PROBLEM SET 6: MORE RANDOM PROBLEMS
DUE: FEBRUARY 18

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Problem 1. * What is

$$(x-a)(x-b)(x-c)\cdots(x-z)?$$

Problem 2. **** Let $a_1, a_2, \ldots, a_n$ be nonzero real numbers, and let $b_1, b_2, \ldots, b_n$ be real numbers with $b_1 < b_2 < \cdots < b_n$.

(a) Show that

$$f(x) = a_1 e^{b_1 x} + a_2 e^{b_2 x} + \cdots + a_n e^{b_n x}$$

has at most $n-1$ real zeroes.

(b) Let m be the number of sign changes, which is the number of i for with $1 \leq i < n$ and $a_i a_{i+1} < 0$. Prove that $f(x)$ has at most m real zeroes.

Problem 3. * Prove that the product of four consecutive terms of an arithmetic progression of integers, plus the fourth power of the common difference, is a perfect square. (An arithmetic progression is a sequence of integers of the form $a, a+d, a+2d, a+3d, \ldots$.)

Problem 4. *** Find all solutions of nonzero positive integers x, y for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$$

Problem 5. ** Let S be a set with 75 elements. Let A, B, C, D be subsets each having at least 25 elements. Prove that some two of these have at least 5 elements in common.

Problem 6. *** Let a, b, c be integers with $a^6 + 2b^6 = 4c^6$. Show that $a = b = c = 0$.

Problem 7. ** Prove that

$$\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}} = \sqrt{6}.$$

Problem 8. *** Let $S \subseteq [0,1]$ be a union of finitely many disjoint closed intervals of total length greater than $4/5$. Prove that the equation $2x + 3y = 1$ has a solution with $x, y \in S$.

Problem 9. *** Find all nonnegative integers n such that $1 + \lfloor\sqrt{2n}\rfloor$ divides $2n$.

Problem 10. ****** Suppose that α, β are angles with $0 < \alpha, \beta < \pi$. There is a round pie on the table. Bob applies the following algorithm: He cuts out a piece with angle α. He take this piece out and turns it upside down. Then he puts this piece back into the cake. Now he turns the whole pie over the angle β (counterclockwise). He cuts out again an α-piece, puts it upside down and moves it back into the pie. He turns the pie again over an angle of β. He keeps repeating this. Show that after a finite number of times, the pie is in its original condition (for example, all the frosting will be on top of the cake again).

Problem 11. ******* For two points $x, y \in \mathbb{R}^2$, let $d(x, y)$ be the Euclidean distance between these two points. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a map that preserves distance 1. This means that for all $x, y \in \mathbb{R}^2$, if $d(x, y) = 1$ then $d(f(x), f(y)) = 1$. Prove that f is an isometrie ($d(f(x), f(y)) = d(x, y)$ for all $x, y \in \mathbb{R}^2$).