PROBLEM SET 8: MORE RANDOM PROBLEMS
DUE: MARCH 10

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Problem 1. ***** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous, and periodic with period 1 (i.e., $f(x+1) = f(x)$ and $g(x+1) = g(x)$ for all x). Prove

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx == \left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right)$$

Problem 2. *** Find the integral part (i.e., greatest integer $\leq$ number) of the number

$$\sum_{n=1}^{10^9} n^{-2/3}$$

Problem 3. *** Let a "continuous" roulette wheel be given which has all numbers from 0 to 1. The weel is spun repeatedly, each spin selecting a random real number between 0 and 1. Stop when the sum of the selected numbers is greater than or equal to 1. What is the expected number of spins for this to happen.

Problem 4. *** Let S be a non-empty set with a binary operation $\star$. Given that $x \star (x \star y) = y$ and $(y \star x) \star x = y$ for all $x, y \in S$. Prove: $\star$ is commutative (i.e., $x \star y = y \star x$ for all $x, y \in S$).

Problem 5. **** A chord of constant length moves in a semi-circle. The midpoint of th chord and the projections of its endpoints onto the base form a triangle. Prove that this triangle is ososceles, and it never changes its shape.

Problem 6. *****

(1) FInd a real number $x \neq 0$ such that $x, 2x, \ldots, 34x$ have no 7's in their decimal expansion.
(2) Prove that for any real $x \neq 0$ at least one of $x, 2x, \ldots, 79x$ has a 7 in its decimal expansion.

Problem 7. **** Let $a_1, a_2, \ldots$ be a nondecreasing sequence of positive itnegers such that $x_n = n/a_n \to \infty$ as $n \to \infty$. Prove that x_n is an integer infinitely often.

Problem 8. * A bath has 3 faucets. With faucet 1 open, the bath fills up in 1 minute. With faucet 2 open, the bath fills up in 2 minutes. With faucet 3 open, the bath fills up in 3 minutes. How long will it take to fill up the bath if all three faucets are open?

Problem 9. *** Find integers p, q, r such that

$$2005 = p^3 + q^3 + r^3 - 3pqr$$

(Hint: The polynomial $p^3 + q^3 + r^3 - 3pqr$ factors into 3 linear forms). Can you find *all* integer solutions?

Problem 10. ** Let a, b, c, d, u be integers such that ac, $bc + ad$ and bd are divisible by u. Prove that bc and ad are divisible by u.