Computing Hypergeometric Solutions of Second Order Linear Differential Equations using Quotients of Formal Solutions

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ABSTRACT

Let L be a second order differential equation with coefficients in $\mathbb{C}(x)$. The goal of this paper is to find solutions of L in the form

$$\exp(\int r\,dx) \cdot {}_2F_1(a_1, a_2; b_1; f) \tag{1}$$

where $r, f \in \overline{\mathbb{Q}(x)}$, and $a_1, a_2, b_1 \in \mathbb{Q}$.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms; G.4 [**Mathematics of Computing**]: Mathematical Software

General Terms

Algorithms

Keywords

Symbolic Computation, Differential Equations, Closed Form Solutions, Hypergeometric Solutions

1. INTRODUCTION

Consider a second order homogenous linear differential equation with rational function coefficients $A_i \in \mathbb{C}(x)$

$$A_2 y'' + A_1 y' + A_0 y = 0 \tag{2}$$

which corresponds to the differential operator

$$L = A_2 \partial^2 + A_1 \partial + A_0 \in \mathbb{C}(x)[\partial]$$

where $\partial = \frac{d}{dx}$. Then (2) is the equation $L(y) = 0$.

This paper gives a (heuristic[1]) algorithm to find a solution of (2) in the form of (1). This form is both more and less general than in prior work. Less general in the sense that papers [2], [7] considered 3 transformations instead of the 2 in section 2.3 and more general in the sense that prior work was restricted to either a specific number of singularities (4 in [9] and 5 in [6]) or specific degrees (degree 3 in [7] and a degree-2 decomposition in [2]). Moreover, our program can also find algebraic functions f in (1) (although at the moment this requires additional user inputs).

We assume that (2) has no Liouvillian solutions (this implies it is irreducible), otherwise one can solve it with Kovacic's algorithm [5]. The goal of this paper is: *Given a second order operator $L_{inp} \in \mathbb{C}(x)[\partial]$, regular singular[2] without Liouvillian solutions, find a solution of form (1) if it exists.* This means finding $a_1, a_2, b_1 \in \mathbb{Q}$ and finding transformations (sections 2.3 and 3.2) that send L_B to the input equation L_{inp}, where L_B is the minimal operator of ${}_2F_1(a_1, a_2; b_1; x)$.

Two crucial steps of this task are: (1) find (candidates for) a_1, a_2, b_1 and (2) find the pullback function f (after that, finding r becomes easy). Given a_1, a_2, b_1 (or equivalently, L_B), by comparing *quotients of formal solutions* of L_B and L_{inp}, we can compute f *if we know the value of a certain constant c*. We have no direct formula for c; to obtain it with a finite computation, we take a prime number ℓ. Then, for each $c \in \{1, \ldots, \ell - 1\}$ we try to compute f modulo ℓ. If this succeeds, then we lift f modulo a power of ℓ, and try reconstruction.

Example 1. **Rational Pullback Function**

$$L = 21x(x-1)(x+1)\partial^2 + (38x^2 - 6x - 14)\partial + \frac{20x - 5}{7}$$

has a ${}_2F_1$-type solution

$$Y(x) = \exp\,(\int r\,dx) \cdot {}_2F_1\left(\frac{5}{42}, \frac{11}{42}; \frac{2}{3}; f\right)$$

where

$$\exp(\int r\,dx) = (x+1)^{-\frac{5}{21}} \quad \text{and} \quad f = \frac{4x}{(x+1)^2} \tag{3}$$

Here the degree of the pullback function f is 2. We can find this solution with the quotient method in remark 1 below. In the quotient method, the parameters a_1, a_2, b_1 (here $\frac{5}{42}, \frac{11}{42}, \frac{2}{3}$) and the degree of f (here 2) are taken as an input. We implemented section 3.2 which computes candidates for

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[1] For completeness we still need a theorem for "good primes" and address remark 2.

[2] For details see the section 2.1.

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a_1, a_2, b_1 and $\deg(f)$ so that a_1, a_2, b_1, and $\deg(f)$ no longer need to be part of the input.

Remark 1. **The Quotient Method**
The hypergeometric function ${}_2F_1(\frac{5}{42}, \frac{11}{42}; \frac{2}{3}, x)$ is a solution of the operator

$$L_B = \partial^2 + \frac{(29\,x-14)}{21x\,(x-1)}\partial + \frac{55}{1764\,x\,(x-1)}.$$

L_B has two solutions at $x = 0$:

$$y_1(x) = {}_2F_1\left(\frac{5}{41}, \frac{11}{42}; \frac{2}{3}, x\right) = 1 + \frac{55}{1176}x + \ldots,$$
$$y_2(x) = x^{\frac{1}{3}}\left(1 + \frac{475}{2352}x + \frac{1941325}{19361664}x^2 + \ldots\right).$$

The so-called *exponents* of L_B at $x = 0$ are the exponents of x in the dominant terms of y_1 and y_2, so the exponents are $e_{0,1} = 0$ and $e_{0,2} = \frac{1}{3}$. The minimal operator for $y(f)$ has these solutions at $x = 0$:

$$y_1(f) = 1 + \frac{55}{294}x - \frac{4939}{86436}x^2 + \frac{16135823}{304946208}x^3 + \ldots,$$
$$y_2(f) = c_f \cdot x^{\frac{1}{3}}\left(1 + \frac{83}{588}x + \frac{6805}{1210104}x^2 + \ldots\right)$$

for some constant c_f that depends on f. Here the exponents are again $0, \frac{1}{3}$. This is because $x = 0$ is a root of f with multiplicity $e = 1$. Let

$$Y_1(x) = \exp(\int r\,dx)y_1(f) = 1 - \frac{5}{98}x + \frac{439}{9604}x^2 + \ldots, \quad (4)$$
$$Y_2(x) = \exp(\int r\,dx)y_2(f) = c_f \cdot x^{\frac{1}{3}}\left(1 - \frac{19}{196}x + \ldots\right). \quad (5)$$

(4) and (5) form a basis of solutions of L. Here $\exp(\int r\,dx)$ is as the same as in (3). Denote the quotients of the formal solutions of L_B and L by

$$q = \frac{y_1(x)}{y_2(x)}, \qquad Q = \frac{Y_1(x)}{Y_2(x)} = \frac{y_1(f)}{y_2(f)} = q(f),$$

respectively. It follows that $q^{-1}(Q(x))$ gives an expansion of f at $x = 0$. Given enough terms we can reconstruct f. However, the following questions occur:

- Q1. How many terms are needed to reconstruct f? This is equivalent to finding a degree bound for f.
- Q2. How to find the parameters a_1, a_2, b_1?
- Q3. The exponents $0, \frac{1}{3}$ of L at $x = 0$ only determine $\frac{Y_1}{Y_2}$ up to a constant factor (see remark 3 in section 2.3). This means $\frac{y_1(f)}{y_2(f)}$ is only known up to a constant c_f. How to find this constant?
- Q4. What if L has logarithmic solutions at $x = 0$?
- Q5. What if f is an algebraic function?

We will address these questions in section 3, which contains the main new results in this paper (the method illustrated in this remark was already used in [9, section 5.1]).

Example 2. **Algebraic Pullback Function**

$$L = \partial^2 + \frac{1}{4}\,\frac{x^4 - 44\,x^3 + 1206\,x^2 - 44\,x + 1}{\left(x^2 - 34\,x + 1\right)^2 x^2}$$

has a ${}_2F_1$-type solution

$$Y(x) = \exp(-\frac{1}{2}\int r\,dx)\,{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; f)$$

where $r =$

$$\frac{-x^5 + 22\,x^4 - 55\,x^3 - 343\,x^2 + 6\,x\left(x^2 - 7\,x + 1\right)\sqrt{x^2 - 34\,x + 1} + 58\,x - 1}{x\left(x^4 - 41\,x^3 + 240\,x^2 - 41\,x + 1\right)(x+1)}$$

and $f =$

$$-\frac{1}{2}\,\frac{-1 - 30\,x + 24\,x^2 - x^3 + \left(x^2 - 7\,x + 1\right)\sqrt{x^2 - 34\,x + 1}}{1 + 3\,x + 3\,x^2 + x^3}.$$

Here the pullback function f is an algebraic function. The algorithm given in this paper can find this solution.

Equations with such solutions are remarkably common, for instance in the OEIS, the *Online Encyclopedia of Integer Sequences* (`oeis.org`). The implementations of Fang [2] and Kunwar [6] solve many but not all such equations, which forms the motivation for this work.

Remark 2. Our current implementation of recovering pullback functions should terminate if there is a pullback function in $\mathbb{Q}(x)$. If there is a pullback in $\overline{\mathbb{Q}(x)}$ but not in $\mathbb{Q}(x)$, without additional inputs, the current version of our program may enter an infinite loop.

2. PRELIMINARIES

2.1 Differential Operators

Let $L = \sum_{i=0}^{n} a_i\partial^i \in \mathbb{C}(x)[\partial]$. A point $p \in \mathbb{C}$ is called a *singularity* of L if it is a zero of the leading coefficient of L or a pole of any other coefficients of L. The point $p = \infty$ is called a singularity if $p = 0$ is a singularity of $L_{1/x}$. Here $L_{1/x}$ is the differential operator obtained from L via a change of variables $x \mapsto \frac{1}{x}$ (note: $x \mapsto f$ sends ∂ to $\frac{1}{f'}\partial$). If $x = p$ is not a singularity, it is called a *regular point* of L. A singularity $p \in \mathbb{C}$ is called a *regular singularity* if $(x-p)^i\,\frac{a_{n-i}}{a_n}$ is analytic at $x = p$ for $1 \leq i \leq n-1$. The point $p = \infty$ is a regular singularity if $p = 0$ is a regular singularity of $L_{1/x}$. The differential operator L is said to be *regular singular* if all singularities of L are regular singular.

The *local parameter* of a point $p = x \in \mathbb{C} \cup \{\infty\}$ is defined by $t_p = x - p$ if $x \neq \infty$, and $t_p = \frac{1}{x}$ otherwise. The exponents $e_{p,1}$ and $e_{p,2}$ at $x = p$ are the powers of t_p in the dominant terms of the formal solutions at $x = p$, as illustrated in remark 1. In this paper we restrict to rational exponents. The *exponent difference* of L at $x = p$ is $\Delta(L, p) = |e_{p,1} - e_{p,2}|$. If a formal solution at $x = p$ involves a logarithm (a logarithmic singularity), then $\Delta(L, p)$ must be an integer [11, 12].

2.2 Gauss Hypergeometric Function

Let $a_1, a_2, b_1 \in \mathbb{Q}$. The operator $L_B = x(1-x)\partial^2 + (b_1 - (a_1 + a_2 + 1)x)\partial - a_1a_2$ is called *Gauss hypergeometric differential operator (GHDO)*. The solution space has dimension 2 because the order is 2. One of the solutions at $x = 0$ is the *Gauss hypergeometric function*, denoted by ${}_2F_1$, defined by the Gauss hypergeometric series

$${}_2F_1(a_1, a_2; b_1; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k}{(b_1)_k k!}x^k.$$

Here $(\lambda)_k$ denotes the *Pochammer symbol*. It is defined as $(\lambda)_k = \lambda(\lambda+1)\dots(\lambda+k-1)$ and $(\lambda)_0 = 1$. L_B has three regular singularities: $x=0$, $x=1$, and $x=\infty$ with exponents $\{0, 1-b_1\}$, $\{0, b_1-a_1-a_2\}$, and $\{a_1, a_2\}$ respectively. We denote the exponent differences as $\alpha_0 = |1-b_1|, \alpha_1 = |b_1-a_1-a_2|, \alpha_\infty = |a_1-a_2|$. Let d_i be ∞ if $\alpha_i \in \mathbb{Z}$, and the denominator of α_i if $\alpha_i \in \mathbb{Q}-\mathbb{Z}$. The so-called Schwarz list [8] classifies a_1, a_2, b_1 for which L_B has Liouvillian solutions. We will only consider a_1, a_2, b_1 for which L_B has no Liouvillian solutions. From the Schwarz list [8] one finds that this is equivalent to $\frac{1}{d_0}+\frac{1}{d_1}+\frac{1}{d_\infty} < 1$.

2.3 Transformations and Singularities

Let $L_1, L_2 \in \mathbb{C}(x)[\partial]$ be two differential operators of order 2. We consider the following *transformations* that send solutions of L_1 to solutions of L_2.

1. Change of variables: $y(x) \longrightarrow y(f)$, $f \in \overline{\mathbb{Q}(x)}$.
   For L this means substituting $(x, \partial) \mapsto (f, \frac{1}{f'}\partial)$.
2. Exp-product: $y(x) \longrightarrow \exp(\int r\,dx)y(x)$, $r \in \overline{\mathbb{Q}(x)}$.
   For L this means $\partial \mapsto \partial - r$.

These transformations are denoted by $\xrightarrow{f}_C$ and $\xrightarrow{r}_E$ respectively. A third transformation, called gauge transformation, was allowed in the algorithms in [2] and [6]. We hope to use [4] to reduce an equation L that requires a gauge transformation to an equation $\tilde{L}$ that doesn't.

Transformations can affect singularities and exponents. If a transformation $\xrightarrow{r}_E$ can send a singular point $x=p$ to a regular point $x=p$, then we call $x=p$ a *false singularity*. We denote $\mathrm{Sing}(L_1)$ as the set of singularities of L_1 except these false singularities. A singularity $x=p$ is a false singularity if and only if $x=p$ is not logarithmic and the exponent difference is 1.

If $x=p$ is a singularity of L_1 and if transformation $\xrightarrow{r}_E$ can send L_1 to an equation L_2 for which all solutions of L_2 are analytic at $x=p$, then we call $x=p$ a *removable singularity*. A point $x=p$ is removable if and only if $x=p$ is not logarithmic and the exponent difference is an integer. Non-removable singularities are called *true singularities*. A point $x=p$ is a true singularity if and only if the exponent difference is not an integer *or* $x=p$ is logarithmic.

Remark 3. The quotient method (remark 1 in section 1) can only use true singularities, otherwise $\frac{Y_1}{Y_2}$ would only be known up to a Möbius transformation instead of a constant.

THEOREM 1. *[1] Let the GHDO L_B have exponent differences α_0 at $x=0$, α_1 at $x=1$, and α_∞ at $x=\infty$. Let $L_B \xrightarrow{f}_C L_{inp}$. If $f(p) \in \{0,1,\infty\}$, then L_{inp} has the following exponent difference at $x=p$:*

1. *$\alpha_0 \cdot e_p$ if f has a zero at $x=p$ with multiplicity e_p,*
2. *$\alpha_1 \cdot e_p$ if $f-1$ has a zero at $x=p$ with multiplicity e_p,*
3. *$\alpha_\infty \cdot e_p$ if f has a pole at $x=p$ with order e_p.*

If $f(p) \notin \{0,1,\infty\}$, then f maps p to a regular point of L_B (exponent difference 1). Then the exponent difference of L_{inp} at $x=p$ is $1\cdot e_p$ where e_p is the ramification index of f at $x=p$ (i.e., $x=p$ is a root of $f(x)-f(p)$ with multiplicity e_p). The Riemann-Hurwitz formula (section 3.1) relates to the sum of all e_p-1 to the degree of f.

3. ALGORITHM

Problem Description: Given a second order linear differential operator $L_{inp} \in \mathbb{C}(x)[\partial]$, irreducible and regular singular, we want to find a ${}_2F_1$-type solution of the differential equation $L_{inp}(y)=0$ of the form of (1). This is equivalent to finding transformations 1 and 2 from a GHDO L_B to L_{inp}. Therefore, we need to find

1. L_B (i.e., find a_1, a_2, b_1),
2. parameters f and r of the change of variables and exp-product transformations such that $L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{inp}$.

The general outline is as follows.

Algorithm Outline: `find_2f1`

Input:

- L_{inp}, a second order differential operator.
- At the moment we only handle coefficients in $\mathbb{Q}(x)$. If f in (1) is algebraic, then our current implementation needs three more inputs which are
  - L_B, a candidate GHDO,
  - a_f, an algebraic degree bound for f,
  - d_f, degree bound for f.

Output:

- A list of basis elements of solutions of L_{inp} in form (1), or an empty list [ ].

1. Try Kovacic's algorithm [5]. If there exists Liouvillian solutions, then return them. The algorithm in [10] computes Liouvillian solutions in form (1) if L_{inp} is irreducible.
2. If L_B, a_f, d_f are not provided in the input, then use section 3.2 (at the moment this only covers rational f's, i.e., $a_f=1$) to compute candidates for L_B and d_f.
3. For a candidate GHDO L_B, compute formal solutions of L_B and L_{inp} at a non-removable singularity (see remark 3 in section 2.3) up to precision $a \geq 2(a_f+1)(d_f+1)+3$. Take the quotients of formal solutions and compute series expansions for q^{-1} and Q (in order to compute $f = q^{-1}(cQ(x))$ in the next step).
4. Choose a good prime number ℓ, and try to find c mod ℓ by looping $c = 1, 2, \dots, \ell-1$ as in section 3.3. If no solution is found, then proceed with the next candidate GHDO (if any) in step 3. If no candidates remain, then return an empty list [ ].
5. Compute f mod (x^a, ℓ) and then use Hensel lifting to find f mod higher powers of ℓ. After each lifting try rational reconstruction. If it does not fail, then we have f.
6. Compute the parameter r of the exp-product transformation (section 3.5).
7. Return a basis of ${}_2F_1$-type solutions of L_{inp}.

Step 2 is explained in sections 3.1 and 3.2. Step 3 is the quotient method, see section 3.3 for more. Steps 5 and 6 are explained in sections 3.4 and 3.5 respectively.

Remark 5, section 3.2, section 3.4, and section 3.3.2 provide answers to Q1, Q2, Q3, and Q4 respectively. Remark 7 and section 3.4 answer Q5. Maple codes can be found at [3].

3.1 General Degree Bound

Let X and Y be two algebraic curves with genus g_X and g_Y, and let $f : X \longrightarrow Y$ be a non-constant analytic map. The *Riemann-Hurwitz* formula says

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \sum_{p\in X}(e_p - 1). \qquad (6)$$

Here p is a branching point and e_p is its ramification order. In this paper $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ so $g_X = g_Y = 0$ and

$$\sum_{p\in\mathbb{P}^1}(e_p - 1) = 2\deg(f) - 2. \qquad (7)$$

In section 3.1.1 and 3.1.2 we compute a degree bound for a rational pullback function f from formula (7). In section 3.1.3 we use it to compute a formula for $\alpha_0 + \alpha_1 + \alpha_\infty$, the sum of the exponent differences of L_B.

3.1.1 *Bound for Logarithmic Cases*

Let L_B be a GHDO with at least one logarithmic singularity. Assume that $L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{inp}$. Let $d_f = \deg(f)$. The number of elements in the set $T = f^{-1}(\{0, 1, \infty\})$ can be at most $3d_f$.

$$\#T = \sum_{p\in T} 1 = \sum_{p\in T} e_p - (e_p - 1) = 3d_f - \sum_{p\in T}(e_p - 1). \quad (8)$$

From (7), we have

$$0 \le \sum_{p\in T}(e_p - 1) \le \sum_{p\in\mathbb{P}^1}(e_p - 1) = 2d_f - 2$$

where the latter sum is taken over all branching points of f. Hence $d_f + 2 \le \#T \le 3d_f$.

The set of true singularities of L_{inp} is a subset of T and these two sets do not need to be equal. Points in T come from (p comes from s when $f(p) = s$) the singular points $\{0, 1, \infty\}$ of L_B. Such points need not be singular, for instance, if L_B has exponents $0, \frac{1}{3}$ at $x = 0$ and f has a root p of order $e_p = 3$, then the exponents at $x = p$ will be $3 \cdot \{0, \frac{1}{3}\} = \{0, 1\}$ and $x = p$ will be a regular point (a "disappeared singularity"). We define the set of *disappeared singularities* as $T - \mathrm{Sing}(L_{inp})$. Logarithmic singularities do not disappear; if $s \in \{0, 1, \infty\}$ is a logarithmic singularity of L_B, then every point p above s is a logarithmic singularity as well.

Let n_{diss} be the number of disappeared singularities of L_{inp}. For a GHDO with exponent differences $[0, \frac{1}{2}, \frac{1}{3}]$ at $0, 1, \infty$ respectively, $n_{diss} \le \frac{1}{2}d_f + \frac{1}{3}d_f$, with equality if and only if every point above s with exponent difference $\alpha = \frac{1}{2}$, respectively $\alpha = \frac{1}{3}$ disappears (i.e., $e_p = 2$, respectively $e_p = 3$). So, if the total number of true singularities of L_{inp} is n_{true}, then

$$\begin{aligned} n_{true} = \#T - n_{diss} &= \left(3d_f - \sum_{p\in S}(e_p - 1)\right) - n_{diss} \\ &\ge [3d_f - (2d_f - 2)] - n_{diss} = d_f + 2 - n_{diss} \\ &\ge d_f + 2 - \left(\frac{1}{2}d_f + \frac{1}{3}d_f\right) = \frac{1}{6}d_f + 2 \end{aligned}$$

and so

$$d_f \le 6(n_{true} - 2). \qquad (9)$$

Inequality (9) is an upper bound for d_f in all cases with at least one logarithmic singularity. This is because $\frac{1}{2}d_f + \frac{1}{3}d_f$ is an upper bound for the number of disappeared singularities in the logarithmic case (the GHDO cannot have two singularities with exponent difference $\frac{1}{2}$ if it is irreducible, this makes $\frac{1}{2}d_f + \frac{1}{3}d_f$ the maximum possible value for n_{diss} in the logarithmic case).

3.1.2 *Bound for Non-Logarithmic Cases*

In the non-logarithmic case one could have disappeared singularities above all three singularities $\{0, 1, \infty\}$ of the GHDO. The maximal degree bound is achieved at exponent differences $[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$. All L_B's with a higher bound such as $[\alpha_0, \alpha_1, \alpha_\infty] = [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}]$, $[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$, etc, are either reducible or appear in Schwarz's list [8], which means they have Liouvillian solutions.

The maximum number of disappeared singularities for $[\frac{1}{2}, \frac{1}{3}, \frac{1}{7}]$ is not $(\frac{1}{2} + \frac{1}{3} + \frac{1}{7})d_f$ because that contradicts the formula (7). The maximum number consistent with (7) is

$$\left(\frac{1}{2} + \frac{1}{3}\right)d_f + \frac{1}{7-1}\left(2d_f - 2 - \frac{2-1}{2}d_f - \frac{3-1}{3}d_f\right)$$

and it leads to

$$d_f \le 36\left(n_{true} - \frac{7}{3}\right). \qquad (10)$$

We use inequality (10) as an a priori upper bound for d_f for all cases with no logarithmic singularity.

Therefore, an a priori degree bound for a rational pullback function f is

$$d_f \le \begin{cases} 6(n_{true} - 2), & \text{logarithmic case,} \\ 36\left(n_{true} - \frac{7}{3}\right), & \text{non-logarithmic case.} \end{cases} \qquad (11)$$

Our algorithm uses this degree bound only as a starting point; additional restrictions are computed during the algorithm that may lower the degree.

3.1.3 *Riemann-Hurwitz Type Formula*

The differential operators L_B and L_{inp} are in $\mathbb{C}(x)[\partial]$, i.e., they are defined on $\mathbb{P}^1$. The function field of $\mathbb{P}^1$ is $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(x)$. Denote $D_{\mathbb{C}(\mathbb{P}^1)} = \mathbb{C}(x)[\partial]$. So $L_B, L_{inp} \in D_{\mathbb{C}(\mathbb{P}^1)}$.

In general, let X be any algebraic curve and $\mathbb{C}(X)$ be its function field. The ring $D_{\mathbb{C}(X)} := \mathbb{C}(X)[\partial_t]$ is the ring of differential operators on X. Here $t \in \mathbb{C}(X)$ with $t' \ne 0$. An element $L \in D_{\mathbb{C}(X)}$ is a differential operator defined on the algebraic curve X.

THEOREM 2. *Let X, Y be two algebraic curves with genus g_X, g_Y; and function fields $\mathbb{C}(X), \mathbb{C}(Y)$. Let $f : X \to Y$ be a non-constant morphism with $\deg(f) = d$. The morphism f corresponds a homomorphism $\mathbb{C}(Y) \to \mathbb{C}(X)$, which induces a homomorphism $D_{\mathbb{C}(Y)} \to D_{\mathbb{C}(X)}$. If $L_1 \in D_{\mathbb{C}(Y)}$ with $\mathrm{ord}(L_1) = 2$ and, L_2 is the corresponding element in $D_{\mathbb{C}(X)}$, then*

$$2 - 2g_X + \sum_{p\in X}(\Delta(L_2, p) - 1) = d(2 - 2g_Y + \sum_{s\in Y}(\Delta(L_1, s) - 1)).$$

PROOF. Let $S \subset Y$ be a finite set and $T = f^{-1}(S)$ such that $\mathrm{Sing}(L_1) \subseteq S$, $\mathrm{Sing}(L_2) \subseteq T$, and all branching points in X are in T. There are infinitely many points in $X \setminus T$ and for each $p \in X \setminus T$, we have $\Delta(L_2, p) = 1$ and $e_p = 1$. There are infinitely many points in $Y \setminus S$ and for each $s \in Y \setminus S$, we have $\Delta(L_1, s) = 1$.

$$\#T = \sum_{p \in T} 1 = \sum_{p \in T} e_p - \sum_{p \in T} (e_p - 1) \tag{12}$$

$$= d \cdot \#S - \sum_{p \in X} (e_p - 1) \tag{13}$$

$$= d \cdot \#S - (2g_x - 2 - d(2g_Y - 2)). \tag{14}$$

From (13) to (14) we used (6). Then,

$$\sum_{p \in X} (\Delta(L_2, p) - 1) = \sum_{p \in T} (\Delta(L_2, p) - 1) \tag{15}$$

$$= \sum_{p \in T} \Delta(L_2, p) - \sum_{p \in T} 1 \tag{16}$$

$$= d \sum_{s \in S} \Delta(L_1, s) - \#T. \tag{17}$$

Combine (14) and (17) to obtain

$$\sum_{p \in X} (\Delta(L_2, p) - 1)$$
$$= d \sum_{s \in S} \Delta(L_1, s) - d \cdot \#S + (2g_x - 2 - d(2g_Y - 2)).$$

Therefore,

$$2 - 2g_X + \sum_{p \in X} (\Delta(L_2, p) - 1) = d(2 - 2g_Y + \sum_{s \in Y} (\Delta(L_1, s) - 1)). \tag{18}$$

□

We use differential operators $L_B, L_{inp} \in \mathbb{C}(x)[\partial]$. So $X = Y = \mathbb{P}^1$ and $g_X = g_Y = g_{\mathbb{P}^1} = 0$. Suppose that

$$L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{inp}$$

where $f : \mathbb{P}^1 \to \mathbb{P}^1$ and L_B is a GHDO with exponent differences $[\alpha_0, \alpha_1, \alpha_\infty]$ at $\{0, 1, \infty\}$. Since the exp-product transformation does not affect exponent differences, formula (18) gives us:

$$2 + \sum_{p \in \mathbb{P}^1} (\Delta(L_{inp}, p) - 1) = \deg(f)(2 + \sum_{i \in \{0,1,\infty\}} (\alpha_i - 1)). \tag{19}$$

We will use formula (19) in section 3.2.

3.2 Candidate Exponent Differences

This section explains a method of computing exponent differences for candidate GHDOs.

Remark 4. Consider the operator L_{inp} in example 1. It has 4 true singularities, so (11) gives us $d_f = 60$. For a candidate L_B having exponent differences $[\alpha_0, \alpha_1, \alpha_\infty]$, we have

$$\alpha_0, \alpha_1, \alpha_\infty \in \{\frac{a}{b} : a \in S_T \cup S_R \cup \{1\}, 1 \le b \le d_f\}. \tag{20}$$

Here S_T is the set of exponent differences of L_{inp} at its true singularities and S_R is the set of exponent differences of L_{inp} at its removable singularities. There are 176 elements in the set (20). This leaves too many candidates for $[\alpha_0, \alpha_1, \alpha_\infty]$. Algorithm `find_expdiffs` is designed to skip most combinations (formula (19) is particularly effective). In about 0.25 seconds `find_expdiffs` returns all different candidates: $[\frac{2}{7}, \frac{1}{3}, \frac{1}{7}, 2]$, $[\frac{1}{7}, \frac{1}{3}, \frac{1}{2}, 20]$. The first candidate gives a pullback function of degree 2 and the second candidate gives a pullback function of degree 20.

Algorithm: `find_expdiffs`

Input:

- e_{inp}, a list of exponent differences of L_{inp} at its true singularities.
- e_{rem}, a (possibly empty) list of exponent differences of L_{inp} at its removable singularities.

Output:

- List of candidate exponent differences for candidate GHDOs.

Output is a list of all lists $e_B = [\alpha_0, \alpha_1, \alpha_\infty, d]$ of integers or rational numbers where $[\alpha_0, \alpha_1, \alpha_\infty]$ is a list of candidate exponent differences and d is a candidate degree for f such that:

- For every exponent difference m in e_{inp} there exists $e \in \{1, 2, \ldots, d\}$ such that $m = e\alpha_i$ for some $i \in \{0, 1, \infty\}$.
- The multiplicities e are consistent with (7), and their sums are compatible with d, see the last paragraph in step 2.

1. Let $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 = \alpha_0, \alpha_1, \alpha_\infty$. After reordering we may assume that $\overline{\alpha}_1, \ldots, \overline{\alpha}_k \in \mathbb{Z}$ and $\overline{\alpha}_{k+1}, \ldots, \overline{\alpha}_3 \notin \mathbb{Z}$ for $k \in \{0, 1, 2, 3\}$. For each $k \in \{0, 1, 2, 3\}$ we use `CoverLogs` in [3] to compute candidates for $\overline{\alpha}_1, \ldots, \overline{\alpha}_k \in \mathbb{Z}$. If $\overline{\alpha}_1 + \cdots + \overline{\alpha}_k \neq 0$ then algorithm `CoverLogs` also returns the exact degree d_f of f (theorem 1 shows that $d_f(\overline{\alpha}_1 + \cdots + \overline{\alpha}_k)$ must be the sum of the logarithmic exponent differences of L_{inp}). Otherwise, it uses (11) to compute a degree bound d_f for f.

2. We will explain only the case where $k = 1$, which is the case $[\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3] = [\alpha_0, \alpha_1, \alpha_\infty]$, where $\alpha_0 \in \mathbb{Z}$ and $\alpha_1, \alpha_\infty \notin \mathbb{Z}$. For other cases ($k = 0, 1, 3$) see [3].

   Let $k = 1$. So we have $\alpha_0 \in \mathbb{Z}$. We need to find rational numbers α_1 and α_∞.

   The logarithmic singularities of L_{inp} come from the point 0. Non-integer exponent differences of L_{inp} must be multiples of α_1 or α_∞. Let S_N be the set of non-logarithmic exponent differences of L_{inp} and S_R be the set of exponent differences of L_{inp} at its removable singularities. Consider the set

$$\Gamma_1 = \begin{cases} \Gamma_A = \{\frac{\max(S_N)}{b} : b = 1, \ldots, d_f\} & \text{if } S_N \neq \emptyset, \\ \Gamma_B = \{\frac{a}{b} : a \in S_R \cup \{1\}, b = 1, \ldots, d_f\} & \text{otherwise.} \end{cases}$$

   α_1 (or α_∞, but if so, we may interchange them) must be one of the elements of Γ_1. We loop over all elements of Γ_1. Assume that a candidate for α_1 is chosen. Let $\Omega = S_N \setminus \alpha_1 \mathbb{Z}$. Now consider the set

$$\Gamma_\infty = \begin{cases} \Gamma_A \cup \Gamma_B & \text{if } \Omega = \emptyset, \\ \{\frac{g}{b} \ : \ g = \gcd(\Omega) : b = 1, \ldots, d_f\} & \text{otherwise.} \end{cases}$$

   Now take all pairs (α_∞, d) satisfying (19), $\alpha_\infty \in \Gamma_\infty$, $1 \le d \le d_f$, with additional restrictions on d, as follows:

   For every potential non-zero value v for one of the α_i's we pre-compute a list of integers N_v by dividing all exponent differences of L_{inp} by v and then selecting the quotients that are integers. Next, let D_v be the set of all $1 \le d \le d_f$ that can be written as the sum of a sublist of N_v. Each time a non-zero value v is taken for one of the α_i, it imposes the restriction $d \in D_v$. This means that we need not run a loop for $\alpha_\infty \in \Gamma_\infty$, instead, we run a (generally much shorter) loop for d (taking values in the intersection of the D_v's so far) and then for each such d compute α_∞ from (19). We also check if $d \in D_{\alpha_\infty}$.

3. Return the list of candidate exponent differences with a candidate degree, the list of lists $[\alpha_0, \alpha_1, \alpha_\infty, d]$, for candidate GHDOs.

Once we have the list of candidate exponent differences, then each of the elements of this list gives a candidate GHDO. If L_{inp} has a ${}_2F_1$-type solution in form (1), then it is among the candidate GHDOs that we computed, via a change of variables and exp-product transformations. This answers question Q2.

3.3 Quotient Method

In this section, we explain a method to recover the pullback function f, which is the most crucial part of our algorithm. We will explain our algorithm for rational pullback functions. For algebraic pullback functions, the only difference is the lifting algorithm, which is explained in section 3.4. Before starting this section, note that we can always compute the formal solutions of a given differential equation $L_{inp}(y) = 0$ up to a finite precision.

3.3.1 Non-Logarithmic Case

Let the second order differential equation $L_{inp}(y) = 0$ be given. Let L_B be a GHDO such that $L_B \xrightarrow{f}_C \xrightarrow{r}_E L_{inp}$. Let $f : \mathbb{P}^1_x \mapsto \mathbb{P}^1_z$ and $L_1 \xrightarrow{f}_C L_2$. If $x = p$ is a singularity of L_2 and $z = s$ is a singularity of L_1, then we say that p comes from s when $f(p) = s$.

After a change of variables we can assume that $x = 0$ is a singularity of L_{inp} that comes from the singularity $z = 0$ of L_B. This means $f(0) = 0$ and we can write $f = c_0 x^{v_0(f)} (1 + \dots)$ where $c_0 \in \mathbb{C}$, $v_0(f)$ is the multiplicity of 0, and the dots refer to an element in $x\mathbb{C}[[x]]$.

Let y_1 and y_2 be the formal solutions of L_B at $x = 0$. The following diagram shows the effects of the change of variables and exp-product transformations on the formal solutions of L_B,

$$y_1(x) \xrightarrow{f}_C y_1(f) \xrightarrow{r}_E Y_1(x) = \exp(\int r dx) y_1(f),$$

$$y_2(x) \xrightarrow{f}_C y_2(f) \xrightarrow{r}_E Y_2(x) = \exp(\int r dx) y_2(f),$$

where Y_1 and Y_2 are solutions of L_{inp}.

Let $q = \frac{y_1}{y_2}$ be a quotient of formal solutions of L_B. The change of variables transformation sends x to f, and so q to $q(f)$. Therefore, $q(f)$ will be a quotient of formal solutions of L_{inp}.

The effect of exp-product transformation disappears under taking quotients. In general, a quotient of formal solutions of L_B at a point $x = p$ is only unique up to Möbius transformations $\frac{y_1}{y_2} \mapsto \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \eta y_2}$.

If $x = p$ has a non-integer exponent difference, then we can choose q uniquely up to a constant factor c. So if we likewise compute a quotient Q of formal solutions of L_{inp}, then we have $q(f) = c \cdot Q(x)$ for some unknown constant c. Then

$$f(x) = q^{-1} (c \cdot Q(x)). \tag{21}$$

If we know the value of this constant c, then we can compute an expansion for the pullback function f from expansions of q and Q. To obtain c with a finite computation, we take a prime number ℓ. Then, for each $c \in \{1, \dots, \ell - 1\}$ we try to compute f modulo ℓ. If this succeeds, then we lift f modulo a power of ℓ, and try reconstruction. Details of lifting is explained in section 3.4.

Remark 5. Here we should compute the formal solutions up to a precision $a \ge (a_f + 1)(d_f + 1) + 3$. This precision is enough to recover the correct pullback function with a few extra terms for checking. This answers Q1.

Algorithm: `case1` (non-logarithmic case)

Input:

- L_{inp}, a second order differential operator with non-logarithmic solutions,
- L_B, a candidate GHDO,
- d_f, degree bound for f.

Output:

- The rational pullback function f, or 0 (in this case there is no rational pullback function).

1. Compute expansions of the formal solutions y_1, y_2 of L_B and Y_1, Y_2 of L_{inp} up to precision $a \ge 2d_f + 5$. Select a prime ℓ for which these expansions can be reduced mod ℓ.
2. $q \leftarrow \frac{y_2}{y_1}$, $Q \leftarrow \frac{Y_2}{Y_1}$, then compute q^{-1}.
3. Search for c_0 such that $c \equiv c_0 \bmod \ell$ by looping over $c_0 = 1, \dots, \ell - 1$. If there is no such c_0, then return 0.
4. Compute $f_1 = q^{-1}(c_0 \cdot Q) \in \mathbb{Z}[x]/(\ell, x^a)$.
5. Lift[3] f_1 to $f_l \in \mathbb{Z}[x]/(\ell^l, x^a)$ for a suitable $l \in \mathbb{N}$, and then reconstruct the rational pullback function f from f_l (we still need to address remark 2).
6. Return f.

3.3.2 Logarithmic Case

A logarithm may occur in one of the formal solutions of L_{inp} at $x = p$ if exponents at $x = p$ differ by an integer. We may assume that L_{inp} has a logarithmic solution at the singularity $x = 0$.

Let y_1, y_2 be the formal solutions of L_B at $x = 0$. Let y_1 be the non-logarithmic solution (it is unique up to a multiplicative constant). Then $\frac{y_2}{y_1} = c_1 \cdot \log(x) + h$ for some $c_1 \in \mathbb{C}$ and $h \in \mathbb{C}[[x]]$. We can choose y_2 such that

$$c_1 = 1 \quad \text{and} \quad \text{constant term of } h = 0. \tag{22}$$

That makes $\frac{y_2}{y_1}$ unique. If h does not contain negative powers of x then define

$$g = \exp\left(\frac{y_2}{y_1}\right) = x \cdot (1 + \dots) \tag{23}$$

where the dots refer to an element of $x\mathbb{C}[[x]]$.

[3] For details see the section 3.4.

Remark 6. If we choose y_2 differently, then we obtain another $\tilde{g} = \exp\left(\frac{y_2}{y_1}\right)$ that relates to g in (23) by $\tilde{g} = c_1 g^{c_2}$ for some constants c_1, c_2. If h contains negative powers of x, then the formula for g is slightly different (we have not implemented this case yet).

We do likewise for the formal solutions Y_1, Y_2 of L_{inp} and denote

$$G = \exp\left(\frac{Y_2}{Y_1}\right) = x \cdot (1 + \dots). \tag{24}$$

Write $f \in \mathbb{C}(x)$ as $c_0 x^{v_0(f)} \cdot (1 + \dots)$. Then $g(f) = c \cdot x^{v_0(f)} (1 + \dots)$. Note that g, G are not intrinsically unique, the choices we made in (22) implies that

$$g(f) = c_1 \cdot G^{c_2} \tag{25}$$

for some constants c_1, c_2. Here $c_1 = c$ and $c_2 = v_0(f)$.

If $\Delta(L_{inp}, 0) \neq 0$, then find $v_0(f)$ from $\Delta(L_B, 0) v_0(f) = \Delta(L_{inp}, 0)$. Otherwise we loop over $v_0(f) = 1, 2, \dots, d_f$. That leaves one unknown constant c. We address this problem as before, choose a good prime number ℓ, try $c = 1, 2, \dots, \ell - 1$. Then calculate an expansion for f with the formula

$$f = g^{-1}\left(c \cdot G^{v_0(f)}\right). \tag{26}$$

Then we lift f modulo a power of ℓ, and try reconstruction. The discussion in this section answers Q4.

Algorithm: `case2` (logarithmic case)

Input:

- L_{inp}, a second order differential operator with at least one logarithmic solution,
- L_B, a candidate GHDO,
- d_f, degree bound for f.

Output:

- The rational pullback function f, or 0 (in this case there is no rational pullback function).

1. Compute the exponents of L_{inp} and L_B.
2. Compute expansions of the formal solutions y_1, y_2 of L_B and Y_1, Y_2 of L up to precision $a \geq 2d_f + 5$. Select a prime ℓ for which these expansions can be reduced mod ℓ.
3. $q \leftarrow \frac{y_2}{y_1}$, $Q \leftarrow \frac{Y_2}{Y_1}$, and compute g and G from (23) and (24) respectively. Then compute g^{-1}.
4. Select (compute if $\Delta(L_{inp}, 0) \neq 0$, loop otherwise) $v_0(f)$ and search for c_0 such that $c \equiv c_0 \mod p$ by looping over $1, \dots, \ell - 1$. If there is no such c_0 (which means there is no rational pullback function for this candidate L_B), then return 0.
5. Compute $f_1 = g^{-1}\left(c_0 \cdot G^{v_0(f)}\right) \in \mathbb{Z}[x]/(\ell, x^a)$.
6. Lift[4] f_1 to $f_l \in \mathbb{Z}[x]/(\ell^l, x^a)$ for a suitable $l \in \mathbb{N}$, and reconstruct the rational pullback function f from f_l (we still need to address remark 2).
7. Return f.

[4]For details see the section 3.4.

Remark 7. **Algebraic Pullback Functions**
Let L_{inp} have a ${}_2F_1$-type solution in the form (1) where f is an algebraic function. We do not have a degree bound for this case, nor the analogue of the algorithm from section 3.2. Therefore, for this case, the current version of our implementation needs extra inputs: a candidate GHDO, a degree bound for f, and an algebraic degree bound for f. Then we can find the algebraic pullback function via the quotient method. The only difference is the lifting algorithm which is explained in section 3.4. An algebraic degree bound is needed for lifting. This remark together with section 3.4 answer question Q5.

3.4 Lifting: Recovering the Pullback Function

We introduce two lifting algorithms, one for rational functions, one for algebraic functions. We explain lifting by using the formula (21) for the pullback function, which occurs in the non-logarithmic case. The algorithm for the formula (26) in the logarithmic case is similar. The discussion in this section answers Q3.

3.4.1 Lifting for a Rational Pullback Function

By using the formula (21), which is $f(x) = q^{-1}(c \cdot Q(x))$, we can recover the rational pullback function f, if we know the value of the constant c. We do not have a direct formula for c. However, if we know c_0 such that $c \equiv c_0 \mod \ell$ for a good prime number ℓ, then we can recover the pullback function f. This can be done via *Hensel lifting techniques.*

Let ℓ be a good prime number and consider

$$h : \mathbb{Q} \longrightarrow \mathbb{Q}[x]/(x^a)$$
$$h(c) \equiv q^{-1}(c \cdot Q(x)) \mod x^a.$$

By looping on $c_0 = 1, \dots, \ell - 1$ and trying rational function reconstruction for $h(c_0) \mod (\ell, x^a)$, we can compute the image of f in $\mathbb{F}_\ell/(x^a)$. If a is high enough, then for correct value(s) of c_0, rational function reconstruction will succeed and return a rational function $\frac{A_0}{B_0} \mod (\ell, x^a)$. This c_0 is the one satisfying $c \equiv c_0 \mod \ell$.

Write $c \equiv c_0 + \ell c_1 \mod \ell^2$ for $0 \leq c_1 \leq \ell - 1$. Taylor series expansion of h gives us

$$h(c) = h(c_0 + \ell c_1) \equiv h(c_0) + \ell c_1 h'(c_0) \mod (\ell^2, x^a). \tag{27}$$

Substitute $c_1 = 0$, $c_1 = 1$, respectively, in (27) and compute

$$h(c_0) \mod (\ell^2, x^a), \tag{28}$$
$$h(c_0 + \ell) \equiv h(c_0) + \ell h'(c_0) \mod (\ell^2, x^a). \tag{29}$$

Subtracting (28) from (29) gives

$$\ell h'(c_0) \equiv [h(c_0 + \ell) - h(c_0)] \mod (\ell^2, x^a).$$

Let

$$S = \left\{h(c_0) + \ell c_1 h'(c_0) \;:\; c_1 = 0, \dots, \ell - 1\right\}. \tag{30}$$

Let $f = \frac{A}{B}$ in characteristic 0. We do not know what A and B are. However, from applying rational function reconstruction for $h(c_0)$, we obtain A_0, B_0 with $f \equiv \frac{A_0}{B_0} \mod (\ell, x^a)$. It follows that $f = \frac{A}{B} \equiv \frac{A_0}{B_0} \equiv E_{c_1} \mod (\ell, x^a)$ for an element $E_{c_1} \in S$ defined in (30). From this equation we have

$$A \equiv B E_{c_1} \mod (\ell, x^a). \tag{31}$$

Now let

$$f = \frac{A}{B} \equiv \frac{A_0 + \ell A_1}{B_0 + \ell B_1} \mod (\ell^2, x^a) \tag{32}$$

where $A_1 = a_0 + a_1 x + \cdots + a_{\deg(A_0)} x^{\deg(A_0)}$ and $B_1 = b_1 x + \cdots + b_{\deg(B_0)} x^{\deg(B_0)}$ are unknown polynomials. Here we are fixing the constant term of B. If we can find the unknowns $\{a_i, b_j\}$, then find $f \bmod (\ell^2, x^a)$. Then, from (31), we have

$$(A_0 + \ell A_1) \equiv (B_0 + \ell B_1)[h(c_0) + \ell c_1 h'(c_0)] \mod (\ell^2, x^a). \quad (33)$$

Now, solve the linear equation (33) for unknowns $\{a_i, b_j, c_1\}$ in $\mathbb{F}_\ell$, and from (32) find $f \bmod (\ell^2, x^a)$ and $c \equiv c_0 + \ell c_1 \bmod \ell^2$. Then try rational number reconstruction. If it succeeds, then check if this rational function is the one that we are looking for or not (apply change of variables transformation and try to find the parameter of the exp-product transformation). If it is not, then use the same algorithm to lift $f \bmod (\ell^2, x^a)$ to mod (ℓ^3, x^a) (or (ℓ^4, x^a) if an implementation for solving linear equations mod ℓ^n is available). After a (finite) (we still need to address remark 2) number of steps, we can recover the rational pullback function f.

3.4.2 Lifting for an Algebraic Pullback Function

We can also recover algebraic pullback functions with a very similar method as explained in the previous section. However, in the algebraic pullback case we need to know an algebraic degree bound for f. The idea here is *to recover the minimal polynomial of the algebraic pullback function f*.

Let d_f be a degree bound, and a_f be an algebraic degree bound for f. Consider the below polynomial in y,

$$\sum_{j=1}^{a_f} A_j y^j \mod (\ell, x^a), \quad (34)$$

with unknown polynomials $A_j = \sum_{i=0}^{d_f} a_{i,j} x^i$, $(j = 1, \ldots, a_f)$.

First we need to find the value of c_0 such that $c_0 \equiv c \bmod \ell$. Similarly, by looping on $c_0 = 1, \ldots, \ell - 1$, we can compute the corresponding $f \equiv f_\ell \in \mathbb{F}_\ell/(x^a)$. For this f_ℓ, the polynomial (34) will be congruent to 0 mod (ℓ, x^a) if we plug f_ℓ in y. So, solve the equation

$$\sum_{j=1}^{a_f} A_j f_\ell^j \equiv 0 \mod (\ell, x^a)$$

in $\mathbb{F}_\ell$ and find the unknown polynomials A_j. After finding $c \equiv c_0 \bmod \ell$ and polynomials A_j, then let $c \equiv c_0 + \ell c_1 \bmod \ell^2$. Then f_ℓ also satisfies the polynomial

$$\sum_{j=1}^{a_f} (A_j + \ell \tilde{A}_j) y^j \mod (\ell^2, x^a).$$

in $\mathbb{F}_\ell$ for unknown polynomials $\tilde{A}_j$. Similarly, find the c_1 and unknown polynomials $\tilde{A}_j = \sum_{i=0}^{d_f} \tilde{a}_{i,j} x^i$, $(j = 1, \ldots, a_f)$. After a finite number of lifting steps, and rational reconstruction, we will have the minimal polynomial of an algebraic pullback function f.

3.5 Recovering the Parameter of Exp-product

After finding f, we can compute the differential operator M, such that $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{inp}$. Then we can compare the second highest terms of M and L_{inp} to find the parameter r of the exp-product transformation: If $M = \partial^2 + B_1 \partial + B_0$ and $L_{inp} = \partial^2 + A_1 \partial + A_0$, then $r = \frac{B_1 - A_1}{2}$.

4. FUTURE WORK

We plan to work on finding a method to compute a degree bound and an algebraic degree bound for an algebraic pullback function as well as finding a method to compute candidate GHDOs for algebraic cases. We also plan to use [4] to find a method to reduce equations involving gauge transformation to equations involving only change of variables and exp-product transformations.

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