PROBLEM SET 7: INEQUALITIES DUE: MARCH 3, 2004

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1. Elementary Inequalities

Perhaps the most fundamental inequality for real numbers is

$$x^2 \geq 0, \quad x \in \mathbb{R}.$$

Using this inequality one can deduce many more inequalities. For example, if we take $x = a - b$ with $a, b \in \mathbb{R}$ we obtain:

$$a^2 - 2ab + b^2 = (a-b)^2 \geq 0.$$

It follows that

$$\frac{a^2 + b^2}{2} \geq ab.$$

This inequality is interesting by itself. If we now substitute $a = \sqrt{y}$ and $b = \sqrt{z}$ we obtain

$$\frac{y+z}{2} \geq \sqrt{yz}.$$

whenever y, z are nonnegative real numbers. Substitution is a very useful method for proving inequalities.

Example 1. Prove that

$$a^2 + b^2 + c^2 \geq ab + ac + bc$$

for all $a, b, c \in \mathbb{R}$. Also prove that equality holds if and only if $a = b = c$.

Discussion. We have to prove that

$$a^2 + b^2 + c^2 - ab - ac - bc \geq 0$$

for all $a, b, c \in \mathbb{R}$. Perhaps we can write $a^2+b^2+c^2-ab-ac-bc$ as a sum of squares. Since $a^2 + b^2 + c^2 - ab - ac - bc = 0$ for $a = b = c = 0$, one should consider squares of functions that vanish whenever $a = b = c = 0$. For example, let's consider the functions $(a-b)^2, (b-c)^2, (c-a)^2$. We have $(a-b)^2 = a^2 - 2ab + b^2$, $(b-c)^2 = b^2 - 2bc + c^2$ and $(c-a)^2 = c^2 - 2ac + a^2$. If we take the sum of these three functions we get

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$$

which is exactly the double of the quantity we are interested in. The proof of the inequality is now easy:

Proof. We have

$$a^2 + b^2 + c^2 - ab - ac - bc = \tfrac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2) \geq 0$$

and it is now obvious that equality holds if and only if $a = b = c = 0$. ☺

Another obvious but important inequality is:

$$xy \geq 0, \quad \text{if } x, y \in \mathbb{R} \text{ and } x \geq 0 \text{ and } y \geq 0.$$

This can be used in many ways. For example if $0 \leq x \leq 1$ then

$$x \geq x^2$$

because

$$x - x^2 = x(1-x) \geq 0$$

and both x and $1 - x$ are nonnegative.

Example 2. Suppose that $x_1, x_2, \ldots, x_n$ are real numbers such that $0 \leq x_i \leq 1$ for all i. Prove that

$$x_1 + x_2 + \cdots + x_n \geq x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_nx_1.$$

When do we have equality?

Discussion. The inequality is not so hard to see, because $x_1 \geq x_1x_2$, $x_2 \geq x_2x_3$, etc. So the inequality is equivalent to

$$x_1(1-x_2) + x_2(1-x_3) + \cdots + x_n(1-x_1) \geq 0.$$

If we have equality then

$$x_1 = 0 \text{ or } x_2 = 1, \quad x_2 = 0 \text{ or } x_3 = 1, \ldots \quad , x_n = 0 \text{ or } x_1 = 1.$$

If $x_1 \neq 0$ then $x_2 = 1$ and in particular $x_2 \neq 0$. From this it follows that $x_3 = 1$. But then $x_3 \neq 0$, so $x_4 = 1$, etc. This way we see that $x_2 = x_3 = x_4 = \cdots = x_n = x_1 = 1$. In a similar way we see that if $x_i \neq 0$ for some i, then $x_1 = x_2 = x_3 = \cdots = x_n = 1$. The only other case where equality holds is when $x_1 = x_2 = \cdots = x_n = 0$.

Example 3. Suppose that $x_1, x_2, \ldots, x_n$ are real numbers such that $0 \leq x_i \leq 1$ for all i. What is the maximum possible value of

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2.$$

Discussion. We are trying to maximalize the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2.$$

(With some analysis one can see that f must have a maximum value, because f is a continuous function on a compact set. Don't worry if you do not understand this. Perhaps we will discuss it later, but we will not use it now.) Let us fix $x_2, x_3, \dots, x_n$, and consider f as a function of one variable x_1. Say $f = ax_1^2 + bx_1 + c$ where $a = n^2 > 0$ and b, c are constants depending on $x_2, x_3, \dots, x_n$. Now f could have a local extremum, but this would always be a local minimum because $a > 0$. The maximum of f is therefore at $x_1 = 0$ or at $x_1 = 1$.

From this discussion it is clear that we can replace x_1 by 0 or by 1 without decreasing the value of $f(x_1, x_2, \dots, x_n)$. Similary, we can replace x_2 by 0 or by 1 without decreasing the value of f etc. So

$$f(x_1, x_2, \dots, x_n) \le f(y_1, y_2, \dots, y_n)$$

for some choices $y_1, y_2, \dots, y_n \in \{0, 1\}$.

So we are looking for the maximum value of

$$f(y_1, y_2, \dots, y_n)$$

where $y_1, y_2, \dots, y_n \in \{0, 1\}$. By symmetry we may assume that $y_1 = y_2 = \dots = y_k = 0$ and $y_{k+1} = y_{k+2} = \dots = y_n = 1$. In that case, the value of $f(y_1, \dots, y_n)$ is $k(n-k) + (n-k)k = 2k(n-k)$. The function $2k(n-k)$ is again a parabola with the maximum at $k = \frac{n}{2}$. But k has to be an integer. It follows that the maximum value of $f(x_1, x_2, \dots, x_n)$ is

$$2\frac{n}{2}(n - \frac{n}{2}) = \frac{n^2}{2}$$

if n is even and

$$2\frac{n-1}{2}(n - \frac{n-1}{2}) = \frac{n^2-1}{2}$$

if n is odd.

Making the right substitutions can be very helpful as the following example shows.

Example 4. Suppose that $a_1, a_2, \dots, a_n$ are real numbers such that $a_i \ge 1$ for all i. Prove the inequality

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \ge \frac{2^n}{n+1}(1 + a_1 + a_2 + \dots + a_n).$$

Discussion. Let us write $a_i = x_i + 1$. Then $x_i \ge 0$ for all i. It is easier to deal with the inequality $x_i \ge 0$ than with the inequality $a_i \ge 1$. The inequality transforms to

$$(2 + x_1)(2 + x_2) \cdots (2 + x_n) \ge \frac{2^n}{n+1}(x_1 + x_2 + \dots + x_n + (n+1)) =$$

$$= 2^n + \frac{2^n}{n+1}(x_1 + x_2 + \dots + x_n).$$

This inequality follows already if we only look at the constant and linear part of the left-handside:

$$(2+x_1)(2+x_2)\cdots(2+x_n) \geq 2^n+2^{n-1}(x_1+x_2+\cdots+x_n) \geq 2^n+\frac{2^n}{n+1}(x_1+\cdots+x_n).$$

because

$$2^{n-1} \geq \frac{2}{n+1}2^{n-1} = \frac{2^n}{n+1}.$$

Problem 1. ** Use the inequality $\frac{x+y}{2} \geq \sqrt{xy}$ repeatedly to prove

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

for all $x, y, z, w \geq 0$.

Problem 2. ** Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{2}{n-1} \sum_{1\leq i<j\leq n} x_i x_j$$

for all positive integers n.

Problem 3. * If $x \leq y \leq z$ and $y > 0$, prove that

$$x + z - y \geq \frac{xz}{y}$$

2. Convexity

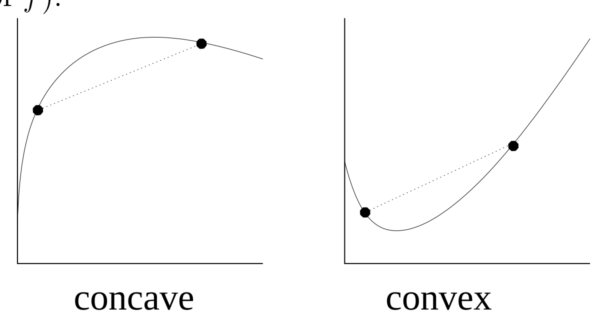
Let f be a real-valued function on an interval $I \subseteq \mathbb{R}$. Now f is said to be *convex* if

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

for all $t \in [0, 1]$ and all $a, b \in I$ (the chord between $(a, f(a))$ and $(b, f(b))$ lies above the graph of f). The function f is said to be *concave* if

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$$

for all $t \in [0, 1]$ and all $a, b \in I$ (the chord between $(a, f(a))$ and $(b, f(b))$ lies below the graph of f).



(You may well be used to a different terminology, for example "concave up" and "concave down" instead of "convex" and "concave".)

Theorem 1. *Suppose that f is a real-valued function on $I \subseteq \mathbb{R}$, $x_1, x_2, \ldots, x_n \in I$, and $t_1, t_2, \ldots, t_n \in [0,1]$ with $t_1 + t_2 + \cdots + t_n = 1$. If f is convex, then*

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n). \tag{1}$$

If f is concave, then

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \geq t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n). \tag{2}$$

Proof. Suppose that f is convex. We will prove the statement by induction on n, the case $n = 1$ being trivial. Suppose that we already have proven that

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n).$$

for all $x_1, x_2, \ldots, x_n \in I$ and all $t_1, t_2, \ldots, t_n \in [0,1]$ with $t_1 + t_2 + \cdots + t_n = 1$.

Suppose now that $x_1, x_2, \ldots, x_{n+1} \in I$ and $t_1, \ldots, t_{n+1} \in [0,1]$ with $t_1+t_2+\cdots+t_{n+1} = 1$. Define $s_i = t_i/(1-t_{n+1})$ for $i = 1, 2, \ldots, n$. Note that $s_1+s_2+\cdots+s_n = 1$. Take $a = s_1x_1+s_2x_2+\cdots+s_nx_n$, $b = x_{n+1}$ and $t = 1-t_{n+1}$. From the definition of convexity and the induction hypothesis follows that

$$\begin{aligned} f(t_1x_1 + \cdots + t_{n+1}x_{n+1}) = f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) = \\ = (1-t_{n+1})f(s_1x_1 + \cdots + s_nx_n) + t_{n+1}f(x_{n+1}) \leq \\ \leq (1-t_{n+1})(s_1f(x_1) + s_2f(x_2) + \cdots + s_nf(x_n)) + t_{n+1}f(x_{n+1}) = \\ = t_1f(x_1) + \cdots + t_{n+1}f(x_{n+1}). \end{aligned}$$

To prove the second statement, observe that f is concave if and only if $-f$ is convex. Then apply the first statement to $-f$. □

In particular the case $t_1 = t_2 = \cdots = t_n = 1/n$ is interesting.

Corollary 1. *If f is convex on I, then*

$$f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

for all $x_1, \ldots, x_n \in I$.

If f is concave on I, then

$$f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) \geq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

for all $x_1, \ldots, x_n \in I$.

Theorem 2. *Suppose that f is a real-valued function on an interval $I \subseteq \mathbb{R}$ with a second derivative. If $f''(x) \geq 0$ for all $x \in I$, then f is convex. If $f''(x) \leq 0$ for all $x \in I$, then f is concave. (The converse of these statements are also true).*

Proof. If $f''(x) \geq 0$ for all $x \in I$ then $f'(x)$ is weakly increasing on the interval I. Suppose that $a, b \in I$ and $t \in [0,1]$. Define $c = ta + (1-t)b$. By the Mean Value Theorem, there exist $\alpha \in (a,c)$ and $\beta \in (c,b)$ such that

$$f'(\alpha) = \frac{f(c) - f(a)}{c - a} \text{ and } f'(\beta) = \frac{f(b) - f(c)}{b - c}.$$

Since $\alpha < \beta$ and f' is weakly increasing, we have

$$\frac{f(ta+(1-t)b)-f(a)}{(1-t)(b-a)} = \frac{f(c)-f(a)}{c-a} = f'(\alpha) \leq$$

$$\leq f'(\beta) = \frac{f(b)-f(c)}{b-c} = \frac{f(b)-f(ta+(1-t)b)}{t(b-a)}$$

Multiplying out gives

$$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b).$$

This shows that f is convex.

The second statement follows from the first statement, applied to $-f$. □

Example 5. Suppose that α, β, γ are the angles of a triangle. Prove that

$$\sin(\alpha) + \sin(\beta) + \sin(\gamma) \leq \frac{3\sqrt{3}}{2}$$

Proof. The function $\sin(x)$ is concave on the interval $[0, \pi]$, because its second derivative is $-\sin(x) \leq 0$. Thus we have

$$\frac{\sin(\alpha)+\sin(\beta)+\sin(\gamma)}{3} \leq \sin\left(\frac{\alpha+\beta+\gamma}{\pi}\right) = \sin(\tfrac{1}{3}\pi) = \frac{\sqrt{3}}{2}.$$

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Problem 4. ** For nonnegative real $u_1, \ldots, u_n$, prove that

$$(\sum_{i=1}^{n} u_i)^3 \leq n^2 \sum_{i=1}^{n} u_i^3.$$

(use that x^3 is convex for $x \geq 0$).

Problem 5. *** Suppose that $p_1, p_2, \ldots, p_n$ are nonnegative real numbers such that $\sum_{i=1}^{n} p_i = 1$. Prove that

$$\sum_{i=1}^{n} -p_i \log p_i \leq \log n.$$

(This inequality comes from *information theory*.)

3. Arithmetics, Geometric and Harmonic mean

Theorem 3. *Let $x_1, x_2, x_3, \ldots, x_n > 0$. We define the Arithmetic Mean by*

$$A(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

the Geometric Mean by

$$G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

and the Harmonic Mean by

$$H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Then we have

$$H(x_1, \ldots, x_n) \leq G(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n).$$

Proof. Let $f(x) = \log(x)$. Then $f''(x) = -1/x^2 < 0$ for $x > 0$ so f is concave on the interval $(0, \infty)$. It follows that

$$\log\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \frac{\log(x_1) + \log(x_2) + \cdots + \log(x_n)}{n}.$$

Applying the exponential function (which is an increasing function) to both sides yields

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

If we now take $y_i = \frac{1}{x_i}$ then we get

$$\frac{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}{n} \geq \frac{1}{\sqrt[n]{y_1 y_2 \cdots y_n}}.$$

Taking the reciprocal yields

$$\frac{n}{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}} \leq \sqrt[n]{y_1 y_2 \cdots y_n}.$$

□

Example 6. Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

Proof. Put $y_i = x_i / x_{i+1}$ for all i. We assume that the index is cyclic, so that $x_{n+1} = x_1$. Comparing the arithmetic and geometric average gives:

$$\frac{y_1 + y_2 + \cdots + y_n}{n} \geq \sqrt[n]{y_1 y_2 \cdots y_n} = 1.$$

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Problem 6. ** For positive real a, b, c prove that

$$b^3c^3 + c^3a^3 + a^3b^3 \geq 3a^2b^2c^2.$$

Problem 7. *** Let

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Prove that

$$n((n+1)^{\frac{1}{n}} - 1) \leq s_n \leq n - \frac{n-1}{n^{1/(n-1)}}.$$

(Hint: use the geometric and arithmetic mean for $1+1, 1+\frac{1}{2}, \cdots, 1+\frac{1}{n}$ and for $1-\frac{1}{2}, 1-\frac{1}{3}, \cdots, 1-\frac{1}{n}$.)

4. The Schwarz Inequality

Another important inequality is the Schwarz inequality. For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ one defines

$$x \cdot y = x_1y_1 + \cdots + x_ny_n.$$

Note that $x \cdot y = y \cdot x$, $(x+y) \cdot z = x \cdot z + y \cdot z$ and $(tx) \cdot y = t(x \cdot y)$ for $t \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^n$.

The norm of the vector x is defined by

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Theorem 4. *Suppose that* $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, *then*

$$|x_1y_1 + \cdots + x_ny_n| \leq \sqrt{x_1^2 + \cdots + x_n^2}\sqrt{y_1^2 + \cdots + y_n^2}$$

or in short form:

$$|x \cdot y| \leq \|x\|\|y\|.$$

Proof. For any vector $a \cdot a \geq 0$. In particular, if we take $a = x + ty$ we get

$$(x+ty) \cdot (x+ty) = x \cdot x + 2t(x \cdot y) + t^2(y \cdot y) \geq 0$$

for all $t \geq 0$. Viewed as a quadratic polynomial in t, this polynomial has a nonpositive discriminant. The discriminant is

$$4(x \cdot y)^2 - 4(x \cdot x)(y \cdot y) \leq 0$$

In particular we have

$$(x \cdot y)^2 \leq (x \cdot x)(y \cdot y)$$

and taking square roots gives us

$$|x \cdot y| \leq \sqrt{x \cdot x}\sqrt{y \cdot y} = \|x\|\|y\|.$$

□

The Schwarz inequality is important in Euclidean geometry in dimension $2, 3$ or higher. In particular, one often defines the angle ϕ between two vectors x, y by

$$\cos(\varphi) = \frac{x \cdot y}{|x||y|}, \quad 0 \leq \varphi \leq \pi.$$

The Schwarz inequality tells us that this definition makes sense, since the right-hand side has absolute value at most 1.

Problem 8. **** Prove the Hölder inequality: If $1/p + 1/q = 1$ and $x, y \in \mathbb{R}^n$ then

$$|x \cdot y| \leq \|x\|_p \|y\|_q$$

where $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. (Hint: Use that $\log(x)$ is convex and prove $x_i y_i \leq x_i^p/p + y_i^q/q$. Then prove the inequality in the special case that $\|x\|_p = \|y\|_q = 1$. Reduce the general case to this special case.)

5. The triangle inequality

Another famous geometric inequality is the triangle inequality. If a, b, c are the lengths of the sides of a triangle, then $a + b \geq c$ (and also $a + c \geq b$ and $b + c \geq a$).

Problem 9. * Let Q be a convex quadrilateral (i.e., the diagonals lie inside the figure). Let S be the sum of the lengths of the diagonals and let P be the perimeter. Prove

$$\frac{1}{2} P < S < P.$$

Problem 10. ** Suppose that we have an triangle with sides a, b, c such that for every positive integer n there exists a triangle with sides a^n, b^n and c^n. Prove that the triangle must be equilateral.

6. One more useful inequality

Theorem 5. *Suppose that $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Suppose that $z_1, z_2, \ldots, z_n$ are the same as $y_1, y_2, \ldots, y_n$, but possibly in a different order. Then we have*

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_n \leq x_1 z_1 + x_2 z_2 + \cdots + x_n z_n \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Proof. Suppose that $z_1, z_2, \ldots, z_n$ is a rearrangement of $y_1, y_2, \ldots, y_n$. Let m be the number of displacements of the sequence $z_1, z_2, \ldots, z_n$, so m is the number of pairs (i, j) with $i < j$ and $z_i > z_j$. We prove the right inequality by induction on m. If $m = 0$ then $z_i = y_i$ for all i and we have inequality. Suppose $m > 0$. Then $z_i > z_{i+1}$ for some i. Note that the sequence

$$z_1, z_2, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_n$$

(exchange z_i and z_{i+1}) has only $m-1$ displacements, so by induction

$$x_1z_1 + x_2z_2 + \cdots x_iz_{i+1} + x_{i+1}z_i + \cdots + x_nz_n \geq x_1y_1 + x_2y_2 + \cdots + x_nz_n.$$

We have

$$(x_{i+1} - x_i)(z_i - z_{i+1}) \geq 0,$$

so

$$x_iz_i + x_{i+1}z_{i+1} \geq x_iz_{i+1} + x_{i+1}z_i$$

and

$$x_1z_1 + x_2z_2 + \cdots + x_iz_i + x_{i+1}z_{i+1} + \cdots + x_nz_n \geq$$
$$\geq x_1z_1 + x_2z_2 + \cdots + x_iz_{i+1} + x_{i+1}z_i + \cdots + x_nz_n \geq x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The left inequality in the Theorem follows from the right inequality. Note that $-y_n \leq -y_{n-1} + \cdots \leq -y_1$ and that $-z_1, -z_2, \ldots, -z_n$ is a rearrangement of $-y_1, -y_2, \ldots, -y_n$. So we have

$$x_1(-z_1) + x_2(-z_2) + \cdots + x_n(-z_n) \leq x_1(-y_n) + x_2(-y_{n-1}) + \cdots + x_n(-y_1).$$

☺

Problem 11. ** Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Prove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \cdots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq x_1 + x_2 + \cdots + x_n.$$

Problem 12. *** Prove that

$$a^a b^b c^c \geq a^b b^c c^a$$

for all positive real numbers a, b, c.

7. Extra problems

Problem 13. * Prove that

$$\frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-1} + x_n} + \frac{x_n}{x_n + x_1} \geq 1$$

Problem 14. **** Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Problem 15. **** Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$= \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Problem 16. **** Let $p_1, p_2, \ldots, p_n$ be any n points on the sphere

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

Prove that the sum of the squares of the distances between them is at most n^2.

Problem 17. ****** [USSR olympiad] Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Prove that

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_n}{x_1+x_2}\geq\frac{n}{4}$$

(indices go cyclic).