathrm{e}_d)y(z_1, z_2, \dots, z_i + 1, \dots, z_d) = \tag{19}$$

$$= q_i(z_1 + \varepsilon_1, z_2 + \varepsilon_2, \ldots, z_d + \mathrm{e}_d)y(z_1, z_2, \ldots, z_i, \ldots, z_d),$$

$i = 1, 2, \ldots, d$. We will consider formal solutions of such systems, i.e., formal sequences F such that

$$p_i(n_1 + \varepsilon_1, n_2 + \varepsilon_2, \ldots, n_d + \mathrm{e}_d)F_{n_1,n_2,\ldots,n_i+1,\ldots,n_d} =$$
$$= q_i(n_1 + \varepsilon_1, n_2 + \varepsilon_2, \ldots, n_d + \mathrm{e}_d)F_{n_1,n_2,\ldots,n_i,\ldots,n_d},$$

$i = 1, 2, \ldots, d$, for any $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. Notice that the coefficients of (19) themselves can be considered as formal d-dimensional sequences.

The set $V(\hat{\mathcal{H}})$ of formal solutions of $\hat{\mathcal{H}}$ is evidently a $K((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d))$-linear space of dimension 1, since if for a concrete $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ we know $F_{n_1,n_2,\ldots,n_d} \in K((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d))$, then, using the system (19), we can define the values of the sequence elements everywhere on $\mathbb{Z}^d$.

A formal solution $G : \mathbb{Z}^d \to K[[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d]]$ of (19) will be called a *formal Taylor solution* of (19) (similarly to the one-dimensional case). It is clear that if G is a formal Taylor solution of $\mathcal{H}$ then $\mathrm{cts}(G)$ is a sequential solution of $\mathcal{H}$. We say that the sequential solution g of (18) is *subformal* if there exists a formal Taylor solution G of (19) such that $g = \mathrm{cts}(G)$. The set $V_{\mathrm{sf}}\ (\mathcal{H})$ of all subformal solutions of an H-system $\mathcal{H}$ is a K-linear space.

Theorem 15. *The following statements on dimension of K-linear spaces of the form $V_{\mathrm{sf}}\ (\mathcal{H})$ hold:*

(i) $\dim V_{\mathrm{sf}}\ (\mathcal{H}) \leq 1$ *for every H-system $\mathcal{H}$;*

(ii) for any integer $d > 1$ there exists a d-dimensional H-system $\mathcal{H}$ such that $\dim V_{\mathrm{sf}}\ (\mathcal{H}) = 0$.

Proof. (i) Let $F, G \in V(\hat{\mathcal{H}}) \bigcap K[[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d]]$. Suppose that the sequences $\mathrm{cts}(F)$ and $\mathrm{cts}(G)$ are K-linearly independent. Then these sequences contain non-zero elements (otherwise they are K-linearly dependent). Since $V(\hat{\mathcal{H}})$ is a one-dimensional space over $K((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d))$, there exist $r, s \in K((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)) \setminus \{0\}$ such that $rF + sG = 0$. We have $r \geq_{lex} s$ since $\mathrm{cts}(F)$ contains a non-zero element, and, resp., $s \geq_{lex} r$ since $\mathrm{cts}(G)$ contains a non-zero element. Hence $r \sim s$ and $\mathrm{mm}(r)\,\mathrm{cts}(F) + \mathrm{mm}(s)\,\mathrm{cts}(G) = 0$. A contradiction.

(ii) It is sufficient to prove the statement for the case $d = 2$, since the corresponding H-systems for the case of an arbitrary $d > 1$ can be obtained from the system with $d = 2$ by adding equations

$$y(z_1, z_2, \ldots, z_i + 1, \ldots, z_d) = y(z_1, z_2, \ldots, z_i, \ldots, z_d)$$

for $i = 3, 4, \ldots, d$ to the systems with $d = 2$.

Consider the system $\mathcal{H}$:

$$(z_1 + 1 - z_2^2)y(z_1 + 1, z_2) = (z_1 - z_2^2)y(z_1, z_2), \tag{20}$$

$$(z_1 - (z_2 + 1)^2)y(z_1, z_2 + 1) = (z_1 - z_2^2)y(z_1, z_2).$$

If we substitute

$$W(z_1, z_2) = \frac{1}{z_1 - z_2^2} \tag{21}$$

into this system for $y(z_1, z_2)$, then we get equalities in the rational function field. This implies that the sequence $F = \hat{W}$:

$$F_{n_1,n_2} = \frac{1}{n_1 + \varepsilon_1 - (n_2 + \varepsilon_2)^2},$$

$(n_1, n_2) \in \mathbb{Z}^2$, is a formal solution of $\hat{\mathcal{H}}$. We can show that $\hat{\mathcal{H}}$ has no non-zero formal solution whose elements belong to $K[[\varepsilon_1, \varepsilon_2]]$. Indeed, if such a formal solution, say G, exists then since $\dim V(\mathcal{H}) = 1$ over $K((\varepsilon_1, \varepsilon_2))$ there exists $u(\varepsilon_1, \varepsilon_2) \in K((\varepsilon_1, \varepsilon_2))$ such that $G = uF$ and for all $(n_1, n_2) \in \mathbb{Z}^2$ we have

$$u(\varepsilon_1, \varepsilon_2) \cdot \frac{1}{n_1 + \varepsilon_1 - (n_2 + \varepsilon_2)^2} \in K[[\varepsilon_1, \varepsilon_2]].$$

This implies that for some $s(\varepsilon_1, \varepsilon_2) \in K[[\varepsilon_1, \varepsilon_2]]$ and $g : \mathbb{Z}^2 \to K[[\varepsilon_1, \varepsilon_2]]$, we have

$$s(\varepsilon_1, \varepsilon_2) = (n_1 + \varepsilon_1 - (n_2 + \varepsilon_2)^2) g_{n_1,n_2}$$

for all $(n_1, n_2) \in \mathbb{Z}^2$. Consider this equality for the elements of $\mathbb{Z}^2$ which have the form (v^2, v), $v \in \mathbb{N}$, setting $r_v(\varepsilon_1, \varepsilon_2) = g_{v^2,v}$. We get

$$s(\varepsilon_1, \varepsilon_2) = (\varepsilon_1 - \varepsilon_2^2 - 2v\,\mathrm{e}_2) r_v(\varepsilon_1, \varepsilon_2),$$

$r_v(\varepsilon_1, \varepsilon_2) \in K[[\varepsilon_1, \varepsilon_2]]$, $v = 0, 1, \ldots$ Therefore

$$s(\varepsilon_2^2 + 2v\varepsilon_2, \varepsilon_2) = 0$$

for any $v \in \mathbb{N}$. Let $s(\varepsilon_1, \varepsilon_2) = \sum_{(i,j) \in \mathbb{N}^2} s_{i,j} \varepsilon_1^i \varepsilon_2^j$, then

$$\begin{aligned} s(\varepsilon_2^2 + 2v\varepsilon_2, \varepsilon_2) = {} & s_{0,0} + (s_{0,1} + 2vs_{1,0})\varepsilon_2 \\ & + (s_{1,0} + s_{0,2} + 2vs_{1,1} + 4v^2 s_{2,0})\varepsilon_2^2 \\ & + (s_{1,1} + 4vs_{2,0} + s_{0,3} + 2vs_{1,2} + 4v^2 s_{2,1} + 8v^3 s_{3,0})\varepsilon_2^3 \\ & + \cdots \end{aligned}$$

It is easy to check that the coefficient of ε_2^k is a sum of a linear combination of products of the form $v^m s_{i,j}$ $(m, i, j \in \mathbb{N}, i + j < k)$ and

$$s_{k,0} + (2v)s_{k-1,1} + (2v)^2 s_{k-2,2} + \cdots + (2v)^k s_{0,k}. \tag{22}$$

Using induction on k we can prove that for any $k \in \mathbb{N}$ and any $i, j \in \mathbb{N}$, $i + j = k$, the equality $s_{i,j} = 0$ holds:

- For $k = 0$ this is correct since the constant term of $s(\varepsilon_2^2 - 2v\varepsilon_2, \varepsilon_2)$ is $s_{0,0}$.
- If (22) is equal to 0 for all v then each of $s_{k,0}, s_{k-1,1}, \ldots, s_{0,k}$ is equal to 0 since taking $v = 0, 1, \ldots, k$ we get for $s_{k,0}, s_{k-1,1}, \ldots, s_{0,k}$ a homogeneous system of linear algebraic equations with a Vandermonde determinant.
  □

Consider the case $K = \mathbb{C}$. By an *analytic function* we will mean single-valued analytic functions of complex variables $z_1, z_2, \ldots, z_d$. If $\varphi(z_1, z_2, \ldots, z_d)$ is an analytic function then we denote by $\mathrm{dom}(\varphi)$ its definition domain. An analytic function $\varphi(z_1, z_2, \ldots, z_d)$ is a solution of (18) if

$$p_i(z_1, z_2, \ldots, z_d)\varphi(z_1, z_2, \ldots, z_i + 1, \ldots, z_d) = \\ = q_i(z_1, z_2, \ldots, z_d)\varphi(z_1, z_2, \ldots, z_i, \ldots, z_d)$$

for any concrete $(z_1, z_2, \ldots, z_i, \ldots, z_d)$ such that

$$(z_1, z_2, \ldots, z_i, \ldots, z_d), (z_1, z_2, \ldots, z_i + 1, \ldots, z_d) \in \mathrm{dom}(\varphi), \quad 1 \leq i \leq d.$$

A sequence $c = (c_{n_1,n_2,\ldots,n_d})$ is a *subanalytic* (sequential) solution of (18) if there exists an analytic solution $\varphi(z_1, z_2, \ldots, z_d)$ of (18) such that $\mathbb{Z}^d \subset \mathrm{dom}(\varphi)$ and $c_{n_1,n_2,\ldots,n_d} = \varphi(n_1, n_2, \ldots, n_d)$, $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. In other words, a subanalytic solution of (18) is a restriction to $\mathbb{Z}^d$ of an analytic solution $\varphi(z_1, z_2, \ldots, z_d)$ such that $\mathbb{Z}^d \subset \mathrm{dom}(\varphi)$.

The set $V_{\mathrm{sa}}\ (\mathcal{H})$ of subanalytic solutions of an H-system $\mathcal{H}$ is a $\mathbb{C}$-linear space. It is obvious that $V_{\mathrm{sa}}\ (\mathcal{H}) \subset V_{\mathrm{sf}}\ (\mathcal{H})$, so Theorem 15 holds for subanalytic solutions.

7. Connection with summation problems

7.1. Summing operators and the discrete Newton-Leibniz formula

Consider again the univariate case. We say that an operator $R \in K(z)[E]$ is a *summing operator* for L of the form (2) if

$$(E - 1) \circ R = 1 + M \circ L$$

for some $M \in K(z)[E]$. In this sense

$$R \equiv (E - 1)^{-1} \pmod{L}.$$

We can assume w.l.g. that $\mathrm{ord}\, R < \mathrm{ord}\, L = d$. In the case $d = 1$ we have $\mathrm{ord}\, R = 0$, i.e., R is a rational function.

If a summing operator exists then it can be constructed by the Accurate Summation algorithm (Abramov, van Hoeij, 1999) or, when $d = 1$, by Gosper's algorithm (Gosper, 1978; Petkovšek, Wilf, Zeilberger, 1996). At a first glance, if a summing operator R for L exists then we can apply both sides of

$$(E - 1) \circ R = 1 + M \circ L$$

to any sequential solution c of L. This gives

$$(E - 1)(R(c)) = c + M(L(c)).$$

Set $b = R(c)$. Taking into account that $L(c) = 0$ we obtain $(E - 1)(b) = c$, i.e.,

$$b_{n+1} - b_n = c_n, \quad n \in \mathbb{Z}.$$

As a consequence, the discrete Newton-Leibniz formula (DNLF) is applicable:

$$\begin{aligned}\sum_{n=v}^{w-1} c_n &= \sum_{n=v}^{w-1} (b_{n+1} - b_n) \\ &= b_w - b_{w-1} + b_{w-1} - b_{w-2} + \cdots + b_{v+1} - b_v \\ &= b_w - b_v\end{aligned}$$

(the telescoping effect). However, if R has rational-function coefficients which have poles in $\mathbb{Z}$ this formula may give incorrect results.

Example 2 (continued) *Gosper's algorithm succeeds on the operator L in (6), returning*

$$R(z) = \frac{2z(z+1)}{z-2}.$$

This might lead an inattentive user to believe that the DNLF

$$\sum_{n=0}^{w-1} c_n \;= R(w)c_w - R(0)c_0 \tag{23}$$

holds for any sequential solution c of (6). It certainly holds for all $w \geq 1$ when $c = c^{(1)}$:

$$\begin{aligned}\sum_{n=0}^{w-1} \lim_{v\to n} \frac{\Gamma(2v-2)}{\Gamma(v+1)\Gamma(v-2)4^v} &= \lim_{v\to w} \frac{2v(v+1)}{(v-2)} \cdot \frac{\Gamma(2v-2)}{\Gamma(v+1)\Gamma(v-2)4^v} \\ &= \frac{(w+1)\Gamma(w-\frac{1}{2})}{4\sqrt{\pi}\,\Gamma(w)},\end{aligned}$$

thanks to the fact that $c^{(1)}$ is a subanalytic solution of (6).

However, the pole at $z = 2$ in $R(z)$ should serve as a warning sign when $w \geq 2$. Indeed, when $c = c^{(2)}$, from (23) we would find

$$\sum_{n=0}^{w-1} \frac{\binom{2n-3}{n}}{4^n} = \frac{2w(w+1)\binom{2w-3}{w}}{(w-2)4^w}$$

which is true only when $w = 1$ (Abramov, Petkovšek, 2005). In fact, it is impossible to define the element b_2 of the sequence $b_n = R(n)c_n^{(2)}$ in such a way that $b_{n+1} - b_n = c_n^{(2)}$ for all $n \in \mathbb{Z}$. By Theorem 17(ii) below, $c^{(2)}$ is not a subanalytic solution of (6).

Proposition 5. *(Abramov, Petkovšek, 2007)* Let $F : \mathbb{Z} \to K((\mathrm{e}))$, $\hat{L}(F) = 0$. If R is a summing operator for L then

(i) $\mathrm{alt}(\hat{R}(F)) = \mathrm{alt}(F)$;

(ii) $(E-1)(\hat{R}(F)) = F$.

Theorem 16. (Abramov, Petkovšek, 2007) *Let $L \in K[z, E]$ and $\hat{L}(F) = 0$, for some $F : \mathbb{Z} \to K[[\mathrm{e}]]$ with $\mathrm{alt}(F) = 0$. If $f = \mathrm{bott}(F)$,*

$$R = r_l(z)E^l + \cdots + r_1(z)E + r_0(z) \in K(z)[E]$$

is a summing operator for L, $G = \hat{R}(F)$, and $g = \mathrm{bott}(G)$, then

(i) $\mathrm{alt}(G) = 0$;

(ii) $g_{n+1} - g_n = f_n$ *for all* $n \in \mathbb{Z}$;

(iii) if $n \in \mathbb{Z}$ is not a pole of any of the coefficients of R then

$$g_n = r_l(n)f_{n+l} + \cdots + r_1(n)f_{n+1} + r_0(n)f_n.$$

When $K = \mathbb{C}$ we have

Theorem 17. (Abramov, Petkovšek, 2007) *Let* $L \in \mathbb{C}[z, E]$ *and* $L(\varphi) = 0$ *where* $\varphi(z)$ *is an analytic function which has no singularity in* $\mathbb{Z}$. *If*

$$R = r_l(z)E^l + \cdots + r_1(z)E + r_0(z) \in \mathbb{C}(z)[E]$$

is a summing operator for L *and* $\psi(z) = R(\varphi(z))$, *then*

(i) $\psi(z)$ *has no singularity in* $\mathbb{Z}$;

(ii) $\psi(n+1) - \psi(n) = \varphi(n)$ *for all* $n \in \mathbb{Z}$;

(iii) if $n \in \mathbb{Z}$ *is not a pole of any of the coefficients of* R *then*

$$\psi(n) = r_l(n)\varphi(n+l) + \cdots + r_1(n)\varphi(n+1) + r_0(n)\varphi(n).$$

Corollary 1. Let f be a subformal or (if $K = \mathbb{C}$) a subanalytic solution of L. Let $R = r_l(z)E^l + \cdots + r_1(z)E + r_0(z)$ be a summing operator for L. If integers v, w, $v < w$, are not among the poles of the coefficients of R, then

$$\sum_{n=v}^{w-1} f_n = r_l(w)f_{w+l} + \cdots + r_1(w)f_{w+1} + r_0(w)f_w$$
$$-(r_l(v)f_{v+l} + \cdots + r_1(v)f_{v+1} + r_0(v)f_v).$$

If, say, v is a pole of a coefficient of R then to compute b_v we can use a truncated power series technique or (in the case $K = \mathbb{C}$) the computation of limits.

Remark 18. Subanalytic solutions of L are safe for applying summation algorithms, but the condition of subanalyticity is not a necessary condition for correct applicability of the summing operator: there exist examples where the dimension of the space of "nice" sequential solutions is $> d$.

Example 9. *(Abramov, Petkovšek, 2007)* If $L = zE - (z+1)$, then Gosper's algorithm produces the one-parametric family of summing operators (rational functions)

$$\frac{z-1}{2} + \frac{\alpha}{z}, \quad \alpha \in \mathbb{C}.$$

If we take $\alpha = 0$ we get $R = \frac{z-1}{2}$. Then any sequential solution of L can be multiplied by R. The dimension of the space of all sequential solutions of L is 2, a basis is

$$c_n^{(1)} = n, \quad c_n^{(2)} = |n|, \quad n \in \mathbb{Z}.$$

The sequential solution $(c_n^{(1)})$ is subanalytic since L has the analytic solution $y(z) = z$. But sequential solution $(c_n^{(2)})$ is not subanalytic since the dimension of the space of subanalytic solutions of a first-order operator is 1.

A description of the whole space of sequential solutions of a given L which are safe for application of the summing operator is given in (Abramov, 2006; Abramov, Petkovšek, 2006).

7.2. Creative telescoping and the discrete Newton-Leibniz formula

Summation problems for d-dimensional hypergeometric sequences also are considered in computer algebra. For example, in the case $d = 2$ one of the integer variables can be the summation variable, while the other one can be a parameter which can appear in the summation bounds.

Considering two-dimensional systems of the form (18) we will write u, v instead of z_1, z_2:

$$p_1(u,v)y(u+1,v) = q_1(u,v)y(u,v), \tag{24}$$

$$p_2(u,v)y(u,v+1) = q_2(u,v)y(u,v).$$

We associate with a given two-dimensional H-system $\mathcal{H}$ of the form (24) two operators

$$\mathcal{H}_1 = p_1(u,v)E_1 - q_1(u,v), \;\; \mathcal{H}_2 = p_2(u,v)E_2 - q_2(u,v),$$

where $E_1(y(u,v)) = y(u+1,v)$, $E_2(y(u,v)) = y(u,v+1)$. System (24) can be rewritten in the form

$$\mathcal{H}_1(y) = \mathcal{H}_2(y) = 0.$$

A pair

$$(R(u,v), L),$$

$R(u,v) \in K(u,v)$, $L \in K[v, E_2]$, is a *Z-pair* of $\mathcal{H}$ of the form (24), if

$$(E_1 - 1) \circ R = L + A \circ \mathcal{H}_1 + B \circ \mathcal{H}_2, \tag{25}$$

$A, B \in K(u,v)[E_1, E_2]$. Zeilberger's algorithm (the "creative telescoping") (Zeilberger, 1991; Petkovšek, Wilf, Zeilberger, 1996) helps quite often to find a closed form of parameterized sums, trying to find a Z-pair of a given two-dimensional H-system. [2]

It was observed that in the case of two-dimensional hypergeometric sequences the combination of the creative telescoping with the discrete Newton-Leibniz formula can produce an incorrect result (see Example 10 below). However we show that creative telescoping with the discrete Newton-Leibniz formula will give the correct result for any subformal or subanalytic solution of any H-system, on which the creative telescoping succeeds.

Concerning summation problems, we consider sequential solutions of two-dimensional H-systems, and the summation variable (which corresponds to u) is denoted by k, while the parameter (which corresponds to v) is denoted by n.

Example 10. *The two-dimensional sequences*

$$t^{(1)}_{k,n} = \lim_{v \to n} \lim_{u \to k} \frac{\Gamma(v+1)\Gamma(2u-2)}{\Gamma^2(u+1)\Gamma(v-u+1)\Gamma(u-2)}, \quad t^{(2)}_{k,n} = \binom{n}{k}\binom{2k-3}{k}$$

are sequential solutions of the H-system

$$(u+1)^2(u-2)y(u+1,v) = 2(u-1)(2u-1)(v-u)y(u,v),$$

$$(v-u+1)y(u,v+1) = (v+1)y(u,v),$$

[2] Zeilberger's algorithm tries to construct for a given two-dimensional H-system a Z-pair (R, L) and succeeds on some of such systems. An algorithm which recognizes if Zeilberger's algorithm succeeds on a given two-dimensional H-system was proposed in (Abramov, 2002, 2003).

or, equivalently, of the system

$$(k+1)^2(k-2)t_{k+1,n} = 2(k-1)(2k-1)(n-k)t_{k,n},$$
$$(n-k+1)t_{k,n+1} = (n+1)t_{k,n}.$$

Starting with this system, Zeilberger's algorithm constructs a Z-pair (R, L), which we write using integer variables k and n:

$$R(k,n) = \frac{k^2(n+1)(3kn-2k-9n+4)}{(k-2)(n-k+1)(n-k+2)},$$
$$L = -(n+2)(3n-5)E_2^2 + 18(n+1)(n-1)E_2 - 5(n+1)(3n-2)$$

($E_1(y_{k,n}) = y_{k+1,n}$, $E_2(y_{k,n}) = y_{k,n+1}$).

The denominator of $R(k,n)$ vanishes when $k \in \{2, n+1, n+2\}$, so we define

$$g^{(1)}_{k,n} = \lim_{v\to n}\lim_{u\to k} R(u,v)\frac{\Gamma(v+1)\Gamma(2u-2)}{\Gamma^2(u+1)\Gamma(v-u+1)\Gamma(u-2)},$$
$$g^{(2)}_{k,n} = \begin{cases} g^{(1)}_{k,n}, & k \in \{2, n+1, n+2\}, \\ R(k,n)t^{(2)}_{k,n}, & \text{otherwise.}\end{cases}$$

It can be checked that the expected equality

$$(E_1 - 1)(g^{(2)}_{k,n}) = L(t^{(2)}_{k,n})$$

is not valid when either $k = 1$, or $k = 0$ and $n \in \{-1, 0\}$. On the other hand, the equality

$$(E_1 - 1)(g^{(1)}_{k,n}) = L(t^{(1)}_{k,n}) \tag{26}$$

is valid for all $(k,n) \in \mathbb{Z}^2$. Note that $t^{(2)}_{k,n} = t^{(1)}_{k,n}$ unless $k \in \{0,1\}$, when $t^{(2)}_{k,n} = 2t^{(1)}_{k,n}$.

Denote $s^{(i)}_n = \sum_{k=0}^{n} t^{(i)}_{k,n}$, for $i = 1,2$ and $n \geq 0$. It is easy to check that $t^{(i)}_{k,n} = 0$ when $k < 0$ or $k > n \geq 0$, therefore $s^{(i)}_n = \sum_{k=-\infty}^{\infty} t^{(i)}_{k,n}$, and we can apply the summation $\sum_{k=-\infty}^{\infty}$ to both sides of (26). Since $L \in \mathbb{C}[n, E_2]$, this gives the equality $L(s^{(1)}_n) = 0$. But the equality $L(s^{(2)}_n) = 0$ is not valid.

It follows from (25) that

$$(E_1 - 1) \circ \hat{R} = \hat{L} + \hat{A} \circ \hat{\mathcal{H}}_1 + \hat{B} \circ \hat{\mathcal{H}}_2. \tag{27}$$

In the rest of the paper $\mathcal{H}$ is an H-system of the form (24), $\hat{\mathcal{H}}$ is its deformation. If $R(u,v) \in K(u,v)$ and $R(u,v) = \frac{a(u,v)}{b(u,v)}$, where $a(u,v)$ and $b(u,v)$ are relatively prime polynomials, then we write $b(u,v) = \mathrm{den}(R(u,v))$. The polynomial $\mathrm{den}(R(u,v))$ is defined up to a non-zero factor belonging to K.

Theorem 19. *Let*
- *F be a formal Taylor solution of $\hat{\mathcal{H}}$,*
- *$(R(u,v), L)$ be a Z-pair of $\mathcal{H}$,*
- *M be the (finite) set of all $s \in K$ such that $v - s$ divides* $\mathrm{den}(R(u,v))$,

- *G be a formal sequence such that $G_{k,n} = \hat{R}_{k,n} F_{k,n}$, $(k, n) \in \mathbb{Z}^2$.*

In this case

(i) if $n \notin M$, $k \in \mathbb{Z}$, then $G_{k,n} \in K[[\varepsilon_1, \varepsilon_2]]$;

(ii) for any integer n, l, m, $l < m$, we have $G_{m,n} - G_{l,n} \in K[[\varepsilon_1, \varepsilon_2]]$;

(iii) for the formal sequences
  - *$H = (H_{l,m,n})$, $H_{l,m,n} = G_{m,n} - G_{l,n}$, $(l, m, n) \in \mathbb{Z}^3$,*
  - *$h = (h_{l,m,n})$, $h = \mathrm{cts}(H)$,*
  - *$f = (f_{k,n})$, $f = \mathrm{cts}(F)$,*

  we have $\sum_{k=l}^{m-1} L(f_{k,n}) = h_{l,m,n}$ for all integer l, m, n, $l < m$.

Proof. (i) Denote $w(u, v) = \mathrm{den}(R(u, v))$. Set

$$\hat{L}(F) = \tilde{F}, \quad \tilde{F} = (\tilde{F}_{k,n}). \tag{28}$$

For all $k, n \in \mathbb{Z}$ we have $\tilde{F}_{k,n} \in K[[\varepsilon_1, \varepsilon_2]]$ since L has polynomial coefficients and F is a formal Taylor sequence. For any $n \notin M$ there exists $k \in \mathbb{Z}$ such that $w(k, n) \neq 0$ and as a consequence $G_{k,n} \in K[[\varepsilon_1, \varepsilon_2]]$. Fix a pair of such n and k and notice that

$$G_{k+1,n} = G_{k,n} + \tilde{F}_{k,n} \in K[[\varepsilon_1, \varepsilon_2]], \; G_{k-1,n} = G_{k,n} - \tilde{F}_{k-1,n} \in K[[\varepsilon_1, \varepsilon_2]].$$

Using induction on i it is easy to derive from this that $G_{k\pm i,n} \in K[[\varepsilon_1, \varepsilon_2]]$ for all $i \in \mathbb{N}$.

(ii) Using the notation (28) we get

$$G_{m,n} - G_{l,n} = \sum_{k=l}^{m-1} \tilde{F}_{k,n} \in K[[\varepsilon_1, \varepsilon_2]]. \tag{29}$$

(iii) We have $\sum_{k=l}^{m-1} \hat{L}(F_{k,n}) = G_{m,n} - G_{l,n}$, and by (ii) the right-hand side of this equality belongs to $K[[\varepsilon_1, \varepsilon_2]]$. Therefore the sequence h is defined correctly. Finally notice that the constant term of $\sum_{k=l}^{m-1} \hat{L}(F_{k,n})$ is equal to $\sum_{k=l}^{m-1} L(f_{k,n})$. $\square$

In addition in the case $K = \mathbb{C}$ the following theorem holds.

Theorem 20. *Let*
- *$\varphi(u, v)$ be a solution of $\mathcal{H}$ which is analytic at any $(u, v) \in \mathbb{C}^2$,*
- *$(R(u, v), L)$ be a Z-pair of $\mathcal{H}$,*
- *M be the (finite) set of all $s \in \mathbb{C}$ such that $v - s$ divides $\mathrm{den}(R(u, v))$,*
- *$\psi(u, v) = R(u, v)\varphi(u, v)$.*

In this case

(i) the function $\psi(u, v)$ is analytic at any point $(u, v) \in \mathbb{C}^2$, $v \notin M$;

(ii) if a function $\chi(v, l, m)$
  - *is defined for all $v \in \mathbb{C}$, $l, m \in \mathbb{Z}$, $l < m$,*
  - *is analytic as a function of v for any fixed $l, m \in \mathbb{Z}$, $l < m$,*
  - *is such that $\chi(v, l, m) = \psi(l, v) - \psi(m, v)$ for all $v \notin M$, $l, m \in \mathbb{Z}$, $l < m$,*

  then $\sum_{u=l}^{m-1} L(\varphi(u, v)) = \chi(v, l, m)$ for any $v \in \mathbb{C}$, $l, m \in \mathbb{Z}$, $l < m$.

Proof. (i) Follows from Theorem 19(i).

(ii) If integer l, m, $l < m$, are fixed then the function

$$\sum_{k=l}^{m-1} L(\varphi(u, v)) \tag{30}$$

is analytic at any $v \in \mathbb{C}$. We see that $\chi(v, l, m)$ coincides with (30) for all $v \notin M$. Since M is a finite set the analytic functions $\chi(v, l, m)$ and (30) coincide for any $v \in \mathbb{C}$. □

Corollary 2. As a consequence of Theorem 20 we have the following. If we find a formula for the difference $\psi(l, v) - \psi(m, v)$ which is correct for the case $v \notin M$ and defines a function holomorphic for all $u \in \mathbb{C}$ when $l < m$ are fixed integers, then this formula represents $\sum_{u=l}^{m-1} L(\varphi(u, v))$ for any complex value of the parameter v.

Remark 21. By Theorems 19 and 20 subformal and subanalytic d-dimensional hypergeometric sequences are safe for applying summation algorithms. However, as demonstrated by the system $\mathcal{H}$ given in (20), not all H-systems have non-zero subformal (and, as a consequence, subanalytic) solutions, although each such system has a non-zero sequential solution by Theorem 14(i). It is even possible that an H-system that has a Z-pair does not have a non-zero subformal solution. If instead of W of the form (21) we take

$$W(z_1 + 1, z_2) - W(z_1, z_2), \tag{31}$$

then the elements of the formal sequence $\hat{S}$ for $S(z_1, z_2) = W(z_1 + 1, z_2)$ are Taylor series on the set of pairs (v^2, v), $v \in \mathbb{Z}$. It follows from the proof of Theorem 15(ii) that the H-system that corresponds to the rational function (31) does not have a non-zero subformal solution.

8. Concluding remarks

We have shown that the discrete Newton-Leibniz formula (DNLF) can be safely applied to the output of indefinite summation algorithms when the summand is a *subanalytic sequence*, i.e., a sequence of values of some single-valued analytic function at integer arguments. This result should be useful to the implementors of such algorithms, since it can potentially improve both the correctness and the efficiency of their implementations. Future research along these lines will concentrate on finding another sufficient condition (weaker than subanalyticity) for correctness of DNLF which would be better suited to the multidimensional case.

In addition to subanalytic sequences, we have also considered *subformal sequences*, whose values are obtained as the bottom coefficients of formal Laurent-series solutions of the deformed operator. Subformal solutions play the analogous role in the case of an arbitrary field K of characteristic zero as subanalytic solutions in the case $K = \mathbb{C}$. An implementation of subformal solutions is expected to be part of the future release of *Maple 14* (procedure `SumTools[Hypergeometric][Bottom]`).

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